The value at risk of the mathematical provision: Critical issues

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Abstract The paper addresses the calculation of the value at risk (VaR) of the mathematical provision applied in a fair valuation context. Following a balance-sheet approach, the classical definition of VaR needs some clarifications. For indentifying worst cases it is opportune to observe that an increase in the value of the liability corresponds to expenses or, better, additional costs, which may result in either a profit shrinkage or a proper loss. Therefore, the classical portfolio return distribution can be redesigned as a liability cost distribution, where critical values lie in the right-hand tail. In the case of the mathematical provision, the expected cost can be easily linked to the expected value of the reserve at the end of the risk horizon. After an overall view on the VaR problems from a managerial perspective, the paper presents, the choice of the VaR model and the number of risk factors to take into account, in addition to describing the calculation technique. The calculation, performed using a simulation approach, is developed as an application case of a life annuity portfolio and provides an estimate of the worst-case loss at a fixed confidence level after a fixed period of time.

Keywords: *value at risk, life insurance, reserve, solvency, fair value.*

INTRODUCTION

At the end of March 2004, the International Accounting Standards Board (IASB) issued the International Financial Reporting Standard 4 Insurance Contracts.[1] This provides guidance on accounting for insurance contracts, and marks the first step in the IASB's project to achieve the convergence of widely varying insurance industry accounting practices around the world. More specifically, the guidance:

> 'permits an insurer to change its accounting policies for insurance contracts only if, as a result, its financial statements present information that is more relevant and no less reliable, or more reliable and no less relevant'.[1]

Moreover:

> 'it permits the introduction of an accounting policy that involves re-measuring designated insurance liabilities consistently in each period to reflect current market interest rates (and, if the insurer so elects, other current estimates and assumptions)'.[1]

This gives rise to a potential reclassification of some or all financial assets as 'at fair value through profit or loss'. As known, the recognition of a fair value disclosure requirement has to comply with the lack of agreement upon a definition of fair value as well as any guidance from the board on how the fair value has to be calculated. According to the majority of commentators, this uncertainty may lead to fair value disclosures that are unreliable and inconsistently measured among insurance entities. As market valuations do not exist for many items on the insurance balance-sheet, it is necessary to rely on entity-specific measurement for determining insurance contracts and fair asset values. As a consequence, such values may be subject to wide ranges of judgment and to significant abuse. Ultimately, they may provide information that is not at all comparable among companies. The cause for this concern is the risk margin component of the fair value. Risk margins are clearly a part of the market value for uncertain assets and liabilities, but with respect to many insurance contracts, their value cannot be reliably calibrated to the market. Hence, a market-based valuation for them would produce irrelevant information.[2]

This statement gives rise to a wider trouble. If there is an amendment in the evaluation criteria for the reserve from one year to another — according to current market yields or even to current mortality tables — there is a possible change in the value of the reserve according to the application of a more stringent or a more flexible criterion. This may turn into a proper fair valuation risk.[3] From an accounting perspective, the introduction of an accounting policy involving re-measuring designated insurance liabilities consistently in each period to reflect current market interest rates (and, if the insurer so elects, other current estimates and assumptions) implies that the fair value of the mathematical provision is properly a current value or a present value. Substantially, the fair value of the mathematical provision could be properly defined as the net present value of the residual debt toward the policy-holders evaluated at current interest rates and, eventually, at current mortality rates. In a sense, this is marking to market. However, this is the crucial point. The International Accounting Standards Committee defines the fair value as 'the amount for which an asset could be exchanged, or a liability settled, between knowledgeable, willing parties in an arm's-length transaction',[4] while the Financial Accounting Standard Board defines the fair value as 'an estimate of an exit price determined by market interactions'.[4] Hence, the CAS Fair Value Task Force defines fair value as 'the market value, if a sufficiently active market exists, or an estimated market value, otherwise'.[5] From the accounting perspective, a fair value is not necessarily an equilibrium price, but merely a market price and, in the case of the mathematical provision, an estimated market price: therefore, there is an assortment of prices and not a unique reference. The variation of such evaluated price gives rise to the recalled fair valuation risk and opens the path to a quantification of contingency provisions.

