Lower-grade municipal bond price risk and sensitivity of price volatility to level of yields

Received (in revised form): 8th January, 2008

S. Lakshmivarahan
is a George Lynn Cross Research Professor in the School of Computer Science at the University of Oklahoma. He has been at the university since 1978 and has had numerous articles published in computer science and meteorology journals.

Duane R. Stock*
is the Price Investments Professor at the University of Oklahoma. He has been at the university since 1979 and has had numerous articles published on fixed income instruments.

*205A Adams Hall, Price College of Business, University of Oklahoma, Norman, Oklahoma 73019, USA
Tel: +1 405 325 5690; Fax: +1 405 325 7688; E-mail: dstock@ou.edu

Abstract Price volatility (variance) is a common measure of bond risk. Previous analysis has shown price volatility to depend upon duration and the volatility of interest rates. A more complete and accurate second-order model shows that price volatility also depends on predicted changes in yield and convexity. Municipal bond price volatility estimated by the more accurate second-order model can be much larger (200 per cent) than estimated first-order volatility. Analysts using only a first-order model can make large errors. Additionally, the effect of yield level on price volatility can be very different in first and second-order models.

Keywords: *volatility, bonds, risk, predictability, ARMA, GARCH, TVPL, municipal*

INTRODUCTION

The US municipal bond market is huge ($2.34 trillion); indeed, according to Hudson and Kim, it is practically half the size of the US government bond market.[1] Banks and mutual funds hold a large proportion of municipal debt. In addition, a large proportion of this type of debt has considerable credit risk and is rated BBB and below. Every investor owning a part of a municipal bond portfolio should be aware of the risks of their particular portfolio. The most common and pervasive municipal bond risk may well be the general interest rate risk that occurs as municipal interest rates naturally change over time. Models of interest rate risk specific to municipal bonds are needed as municipal bond markets are different from other debt markets. This paper briefly gives reasons why yields in other debt markets may change quite differently from municipal yields.

First, consider US Treasury bonds. These have no default risk; as such, their yield changes will not necessarily be strongly correlated with municipal yields, which have significant default risk. Secondly, consider corporate bonds. A major corporate bond default can increase yields on all corporate bonds but may leave municipal yields relatively unaffected. In addition, foreign investors can be major purchasers of corporate bonds, as demonstrated by the surge in foreign buying in the 1990s (eg foreign investors acquired \$79.2bn of corporate debt in 1996[2]). As they cannot enjoy the federal tax exemption, however, foreign investors do not buy significant quantities of municipal bonds. If foreign investors purchase large quantities of corporate bonds, corporate bond yields will tend to decline relative to other debt.[3]

Consistent with the above, municipal bond research has shown the municipal market has unique characteristics that can affect yield changes. Stock and Schrems find that the volatility of short maturity Treasury yields is greater than short municipal yields, but that long maturity Treasury yield volatility is less than long municipal volatility.[4] They suggest this is due to a pattern of bank municipal purchases where banks buy large quantities of short-term municipals during economic expansions and buy much less during recessions (this effect may have weakened with changes in tax codes and other factors; however, this behaviour of municipal yields may still be strong for bank-qualified municipal bonds). Furthermore, Stock has found that the relationship of Treasury and municipal yields depends on maturity, shape of the term structure and default risk.[5]

The purpose of this paper is to develop improved estimates of time-varying municipal bond price volatility where the focus is on lower-grade (BBB) debt. Because predictability in financial asset pricing is becoming increasingly well known, the price volatility model is one that stresses predictability in both expected changes in yields and conditional yield volatility. In fact, the major contribution of this work is to illustrate how predictability in expected yields and conditional volatility have important implications for price volatility models. More specifically, if changes in yields are dependent upon lagged levels of yield, and also display an autoregressive component, then the level of yields has a much enhanced and complex role in models of price volatility. In fact, volatility may be 200 per cent greater than in a simple first-order model. Recent advances in modelling conditional distributions of financial time series have greatly improved models of expected yield changes and conditional variance of yield changes. Cochrane lists return predictability as one of the new facts of finance and maintains that bond yield and returns are predictable.[6] Jones, Lamont and Lumsdaine,[7] and Baker, Greenwood and Wurgler[8] also find that yields are predictable. Furthermore, Kirby, among others, notes how common predictability in mean equations has become in empirical finance.[9] This paper also develops improved models of how price volatility is related to the level of yields in complex ways.

The next section describes simultaneous econometric estimates of changes in BBB municipal bond yields and yield volatility. It is critical to recognise that changes in yields are shown to be autoregressive moving average (ARMA) processes where the first-order autoregressive coefficient plays a critical role. Yield volatility is a special type of generalised autoregressive conditional heteroskedasticity

(GARCH) process where the GARCH process interacts with the yield level. The following section then illustrates how the above econometric estimations can be inserted into a price variance model to give a time series estimate of price volatility. A simple first-order model of price variance is contrasted with a more complex second-order model where the difference is much larger than many market participants would be likely to expect. Then, the following section illustrates that the relation between price volatility and the level of yields is quite complex. Dependent upon the model (first or second-order), maturity and point in time, price volatility may increase or decrease as yields increase. In all cases, the autoregressive aspect of yield changes plays an important role.

EMPIRICAL ESTIMATION OF CHANGES IN YIELD AND YIELD VOLATILITY

The study borrows from Lakshmivarahan and Stock's estimates of daily municipal bond yield changes and yield volatility[10] in which yield data from the Bloomberg financial system (12th February, 1992 to 10th September, 2002) were used to estimate equations. Their results can be summarised as follows (see Table 1). Changes in BBB constant maturity municipal bond yields of five, 15 and 30-year maturities follow an ARMA process. More specifically, yield changes evolve as given below where $\Delta y_{t,M}$ is the yield change for a bond of maturity M. The $a_{1,M}$ value estimates change in yield as dependent on the lagged level of yields (y_{t-1}) and $a_{2,M}$ is the first-order autoregressive coefficient, showing that the change in yield is related to the change in yield for the prior period ($y_{t-1}-y_{t-2} = \Delta y_{t-1}$). Importantly, y_{t-1} is clearly a function of the lagged level of yields. The $a_{1,M}$ and $a_{2,M}$ estimates are clearly significant (except in one case) and play a major role in the price volatility modelling given below. Finally, $b_{1,M}$ reflects a moving average component, ε_t is the error term, and σ_t^2 is the variance of the error term.

$$\Delta y_{t,M} = a_{0,M} + a_{1,M} y_{t-1,M} + a_{2,M} \Delta y_{t-1,M} + b_{1,M} \varepsilon_{t-1,M} + \varepsilon_{t,M}$$
$$\Delta y_{t,M} = y_{t,M} - y_{t-1,M}$$

With respect to volatility of yields, the authors tested two basic theories — the levels theory and the GARCH effect. According to the levels theory, yield volatility is dependent upon the yield level. This expectation is very common because of the scale effect — the greater the level of yields, the greater the base for yield changes. Thus, if the current level of rates is 6 per cent, a proportionate change in rates will generate twice as much of a change in yield than a current level of 3 per cent. According to the popular GARCH effect, however, high volatility tends to persist. The authors found the best model represents yield volatility sensitive to both the level of yields and a GARCH process in a multiplicatively interactive process called TVPL (time-varying parameters, levels). For more details, interested readers are referred to the work of Brenner, Harjes and Kroner,[11] where the basic idea is that conditional volatility (σ_t^2) of yield changes for a particular maturity is given by:

$$\sigma_t^2 = \Psi_t^2 r_{t-1}^{2\gamma}$$

Here Ψ_t^2 follows a GARCH process given as:

$$\psi_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta \psi_{t-1}^2$$

The TVPL process has proven quite popular in modelling interest rate volatility.

In the study by Lakshmivarahan and Stock, the TVPL process was not found to be very applicable for AAA-grade municipal yield volatility because the γ coefficient was not significant.[10] One suggested reason why yield volatility is significantly related to the level of yields for BBB bonds, but not AAA, is that default risk rises with the level of rates (debt service requirements rise), hence bond yields with greater default risk tend to be more volatile. Whatever the underlying theory, there is strong evidence that yield volatility increases with the level of interest rates for BBB yields, and this has important implications for a price volatility model.