From this perspective, this paper addresses the question of calculation and application of the value at risk (VaR) of the mathematical provision in a fair valuation context. It starts with a survey of the issues involved in calculation of the VaR of the mathematical provision and goes on with the mathematical formalisation and an application to a concrete portfolio case fitted out with numerical solutions.

THE RATIONALE OF THE VAR

Typically, VaR is defined as 'the predicted worst-case loss at a specific confidence level over a certain period of time'. This definition is based on the identification of the probability density function for the one-period profit or loss of a portfolio and on the capability of summarising the distribution with a single statistic, often reported as the maximum loss that can occur within a given confidence interval. When this approach is applied to a balance-sheet item and namely to a liability such as the mathematical provision, the classical definition needs some specifications.

To begin with, worst cases coincide with an increase in the value of the liability, because these rises are the counterparty of expenses or, better, additional costs which may result in either a profit shrinkage or a proper loss. Therefore, the classical portfolio return distribution can be redesigned as a liability cost distribution, where critical values lie in the right-hand tail.

Moreover, the portfolio return distribution for assets is centred on the expected return, so the cost distribution for a liability has to be centred on the *expected cost*. In the case of the mathematical provision,[6] the expected cost can be easily linked to the expected value of the reserve at the end of the risk horizon, which is the value of the reserve without any modification of relevant risk factors,[3,7] apart from the time passage. In this perspective the expected reserve is the predicted value of the reserve at the end of the risk horizon calculated using the informative set available at the beginning of the risk horizon.

Finally, the risk horizon choice has to be considered. In a balance-sheet approach, the risk horizon should coincide with the balance period, that is to say with a year-by-year evaluation if the balance is annual or a semester evaluation if the balance is six-monthly. In a risk management approach, the risk horizon could even be different from the balance period.

Therefore, the following is introduced:

Definition 1: The VaR of the mathematical provision is the predicted worst case additional cost at a specific confidence level over a period of time consistent with the risk analysis.

As far as the risk factors are concerned, a full VaR estimation should consider both financial and actuarial risk factors. Setting apart the actuarial risk components, a first-order approximation of the performance variation due to a change in the evaluation interest rate can be obtained through the cash-flow mapping of the reserve. The mathematical provision can be assimilated to a portfolio of zero-coupon bonds — whose nominal values is the certain equivalent of the conditional payment — with maturity equal to each single critical (remaining) instant and with portfolio contribution equal to the ratio of the individual present value of each cash flow to the total present value.

If one concentrates only on the financial risk factor, the expected reserve, in a fair valuation context, is that value of the mathematical provision calculated by means of the forward rates implied by current spot rates. Therefore, the following is given:

Definition 2: The expected cost of the mathematical provision is the algebraic summation of the initial provision and the expected reserve.

In any case in which the summation of the initial and the effective final provision is higher than the expected cost there is an accounting loss. The corresponding potential loss can be effectively measured by means of the reserve VaR, assuming it is possible to produce a significant estimate of the relevant risk factors.[7] In this order of ideas, it is possible to estimate at the beginning of each risk horizon (namely of each year), the maximum additional cost due to a change in the interest rates at a specific confidence level.

The estimation of such VaR can have multiple applications, both internal and external. From a managerial point of view, this measure can serve as a benchmark for the quantification of the contingency provisions and as a guide in the profit distribution. From an institutional perspective, the measure can be effectively applied for comparison among companies, given that a fair valuation system may reduce the comparability of results across different insurance entities. In this perspective, the VaR as a comparability measure could be usefully applied both for accounting standards and for solvency requirements.

THE MATHEMATICAL MODEL

The quantile reserve and the value at risk

The framework in which the life insurance contract analysis will be developed is based on specific assumptions on the financial and the actuarial scenario impacting on the financial situation that will be investigated. The paper poses all the valuations made on the basis of the information flow existing at the business issue time; the behaviour of the financial and demographic risk source will be described according to specific stochastic hypotheses made at certain points during their evolution.

The financial aspect of a life insurance liability flow is a completely fair valuable, but the results are not perfectly replicable for its demographic component. This point is crucial in fair value problems involving life insurance contracts. As Buhlmann states, their existence in the economic reality even if the demographic component is not tradable in an existing market, makes acceptable the fair valuation by means of a portfolio of financial instruments giving origin to a cash flow matching that one underlying the liability flow.[8] As pointed out in literature,[9] the systematic and the accidental components of the demographic uncertainty[10] can be solved using the best estimate of the survival assumptions.

Indicating by $R(t)$ the financial position at time t, that is, in the present case, the stochastic mathematical provision of a life insurance contract or a portfolio of contracts, the quantile reserve at a confidence level α $(0 < \alpha < 1)$, is expressed by the value $R^*_\alpha(t)$ in the following equation:

$$P\{R(t) > R^*_\alpha(t)\} = 1 - \alpha$$

This representation gives rise to a market consistent value of the insurance liability. As a consequence, assessment can be significantly measured by stochastic methods such as the VaR, calculated at the beginning of each year of the life contract, if the balance period is annual.

Consider the time interval $[t,t+h]$ and indicate the financial position at its extremes r(t) and $R(t+h)$ respectively. The potential periodic loss is defined as:

$$L = r(t)\ R(t + h)$$

At the confidence level α, the value at risk, $VaR(\alpha)$ is given by the requirement:

$$P\{L > VaR(\mathrm{a})\} = 1\alpha$$

representing the probable maximum expense; in other words, it means that, in 100 per cent of the cases, the expense is smaller or equal to $VaR(\alpha)$. In terms of the function F, representing the distribution of L, the following expression holds:

$$Var(\alpha) = F^1(\alpha).$$

The valuation scheme

Two probability spaces (Ω,F',P') and (Ω,F'',P'') are introduced, where F' and F'' are the α-algebras containing, respectively, the financial events and the life duration events.

Assuming the independence of the randomness in mortality on the fluctuations of interest rates, (α,F,P) denotes the probability space generated by the preceding two. F contains the information flow about both mortality and financial history, represented by the filtration $\{F_k\} \subset F\left(F_k = F'_k \cup F''_k\right.$ with $\{F'_k \subset F'\}$ and $\left.\{F'_k \subset F''\}\right)$.

If Nj is the number of claims (living or dead according to the kind of life contract) at time j within a portfolio of identical policies and Xj is the amount of the stochastic cash flow referred to the time j, one can evaluate at time t the stochastic stream of cash flow $N_{t+1}X_{t+1}, N_{t+2}X_{t+2}, \ldots, N_nX_n$ that is the stochastic loss at time t.

As usually assumed in a fair valuation framework, the market is frictionless, with continuous trading, no restrictions on borrowing and short sales, the zero bonds and the stocks are both infinitely divisible.

The fair value of the reserve at time t is framed within stochastic assumptions made at time 0, specifically in a perspective valuation context in which the stochastic paths of both the demographic and the financial variables are taken into account.

The following formula gives the fair valuation of the reserve according to a risk-neutral valuation assumption:

$$V_t = E\Big[\sum_{j>t} N_j X_j v(t,j)/F_t\Big] \qquad (1)$$

in which $v(t,j)$ is the present value at time t of one monetary unit due at time j and E is the expectation under the risk-neutral probability measure, whose existence is assumed coherently with the suggestions of FASB 2004,[1] as explained in a general context by Ballotta *et al.*[11]

In the recent literature, several authors have remarked that the demographic valuation is not supported by the hypotheses of the completeness of the market, and to this extent appropriate probability measures have been introduced, as for example recalled in De Felice *et al.*[12] As previously pointed out, the current valuation can be represented practically by means of the expectation, consistent with the best prediction of the demographic scenario. In this order of ideas, a fair valuation procedure involves the latest information on the two main factors bringing risk to the business, ie interest rates and mortality.