MODELS FOR VARIANCE OF PRICE CHANGES

The first-order model

As in Pappu, Lakshmivarahan and Stock, the paper utilises first and second-order models of variance.[12] It is well known that the first-order approximation to change in bond price (P_t) is

$$\Delta P_t = -d_{t-1} \quad \Delta y_t \quad P_{t-1}$$

where d_{t-1} is the modified duration. The bonds initially sell at par ($P_{t-1} = 1{,}000$) consistent with the use of constant maturity yields in this time series analysis. Let $\Delta y_t = z_{t-1} + \varepsilon_t$ be the stochastic dynamic model that describes the evolution of the yield change where expected yield change is

$$E[\Delta y_t | F_{t-1}] = z_{t-1}$$

and F_t is the set of all information available up to time t. Of course, a more specific model for $E\,[\Delta y_t]$ is given as an ARMA model above. The expected error and conditional variance of yield changes are, respectively, $E[\varepsilon_t | F_{t-1}] = 0$ and $E[\varepsilon_t^2 | F_{t-1}] = \sigma_t^2$. The conditional variance of price changes in the first-order model is thus:

$$V_{P1} = \mathrm{Var}(\Delta P_t | F_{t-1}) = d_{t-1}^2 \sigma_t^2 \, P_{t-1}^2$$

Note that first-order conditional variance, V_{P1}, does not depend on the conditional mean, z_{t-1}, from an ARMA model, but only depends on the duration and the conditional variance (σ_t^2) of Δy_t.

The second-order model

The above first-order model for price variance is appealing because it is compact and quite simple. However, it ignores convexity and any predictability of yields. This is critical because, for example, Burghardt, Flores and Johnson stress the importance of convexity in valuing municipal bonds and bond futures.[13] They maintain that market participants need to pay attention to *both* size and direction of yield changes. More specifically, they emphasise that the value of a basis point change in yield depends on both the size of the yield change and the direction of the yield change. Thus, for a futures contract with a face value of $100,000, small errors in a model for variance can lead to expensive errors in hedging and speculation.

The well-known second-order approximation to the change in bond price is given by

$$\Delta P_t = \left\{ -d_{t-1}\ \Delta y_t + \tfrac{1}{2} C_{t-1} (\Delta y_t)^2 \right\} P_{t-1}$$

The paper now reports a more accurate (second order) model of conditional price variance. Substituting and simplifying, one gets

$$\Delta P_t = \left\{ \left[\tfrac{1}{2} C_{t-1} z_{t-1}^2 - d_{t-1} z_{t-1} \right] + \tfrac{1}{2} C_{t-1} \varepsilon_t^2 + g_{t-1} \varepsilon_t \right\} P_{t-1}$$

where C_{t-1} is the convexity of the bond and $g_{t-1} = C_{t-1} z_{t-1} - d_{t-1}$.

Following Pappu, Lakshmivarahan and Stock,[12] the more accurate expression for the second-order conditional variance of ΔP_t is

$$V_{P2} = \left\{ \tfrac{1}{2} C_{t-1}^2 \sigma_t^4 + g_{t-1}^2 \sigma_t^2 \right\} P_{t-1}^2$$

Note that V_{P2} depends on z_{t-1}, d_{t-1}, C_{t-1}, σ_t^2 and σ_t^4, whereas V_{P1} depends only upon d_{t-1} and σ_t^2. Very importantly, z_{t-1}, expected change in yield, depends on the level of yields (y_{t-1}) in the ARMA model.

Difference in first and second-order models of price variance

Municipal bond portfolio managers, municipal bond issuers and those managing futures positions should be compelled to ask whether the extra effort needed to compute the more accurate V_{P2} measure is worthwhile. The answer depends on how far V_{P1} differs from V_{P2}. If the difference is trivial, it is not worth the trouble to estimate the additional parameters needed for V_{P2}. However, the larger the difference, the more compelled a manager should be to make the effort to compute V_{P2}.

Solving for the percentage difference in the first and second-order cases results in

$$\frac{V_{P2} - V_{P1}}{V_{P1}} = Q = r_{t-1}^2 (0.5\ \sigma_t^2 + z_{t-1}^2) - r_{t-1}(2 z_{t-1})$$

where r_{t-1} is C_{t-1}/d_{t-1}.

If there is no predictability in yield changes, $z_{t-1} = 0$ and Q is merely r_{t-1}^2 0.5 σ_t^2. However, in cases where the models of expected changes in yield have predictive power ($z_{t-1} \neq 0$) the difference between the first and second-order model is complex and potentially very large. The mean equation above gives strong evidence of predictability for the ARMA mean equation. In the above definition of Q, r_{t-1}^2 (0.5 $\sigma_t^2 + z_{t-1}^2$) is always positive, but the sign of $-r_{t-1}(2z_{t-1})$ is dependent on the sign of z_{t-1}.

If yields are expected to decline, $z_{t-1} < 0$, this last term always increases a positive difference, especially for long maturities. If yields are expected to increase, $z_{t-1} > 0$, a positive difference is reduced. In fact, a large positive value of z_{t-1} can make Q negative, ie V_{P2} is less than V_{P1}. The sign and precise size of the differences in variance are left to empirical examination as is the impact of z_{t-1}. However, upon analysing the expression for Q, one can develop the expectations of expected sensitivity of Q to r_{t-1}, σ_t^2 and z_{t-1} given below.

First consider r_{t-1}. Given that z_{t-1} values are small compared with r_{t-1}, one expects Q to increase with r_{t-1} (see Appendix A for details). Also, note that convexity is roughly a measure of squared timing of cash flows, whereas duration merely measures the timing. Therefore, one expects r_{t-1} to increase fairly dramatically with maturity and z_{t-1}^2 to increase even more rapidly as maturity increases. Empirical analysis confirms this expectation. Even though

$-r_{t-1}(2z_{t-1})$ can sometimes be negative, one expects the largest absolute Q values to be positive as $r_{t-1}^2\ (0.5\sigma_t^2 + z_{t-1}^2)$ is always positive.

Secondly, consider σ_t^2. It is clear that Q increases with conditional variance estimated from the empirical estimation of yield volatility. High yield volatility is attributed to persistence in past volatility (GARCH) and a high level of yields (γ).

Thirdly, consider the impact of z_{t-1}. Greater negative z_{t-1} values always serve to increase Q as $-r_{t-1}(2z_{t-1})$ is then positive. On the other hand, if z_{t-1} is positive, Q increases (decreases) with z_{t-1} if z_{t-1} is greater (less) than $1/r_{t-1}$. (Appendix B provides the details behind the derivation of $1/r_{t-1}$ as the critical value of z_{t-1}.) Of course, Q is a function of the ARMA mean equation coefficients $a_{1,M}$ and $a_{2,M}$ because these coefficients determine z_{t-1}. However, the relation between Q and these coefficients is very complex given the above. As a simplified example, consider only cases where absolute z_{t-1} is large so that increases in absolute z_{t-1} increase Q. Then, larger absolute values of a_1, a_2, y_{t-1} and Δy_{t-1} all tend to increase Q as they tend to increase absolute z_{t-1}.

Figures 1a, 1b and 1c depict the Q values for five, 15 and 30-year maturities where Q is usually positive but sometimes negative. For the five-year maturity, Figure 1a, the maximum and

Table 1: Results of regressing the change in yield on explanatory variables using the ARMA mean equation for different maturity BBB municipal bonds

	Maturity 5-year	15-year	30-year
$a_{0,M}$	0.0003	*0.006*	0.0066
	(0.889)	*(0.034)*	(0.023)
$a_{1,M}$	−0.0001	−0.0011	−0.0012
	(0.753)	(0.026)	(0.018)
$a_{2,M}$	*0.7122*	*0.6648*	*0.6591*
	(0.000)	*(0.000)*	*(0.000)*
$b_{1,M}$	*−0.5284*	*−0.4557*	*−0.4493*
	(0.000)	*(0.000)*	*(0.000)*
ω	*1.09E−05*	*1.01E−05*	*1.20E−05*
	(0.000)	*(0.001)*	*(0.002)*
$\alpha 1$	*0.0578*	*0.067*	*0.0837*
	(0.000)	*(0.000)*	*(0.000)*
β	*0.9302*	*0.9474*	*0.9433*
	(0.000)	*(0.000)*	*(0.000)*
γ	*0.2409*	*0.4044*	0.3158
	(0.005)	*(0.002)*	(0.002)
L	6,093.841	6,300.286	6,279.302
SBC	−4.5393	−4.6939	−4.6782
$Q(\varepsilon_{12})$	17.208	8.479	5.8222
	(0.142)	(0.747)	(0.925)
$Q(\varepsilon_{12}^2)$	11.771	6.2571	11.171
	(0.464)	(0.903)	(0.514)

$\Delta y_{t,M} = a_{0,M} + a_{1,M}\, y_{t-1,M} + a_{2,M}\, \Delta y_{t-1,M} + b_{1,M}\, \varepsilon_{t-1,M} + \varepsilon_{t,M}$

$\sigma_t^2 = \Psi_t^2\, r_{t-1}^{2\gamma}$

Yield data collected from the Bloomberg financial system, 12th February, 1992 to 10th September, 2002; regressions taken from Lakshmivarahan and Stock.[10] Volatility is estimated simultaneously as a TVPL process.

L, likelihood; SBC, Schwarz-Bayes information criteria. *Q* statistics represent the Ljung-Box tests for serial correlation at lag 12. Numbers in parentheses are significance probabilities (*p*-values); 5 per cent significance shown in italics.

$$Q = r_{t-1}^2(0.5\ \sigma_t^2 + z_{t-1}^2) - r_{t-1}(2z_{t-1})$$

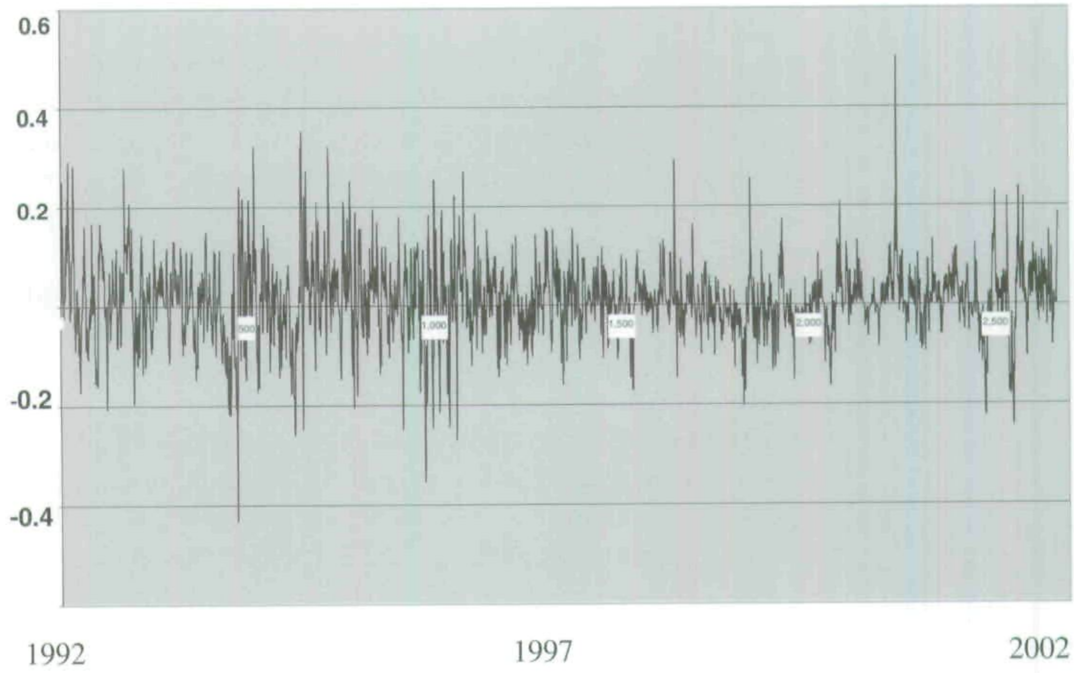


Figure 1a: Percentage difference (Q) between the first and second-order variance, 12th February, 1992 to 10th September, 2002; (a) Five-year maturity

$$Q = r_{t-1}^2(0.5\ \sigma_t^2 + z_{t-1}^2) - r_{t-1}(2z_{t-1})$$

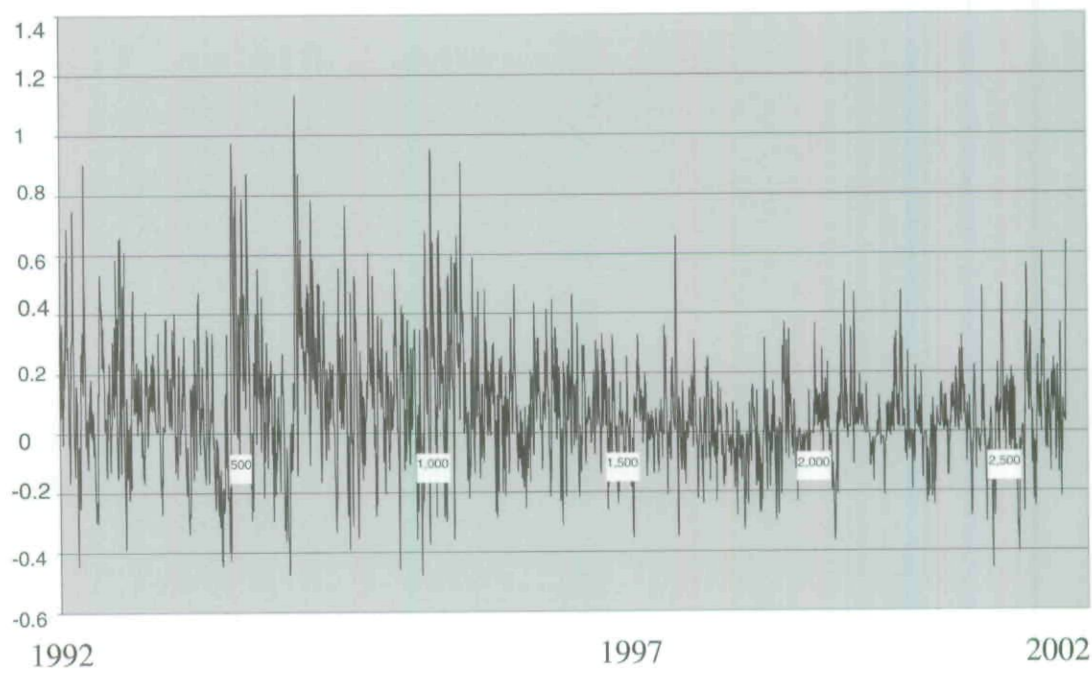


Figure 1b: Percentage difference (Q) between the first and second-order variance, 12th February, 1992 to 10th September, 2002; (b) 15-year maturity

$$Q = r_{t-1}^2(0.5\,\sigma_t^2 + z_{t-1}^2) - r_{t-1}(2z_{t-1})$$

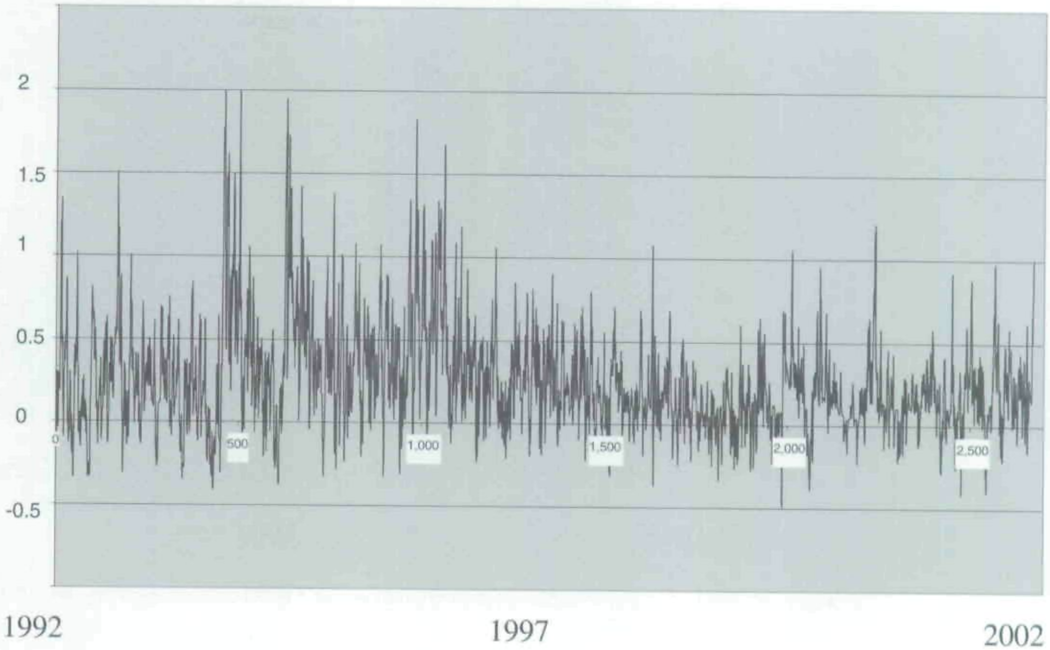


Figure 1c: Percentage difference (Q) between the first and second-order variance, 12th February, 1992 to 10th September, 2002; (c) 30-year maturity