In a portfolio perspective and in the case of surviving benefits, if c is the number of identical policies issued at time *0*, one can write:

$$V_t = E\Big[\sum_{j>t} c1_{\{k_{x,t}>j\}} X_j v(t,j)/F_t\Big] \quad (2)$$

where the indicator function $1_{\{k_{x,t}>j\}}$ takes the value 1 if the curtate future lifetime of the insured aged x at issue takes values greater than $t+j$ (j = *1,2,...*) (equivalently if the insured aged $x+t$ survives up to the time $t+j$), 0 otherwise. It is opportune to point out that the demographic valuations for defining the number of payments due to the living insureds are based on the information flow at time 0 on the initial cohort of policies.

By virtue of the basic assumptions on the risk sources, one arrives at the following:

$$V_t = \sum_{j>t} cX_j E\Big[1_{\{k_{x,t}>j\}}/F_t\Big] E[v(t,j)/F_t] \quad (3)$$

As already pointed out, equation 3 is framed in a general view according to a forward perspective in an essentially actuarial approach, originating from the initial position 0. The model can be recalibrated in a spot perspective, in a year-by-year valuation in an essentially accounting approach. In this last case, equation 3 provides the expected value of the reserve at the end of the year, valued at the beginning of the year itself.

The application: A portfolio of life contracts

In this section the VaR calculation is applied to a portfolio of immediate unitary life annuity contracts issued on c lives aged x at issue. The following are posed: $x = 40$, the ultimate lifetime $\omega = 110$ and the time of valuation $t = 10$. The aim is to quantify the two critical values $R^{\star}_{0.95}(t)$and $R^{\star}_{0.99}(t)$ of the reserve distribution at the beginning of the year, that is the quantile reserves $R^{\star}_{\alpha}(t)$ at confidence levels of 95 and 99 per cent. The threshold rule the two values carry out lies in their meaning of the greatest value the reserve can reach in t with probability α% and that will be overstepped with the probability (1-α)%.

Interest rates and mortality scenario

The liability valuation item, inserted in the fair value framework, leads to the choice of a replicating portfolio composed by zero coupon bonds assuming a term structure of interest rates modelled by the Cox Ingersoll and Ross stochastic differential equation (SDE):

$$dr_t = -\beta(r_t - \mu)dt + \sigma\sqrt{r_t}dW_t \qquad (4)$$

where β and σ are positive constants, β the long-term mean, σ the diffusion coefficient, and W_t is the Wiener process. Concerning the times of valuations of interest within the balance-sheet point of view, a year-by-year approach is considered in the forward assumption.

The interest rate model is adjusted on a data panel deduced by the Bank of Italy official statistics concerning the government three-month treasury bill net interest rates, from January 1996 to January 2004 with the starting point $r_0 = 0.0279$; the process parameters are reported in Table 1.

Concerning the choice of the set of values for the demographic component, in order to describe the survival trend in the best estimate approach, one takes into account the survival probabilities deduced from the Italian Male Survival Table RG48, which is a projected mortality table in which the survival probabilities are calculated taking into account the improvement in the mortality trend due to the longevity phenomenon.

Table 1: Parameter estimates

m	b	s
0.045166619	0.026335972	0.005290885

The computational procedure

The reserve distribution $R(t)$ is obtained by means of a simulation procedure, developed with n simulations (n = 1,000, 10,000, 75,000), and can be schematised in the following three basic steps (Cleur *et al.*, 1999).[13]

Step one: one discretises the SDE in equation 4. The discretised process considered here can be represented by the sequence $\{r\Delta, r2\Delta, ..., rk\Delta,\}$, where k is the number of time steps, Δ is a constant and $k\Delta = T$ is the time horizon. In the perspective of the numerical example proposed, concerning the valuation of the annual reserve, $\Delta = 1$. Moreover, as T represents the residual duration of the contract at time t, the result is: $T = \varpi - (x + t)$, T, ϖ and x being the upper limit age and the age at issue of the policy-holder, respectively.