minimum values are quite large and are fairly similar in magnitude (0.500 and –0.424, respectively). Positive values for the 15 and 30-year cases tend to be much larger magnitudes than the negative Q values. Additionally, the positive Q values are sometimes very striking, indicating that it is well worth taking the extra trouble to estimate a second-order model. At one point, the second-order model is practically 200 per cent (Q = 1.998) greater than the first-order model for the 30-year maturity. An analyst using the first-order model would grossly underestimate the conditional price volatility using a first-order model. Further analysis reveals that the pattern for Q is very similar to that of σ_t^2 as given in Figure 2. Generally speaking, Q has the same pattern as σ_t^2 where Q is σ_t^2 strongly affected by r_{t-1}^2 and also affected by other factors to varying degrees.

These results for Q may be contrasted with those of Lakshmivarahan and Stock's analysis of US and UK government bond price volatility.[10] They only use 30-year maturities, assume no predictability in yield changes, and use a much less sophisticated estimate of yield volatility. The highest Q value for any of 12 time periods is only 0.37 and Q is generally much lower. The lowest is 0.06. Q is thus shown to be dramatically higher for municipal bonds of maturities as short as five years. This is largely due to predictability in the mean equation and strong values of z_{t-1}.

THE RELATION OF V_{P1} AND V_{P2} AS DEPENDENT ON YIELD LEVEL (Y_{t-1})

The paper now turns to the sensitivity of the variance of price changes to the level

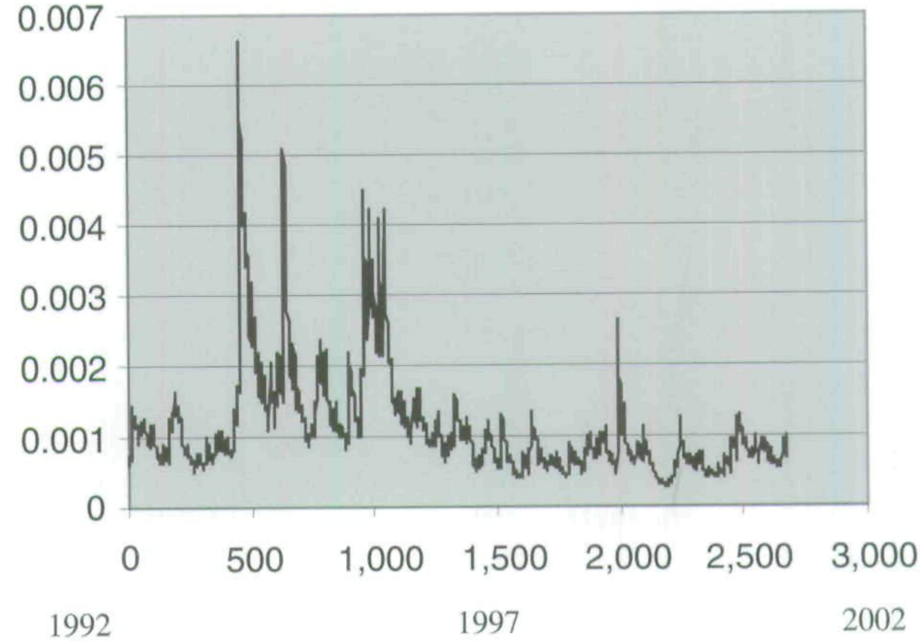


Figure 2: Daily conditional variance of yield changes for the 30-year maturity model, 12th February, 1992 to 10th September, 2002

of yields (y_{t-1}). In the first-order model, the impact of yield on V_{P1} is complex and maturity-dependent in that it can be positive throughout the time period (five-year), change sign at different times (15-year) or be negative throughout (30-year). The more accurate second-order model is complex.

Relation of V_{P1} to yield level

The variance of price change in the first-order model is $d_{t-1}^2 \sigma_t^2 P_{t-1}^2$ where both yield volatility and duration are dependent upon yields. The partial derivatives of σ_t^2 and modified duration with respect to yield are given below. The derivative of yield volatility with respect to level of yield is theoretically positive and numerous empirical results where γ is positive for short rates support the theory.

$$\frac{\partial \sigma_t^2}{\partial y_{t-1}} = \frac{2\gamma\sigma_t^2}{y_{t-1}} > 0$$

Following Brooks and Livingston,[14] a par bond duration is described as:

$$d_{t-1} = \frac{1}{y_{t-1}}\left[1 - \frac{1}{(1+y)^n}\right]$$

Thus, the derivative of par bond duration with respect to yield is:

$$\frac{\partial d_{t-1}}{\partial y_{t-1}} = \frac{(n - d_{t-1})}{y(1+y)} - \frac{(n+1)d_{t-1}}{(1+y)} < 0$$

It is well known that duration declines with yield where the intuition is that the present value (weights) of later cash flows declines more than earlier cash flows when yield increases. Thus, the sign and absolute magnitude of the impact of the level of yields upon variance of price changes depend on the comparative strength of these two effects.

The partial derivative of variance with respect to yield is thus:

$$\frac{\partial [V_{P1}]_t}{\partial y_{t-1}} =$$

$$P_{t-1}^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}} \right] d_{t-1}^2 + 2\, P_{t-1}^2 \sigma_t^2 d_{t-1} \left[\frac{\partial d_{t-1}}{\partial y_{t-1}} \right]$$

$$\textit{Part A} \qquad + \qquad \textit{Part B}$$

One refers to the first term on the right side as Part A, where Part A is clearly positive. Similarly, one refers to the second term as Part B, which is clearly negative.

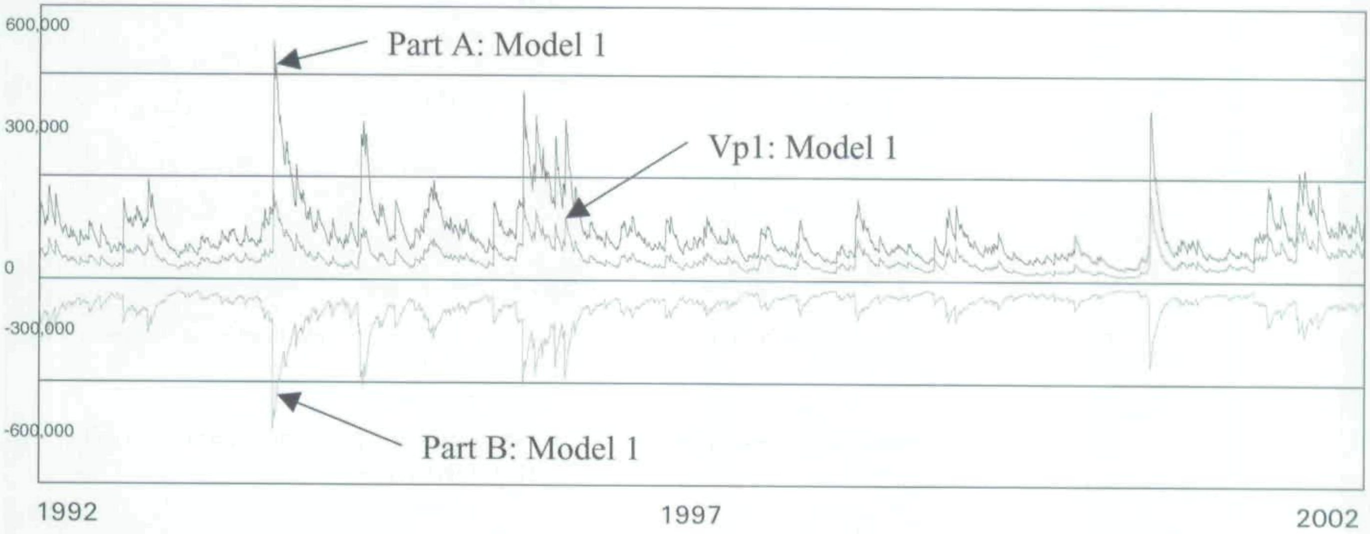
Figures 3a, 4a and 5a plot the first-order empirical results for the three different maturities of BBB municipal bonds. Part A (yield volatility sensitivity) of V_{P1} dominates Part B (duration sensitivity) for the five-year maturity so that price variance always increases with level of yield throughout the time period. In other words, the impact of duration declining with yields is always strong enough for Part A to dominate.

The results for the 15-year maturity are quite different in that Part A does not dominate Part B for the whole time period. That is, Part B has greater magnitude at some times so that the relationship is sometimes negative. In other words, the impact of duration declining with yields is strong enough to generate a negative relation. In general, Parts A and B are roughly offsetting such that net impact of level of yield upon price variance is relatively small throughout.

The results for the 30-year maturity are distinctly different from both the five and 15-year case. Here Part B dominates throughout the time-span so the relation between price volatility and yields is always strongly negative. Table 2 summarises the behaviour of Parts A and B (Model 1 relation, first order) and their sum.

Relation of V_{P2} to yield level

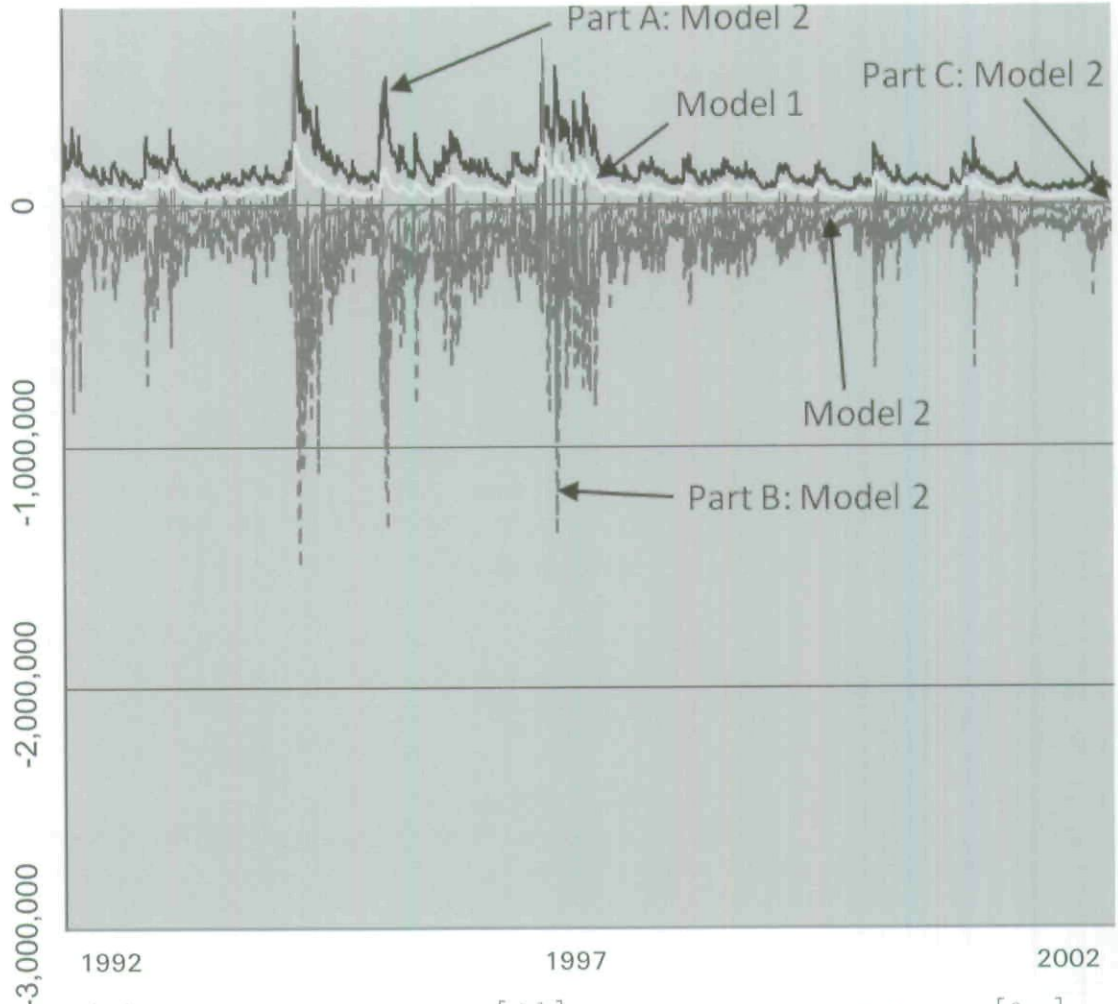
The V_{P2} relation is more complex and it is divided into parts A, B and C. As per



$$\frac{d[V_{P1}]_t}{dy_{t-1}} = P_{t-1}^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}} \right] d_{t-1}^2 + 2\, P_{t-1}^2 \sigma_t^2 d_{t-1} \left[\frac{\partial d_{t-1}}{\partial y_{t-1}} \right]$$

$$\text{Model 1 sensitivity} = \text{Part A} + \text{Part B}$$

Figure 3a: Five-year maturity model: (a) In the first-order model, Part A (yield volatility sensitivity) dominates Part B (duration sensitivity) and price variance increases with level of yields



$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]g_{t-1}^2}_{Part\ A} + \underbrace{2\,P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{Part\ B}$$

$$+ \underbrace{\frac{P_{t-1}^2}{2}\left\{\left[2C_{t-1}\ \frac{\partial C_{t-1}}{\partial y_{t-1}}\right](\sigma_t^2)^2 + 2C_{t-1}^2\ \sigma_t^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]\right\}}_{Part\ C}$$

Model 2 sensitivity = Part A + Part B + Part C

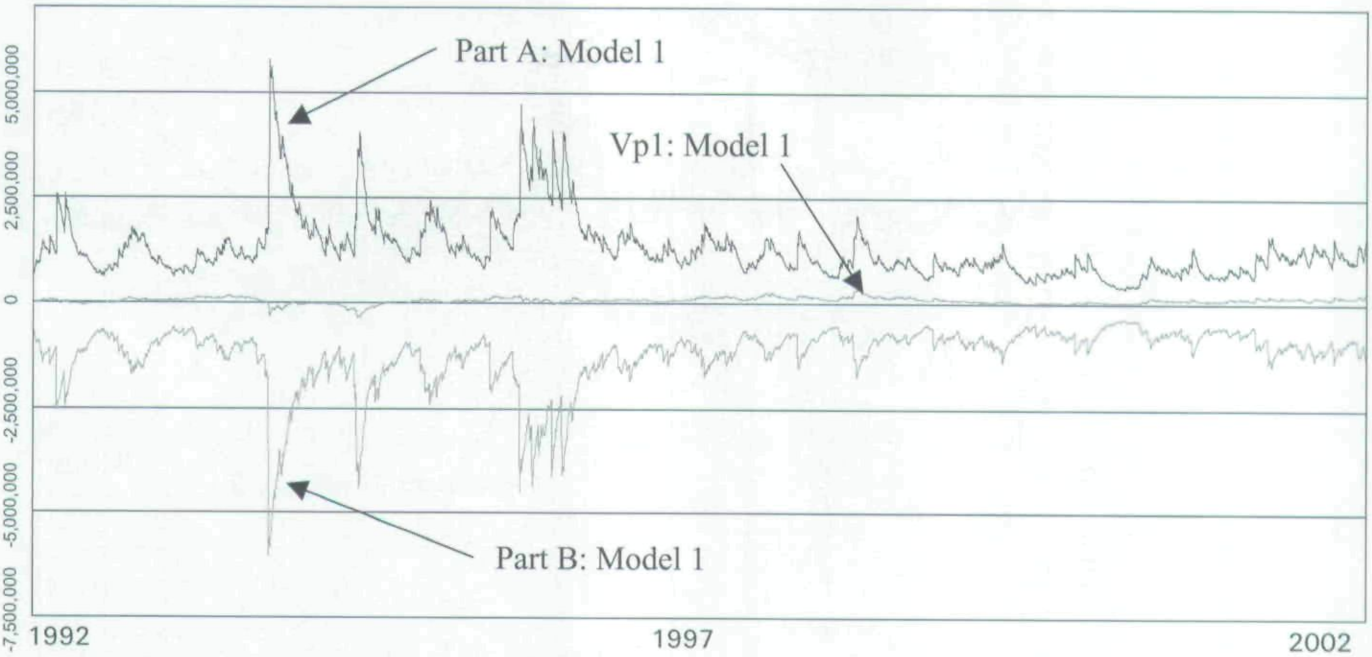
Figure 3b: Five-year maturity model: (b) The sign of the total relation changes over time. In contrast, the sign is always positive in the first order model