Step two: one discretises the model using the Euler scheme,[14] expressed by the following relation:

$$r_k = r_{k-1} + \beta(\mu - r_{k-1})\Delta + \sigma\sqrt{r_{k-1}\Delta}\,\varepsilon_k$$
$$k = 1, 2, ..., T$$
where $\{\varepsilon_k\} \approx N(0,1)$

The T random values needed for $\{\varepsilon_k\}$ are provided by one simulated path for the stochastic process r_t.

Step three: on the basis of these data the computation of the reserve $R(t)$ is possible by means of the simulated path.[15]

Practical evidence

The proposed example is referred to the case of 1,020 contracts existing at the time of valuation $t = 10$. Table 2 reports the results obtained for the fair value of the reserve $V(t)$ and the two values of the quantile reserve $R^{\star}_{0.95}(t)$ and $R^{\star}_{0.99}(t)$

Table 2: Portfolio of immediate life annuity contracts

	n=1,000	n=10,000	n=75,000
$V(10)$	19,519.30	19,520.49	19,521.79
$R0.95^{*}(t)$	19,663.80	19,669.60	19,668.32
$R0.99^{*}(t)$	19,619.81	19,627.04	19,625.58

c = 1000, x = 40, t = 10, m = 0.045166619, s = 0.005290885, b = 0.005290885; RG48 Italian Male Mortality Table, number of simulation paths: 1,000, 10,000, 75,000.

Table 3: Portfolio of immediate life annuity contracts

	n=1,000	n=10,000	n=75,000
$V(10)$	18,600.76	18,612.46	18,613.34
$R0.95^{*}(t)$	18,752.70	18,754.51	18,753.90
$R0.99^{*}(t)$	18,702.87	18,720.73	18,718.46

c = 1000, x = 40, t = 10, m = 0.045166619, s = 0.005290885, b = 0.005290885; SIM 2002 Italian Male Mortality Table, number of simulation paths: 1,000, 10,000, 75,000.

respectively with confidence levels 95 and 99 per cent, with hypotheses and the procedure introduced in the preceding sections.

The values of the quantile reserve feel the impact of the demographic choice: the application has been developed describing the survival trend by means of a mortality table constructed with a degree of projection based on the improvement in the mortality trend known as longevity phenomenon. This translates into increased survival probabilities gathered at different ages as function of time. A survival model characterised by a lower degree of projection involves consequences on the values shown in Table 2. In order to give the numerical values the system implies when the degree of projection varies, Table 3 lists the results obtained for the fair value of the reserve $V(t)$ and the two values of the quantile reserve $R^{\star}_{0.95}(t)$ and $R^{\star}_{0.99}(t)$ respectively with confidence levels 95 and 99 per cent, when the survival table originates from hypotheses on mortality neglecting the longevity phenomenon. The mortality table chosen is the Italian Male Mortality Table SIM 2002; tables like this represent the future as the past (even if recent), without taking into account that future death probabilities will most likely be lower than the corresponding period-based ones.

The values reported in Table 3 are lower than the corresponding ones in Table 2. In the liability valuations contained in the last table the reduced number of payments each year (and contextually the shorter periods in which the payments will be due) affect the results.

CONCLUDING REMARKS

In conclusion, the implementation in January 2005 of International Financial Reporting Standard 4 Insurance Contracts gives rise to many reporting issues. One of the most relevant is the meaning of the fair valuation of mathematical provisions and the comparison among different but equally achievable and acceptable measures. Such values may be subject to wide ranges of judgment and may provide information that is not comparable among time (reporting dates) and/or space (companies). A way to solve the time comparison question is to express separately the expected value of the reserve and the maximum additional change, through VaR. This consideration reinforces the pregnant role of the VaR analysis in the insurance accounting framework.

Further research on this item will involve the net result of the assets and liabilities, as this value will synthesise the financial position of a life business.

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