Brooks and Livingston,[14] convexity for a par coupon bond is

$$C_{t-1} = \frac{1}{y_{t-1}^2}\left(1 - \frac{1}{(1+y_{t-1})^n}\right) - \left(\frac{n}{y_{t-1}(1+y_{t-1})^{n+1}}\right)$$

Appendix C provides details on pieces of the derivative.

$$\frac{\partial[V_{P2}]_t}{\partial y_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial y_{t-1}}\right]g_{t-1}^2}_{Part\ A} + \underbrace{2\,P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{Part\ B}$$



$$\frac{d[V_{P1}]_t}{dy_{t-1}} = P_{t-1}^2\left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right] d_{t-1}^2 + 2\,P_{t-1}^2\sigma_t^2 d_{t-1}\left[\frac{\partial d_{t-1}}{\partial y_{t-1}}\right]$$

Model 1 sensitivity = Part A + Part B

Figure 4a: Fifteen-year maturity model: (a) For this maturity, neither Part A nor Part B dominates throughout. The sign of the relation changes over time. Parts A and B largely neutralise each other so that the net effect is small

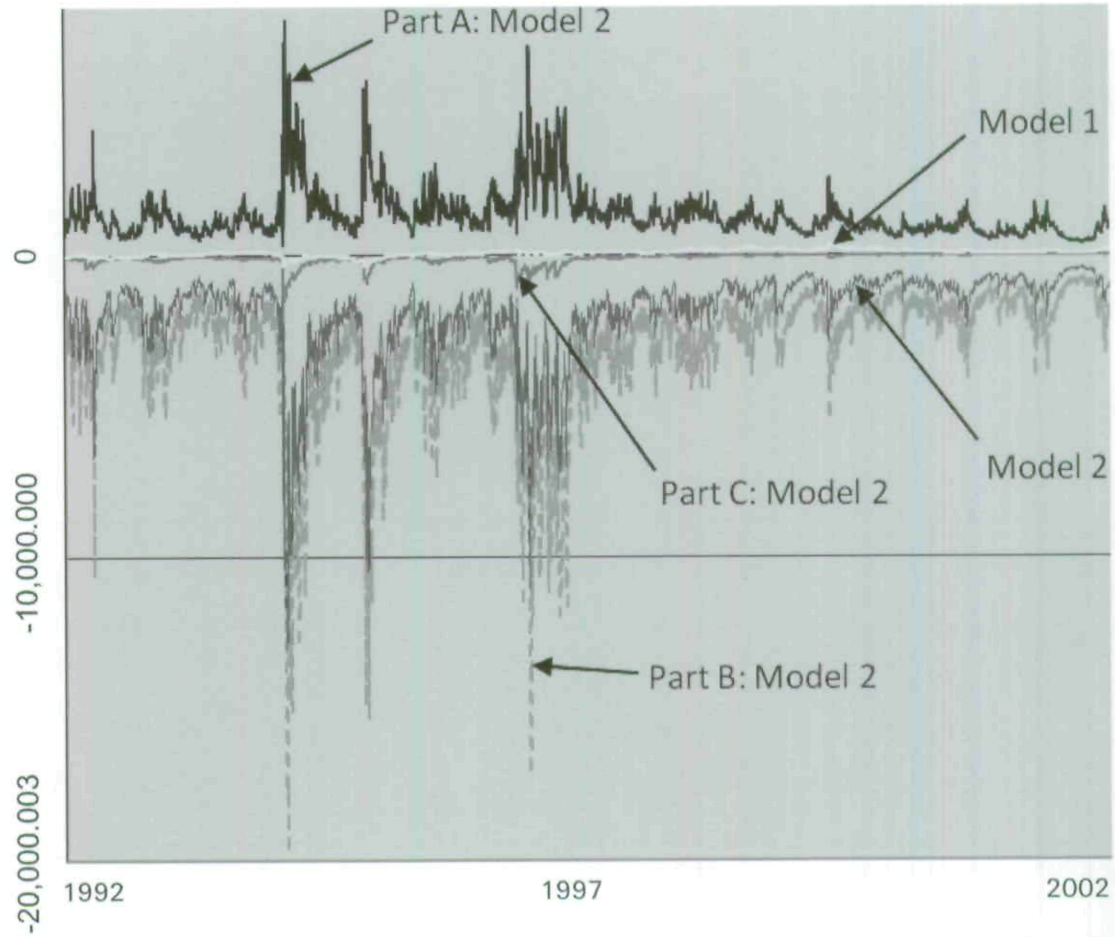
$$+\frac{P_{t-1}^2}{2}\left\{\left[2C_{t-1}\,\frac{\partial C_{t-1}}{\partial y_{t-1}}\right](\sigma_t^2)^2 + 2C_{t-1}^2\,\sigma_t^2\left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right]\right\}$$

Part C

Parts A and B are very similar to V_{P1} where g_{t-1} replaces the role of duration (d_{t-1}) in the second-order derivative. Like V_{P1}, Part A is always positive. The complexity of part B is partially due to the dependence of both g_{t-1} and the derivative of g_{t-1} upon z_{t-1} where z_{t-1} can, of course, be positive or negative. Empirical analysis finds that Part B is always negative, although it has a wide range. The derivative of z_{t-1} with respect to yield, part of the derivative of g_{t-1}, depends upon the coefficients $a_{1,M}$ and $a_{2,M}$ as shown in Appendix C.

Part C represents nonlinear interactive aspects that have no counterpart in the V_{P1} derivative. As others have shown, consistent with the present calculations, the derivative of convexity with respect to yield is negative. Therefore, the first part of the bracketed term of Part C is always negative, while the second term in brackets is always positive. Thus the sign of Part C may be maturity and time-specific.

Figures 3b, 4b and 5b illustrate the estimated second-order empirical relationship. (The figures are best viewed in colour where the total relation for the second-order case is in red.) For all maturities, as in the first-order model, Part A is always positive and Part B is always negative. Part C is always negative for all maturities. Table 2 summarises the behaviour of the total



$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P^2_{t-1}\left[\frac{\partial \sigma^2_t}{\partial y_{t-1}}\right] g^2_{t-1}}_{Part\ A} + \underbrace{2\,P^2_{t-1}\sigma^2_t g_{t-1}\left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{Part\ B}$$

$$+ \underbrace{\frac{P^2_{t-1}}{2}\left\{\left[2C_{t-1}\ \frac{\partial C_{t-1}}{\partial y_{t-1}}\right](\sigma^2_t)^2 + 2C^2_{t-1}\,\sigma^2_t\left[\frac{\partial \sigma^2_t}{\partial y_{t-1}}\right]\right\}}_{Part\ C}$$

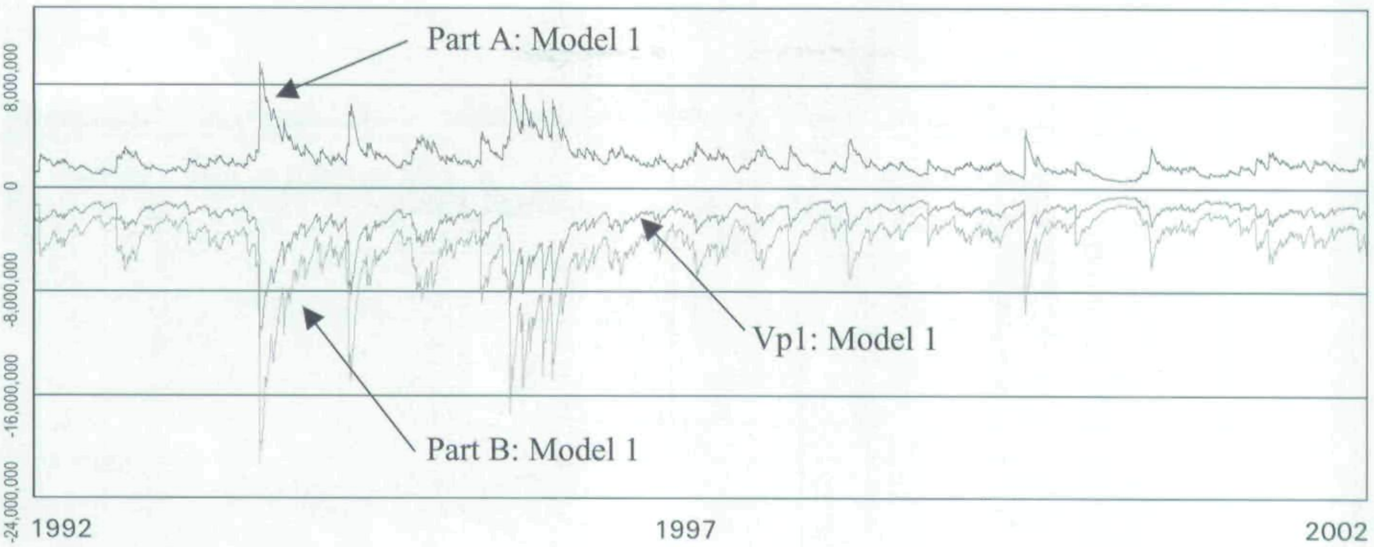
$$\text{Model 2 sensitivity} = \text{Part A} + \text{Part B} + \text{Part C}$$

Figure 4b: Fifteen-year maturity model: (b) For the second order model, the total relation is always negative. This is clearly different from the first order model where the sign varies

relation (Model 2 relation, second order) to level of yields.

With regard to the total relation, first consider five-year (Figure 3b) and 15-year (Figure 4b) maturities. The figures show very different patterns compared with the first order. In fact, the sign of the relation may be different. In the last column of Table 2, the Model 2 entry (+/–) is different from the five-year entry of Model 1 (+). For the 15-year case, the Model 2 total relation entry (–) is also different from Model 1 (+/–). Additionally, the more accurate 15-year Model 2 result shows the magnitude of the total relation to be much greater than Model 1.

In the 30-year case, the sign of the total relation is negative for both first and second-order models, but the magnitudes for Model 2 average approximately 2.5-fold more (see Figure



$$\frac{d[V_{P1}]_t}{d\gamma_{t-1}} = P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial\gamma_{t-1}}\right] d_{t-1}^2 + 2\, P_{t-1}^2\sigma_t^2 d_{t-1}\left[\frac{\partial d_{t-1}}{\partial\gamma_{t-1}}\right]$$

Model 1 sensitivity = Part A + Part B

Figure 5a: Thirty-year maturity model: (a) Part B dominates the relation for this maturity such that the relation is always negative

$$\textit{Model 1}: \quad \frac{\partial[V_{P1}]_t}{\partial\gamma_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial\gamma_{t-1}}\right] d_{t-1}^2}_{\textit{Part A}} + \underbrace{2\, P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial\gamma_{t-1}}\right]}_{\textit{Part B}}$$

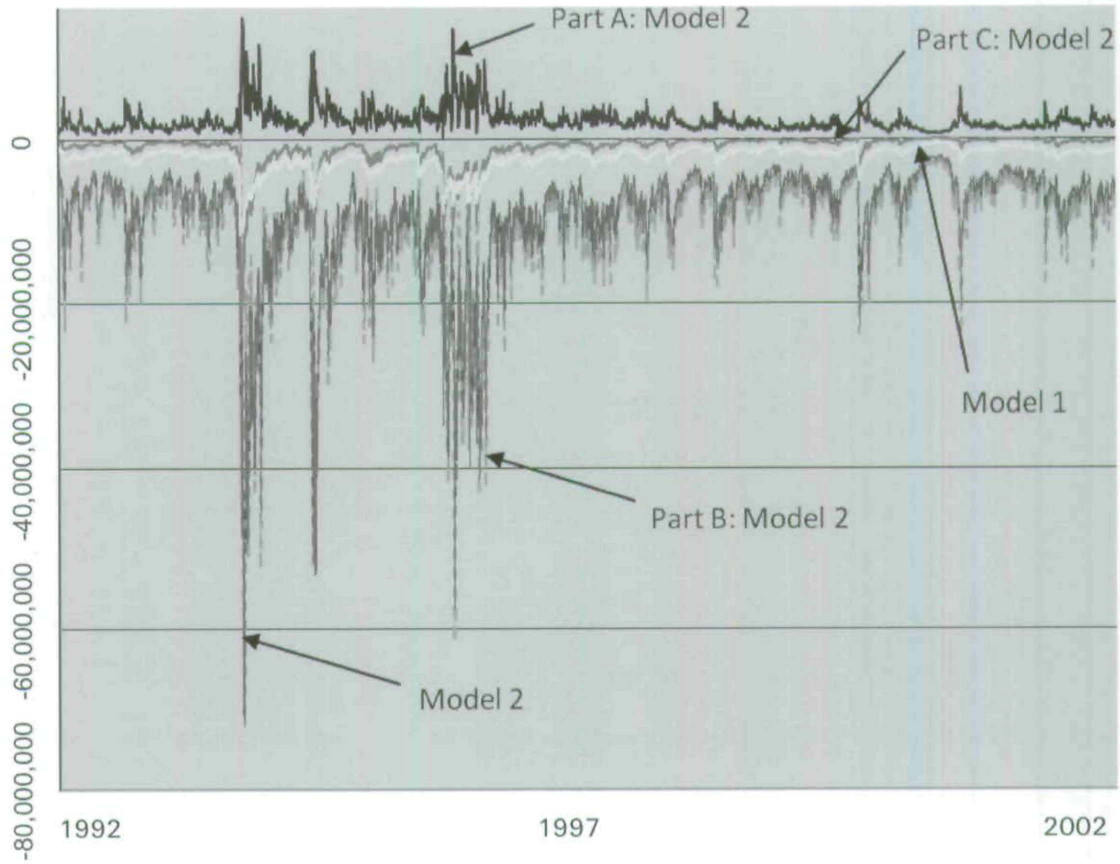
$$\textit{Model 2}: \quad \frac{d[V_{P2}]_t}{d\gamma_{t-1}} = \underbrace{P_{t-1}^2\left[\frac{\partial\sigma_t^2}{\partial\gamma_{t-1}}\right] g_{t-1}^2}_{\textit{Part A}} + \underbrace{2\, P_{t-1}^2\sigma_t^2 g_{t-1}\left[\frac{\partial g_{t-1}}{\partial\gamma_{t-1}}\right]}_{\textit{Part B}}$$

$$+ \underbrace{\frac{P_{t-1}^2}{2}\left\{\left[2C_{t-1}\ \frac{\partial C_{t-1}}{\partial\gamma_{t-1}}\right](\sigma_t^2)^2 + 2C_{t-1}^2\ \sigma_t^2\left[\frac{\partial\sigma_t^2}{\partial\gamma_{t-1}}\right]\right\}}_{\textit{Part C}}$$

Table 2: Signs for parts of model sensitivity

	Model 1 relation (first order)			Model 2 relation (second order)			
Maturity	Part A	Part B	Total relation	Part A	Part B	Part C	Total relation
5-year	+	–	+	+	–	–	+/–
15-year	+	–	+/–	+	–	–	–
30-year	+	–	–	+	–	–	–

5b). Deeper analysis reveals that the much greater sensitivity for Model 2 is due to $\partial g_{t-1}/\partial\gamma_{t-1}$ being much larger than its Model 1 counterpart, $\partial d_{t-1}/\partial\gamma_{t-1}$. In turn, this is due to the large value of $C_{t-1} \times \partial z_{t-1}/\partial\gamma_{t-1}$, where $\partial z_{t-1}/\partial\gamma_{t-1}$ is largely determined by the strong autoregressive nature ($a2$) of yield changes. The difference between Models 1 and 2 would usually be relatively small if not for this autoregressive nature of yield changes.



$$\frac{d[V_{P2}]_t}{dy_{t-1}} = \underbrace{P_{t-1}^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right] g_{t-1}^2}_{Part\ A} + \underbrace{2\, P_{t-1}^2 \sigma_t^2 g_{t-1} \left[\frac{\partial g_{t-1}}{\partial y_{t-1}}\right]}_{Part\ B}$$

$$+ \underbrace{\frac{P_{t-1}^2}{2} \left\{ \left[2C_{t-1} \ \frac{\partial C_{t-1}}{\partial y_{t-1}}\right] (\sigma_t^2)^2 + 2C_{t-1}^2 \ \sigma_t^2 \left[\frac{\partial \sigma_t^2}{\partial y_{t-1}}\right] \right\}}_{Part\ C}$$

$$\text{Model 2 sensitivity} = \text{Part A} + \text{Part B} + \text{Part C}$$

Figure 5b: Thirty-year maturity model: (b) The second order total relation is always negative, as in the first order model; however the magnitude is much larger in the second order

Again, the reader is referred to Appendix C for details of the mathematics.

These results may be contrasted with those of Lakshmivarahan and Stock,[15] where they analyse three-month, six-month, one-year and three-year US Treasury bonds using a more simplistic model of yield volatility based exclusively upon the level of yields. They show that the total relation (total derivative) turns from positive for the shorter maturities to clearly negative at the three-year maturity. The total relation can still be positive at a five-year maturity for a municipal bond.

In summary, analysts can make very large errors in estimating the relation of price variance to yields if only a simple first-order model is used. For the five-year example, an analyst using the first-order model expects a positive relation where the more accurate second-order model reports that the relation can be either positive or negative. For the 30-year case, the negative relation between price variance and yield will be greatly underestimated if the first-order model is used.

CONCLUSION

The municipal bond market is a major part of the US financial system. Many portfolio managers and issuers are responsible for multi-billion dollar positions. Fundamental bond analysis shows that interest rate risk is proportional to duration and volatility of yields. However, more sophisticated analysts should be aware that estimates of conditional variance of prices can be much improved by modelling predictability in both yield changes and conditional volatility of interest rates. The autoregressive nature of yield changes can have a dramatic effect on price volatility. The paper shows that price volatility of the more accurate second-order model reveals volatility that can be as much as 200 per cent greater than the simpler first-order model. Those using a first-order model may well be grossly underestimating price volatility of their portfolio. Furthermore, the relation of price volatility to the level of yields is much different for the more accurate second-order model. For example, the first-order model may suggest a positive relation where the more accurate second-order model reveals a negative.

Acknowledgments

Duane R. Stock wishes to thank the Price College of Business, Center for Financial Studies, and Bank of Oklahoma for support. Matthew Rison, Shengguang Qian, Hamed Bagherpour and Allison Hock provided valuable research support. This paper was previously presented at the University of Groningen in June 2006. Comments by Henk von Eije are appreciated.

References

1 Hudson, M. and Kim, J. J. (2007) 'Should you buy your own city?', *Wall Street Journal*, 20th January, Vol. CC, No. 18, p. B1.

2 Bond Market Association, 'Bonds Outstanding' (2007), January, www.bondmarkets.com.

3 Trzcinka suggests that municipal bonds may be more risky than corporate bonds of equal credit rating because of the dubious value of claims on municipal assets in case of default. Trzcinka also maintains that municipal debt may be more risky because there is no clear objective function for the issue whereas corporate issuers may be assumed to maximise profits. See Trzcinka, C. (1982) 'The pricing of tax-exempt bonds and the Miller hypothesis', *Journal of Finance*, Vol. 37, No. 4, pp. 907–923.

4 Stock, D. and Schrems, E. (1984) 'Municipal bond demand premiums and bond price volatility: A note', *Journal of Finance*, Vol. 39, No. 3, pp. 535–539.

5 Stock, D. (1994) 'Term structure effects on default risk premia and the relationship of default-risky tax exempt yields to risk free taxable yields — A note', *Journal of Banking and Finance*, Vol. 18, No. 6, pp. 1185–1203.

6 Cochrane, J. H. (1999) 'New facts in finance', *Chicago Federal Reserve Bank — Economic Perspectives*, Vol. 23, pp. 36–58.

7 Jones, C. M., Lamont, O. and Lumsdaine, R. L. (1998) 'Macroeconomic news and bond market volatility', *Journal of Financial Economics*, Vol. 47, No. 3, pp. 315–337.

8 Baker, M., Greenwood, R. and Wurgler, J. (2003) 'The maturity of debt issues and predictable variation in bond returns', *Journal of Financial Economics*, Vol. 70, No. 2, pp. 261–291.
9 Kirby, C. (1998) 'The restrictions on predictability implied by rational asset pricing models', *Review of Financial Studies*, Vol. 11, No. 2, pp. 343–382.
10 Lakshmivarahan, S. and Stock, D. (2006) 'Explaining municipal bond volatility: Implications for practice', *Municipal Finance Journal*, Vol. 27, No. 1, pp. 1–33.
11 Brenner, R., Harjes, R. and Kroner, K. (1996) 'Another look at alternative models of the short-term interest rate', *Journal of Financial and Quantitative Analysis*, Vol. 31, No. 1, pp. 85–107.
12 Pappu, N., Lakshmivarahan, S. and Stock, D. (2006) 'Models of conditional variance for bond prices', *Journal of Fixed Income*, Vol. 16, No. 1, pp. 65–70.
13 Burghardt, G., Flores, L. and Johnson, N. (1994) 'Hedging and trading municipal bond futures', *Derivatives Quarterly*, Vol. 1, pp. 49–64.
14 Brooks, R. and Livingston, M. (2001) 'Relative impact of duration and convexity on bond price changes', *Financial Practice and Education*, Vol. 2, No. 2, pp. 93–99.
15 Lakshmivarahan, S. and Stock, D. (2007) 'Less risk for strong returns in bond portfolios', *Journal of Fixed Income*, Fall, Vol. 17, No. 2, pp. 46–62.

APPENDIX A

$$\frac{dQ}{dr_{t-1}} = 2r_{t-1}\left[\frac{\sigma_t^2}{2} + z_{t-1}^2\right] - 2z_{t-1}$$

Given that r_{t-1} values tend to be relatively large compared with z_{t-1}, this derivative is positive for most practical purposes so that Q increases with r_{t-1}.

APPENDIX B

Taking the derivative of R with respect to z_{t-1},

$$\frac{dQ}{dz_{t-1}} = 2r_{t-1}(r_{t-1}z_{t-1} - 1)$$

$$\text{and}\ \frac{d^2Q}{dz_{t-1^2}} = 2r_{t-1}^2 > 0.$$

Q as a function of z_{t-1} is U-shaped. The difference in variance is minimised and the derivative is zero at $z_{t-1} = 1/r_{t-1}$. Thus z_{t-1} reduces (increases) Q when z_{t-1} is less (more) than $1/r_{t-1}$.

Taking the derivative of Q with respect to the AR(1) term, a_2,

$$\frac{dQ}{da_2} = \frac{dQ}{dz_{t-1}}\frac{dz_{t-1}}{da_2} = 2r_{t-1}(r_{t-1}z_{t-1} - 1)(\Delta y_{t-1}).$$

The sign of the impact of a_2 on the percentage difference in variance will be the same as $\frac{dR}{dz_{t-1}}$ if Δy_{t-1} is positive.

APPENDIX C

Details on the second-order relation of price variance are as follows:

$$\frac{\partial g_{t-1}}{\partial y_{t-1}} = \frac{\partial C_{t-1}}{\partial y_{t-1}}(z_{t-1}) + C_{t-1}\frac{\partial z_{t-1}}{\partial y_{t-1}} - \frac{\partial d_{t-1}}{\partial y_{t-1}}$$

where

$$\frac{\partial C_{t-1}}{\partial y_{t-1}} = 2\left\{\frac{2}{y_{t-1}^3}\left[-1 + \frac{1}{(1+y_{t-1})^{2+n}}\right]\right\} + 2\frac{[2(n+1) + n(n+2)y_{t-1}]}{y_{t-1}^2(1+y_{t-1})^{2+n}}.$$

Computations show the derivative of a par coupon convexity is negative for realistic parameters. Finally,

$$\frac{\partial z_{t-1}}{\partial y_{t-1}} = a_{1,M} + a_{2,M}$$