Technical note: Application of non-cooperative game theory to market disequilibria

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Abstract Non-cooperative game theory is the (n-person) generalisation of decision theory which examines the outcomes of rational players pursuing strategies of perceived self-interest in an interactive forum. The financial market is such a forum, where hedge funds and banks' proprietary trading divisions are increasingly deploying risk capital in like-minded strategies. This paper shows how the forum can be modelled under a game theoretical construct when consensual trading strategies cluster to a point of liquidity impairment. The paper discusses the relevance of non-cooperative game theory to the risk management modelling of such behavioural interactions. Commencing with a review of the seminal paradox of the Prisoner's Dilemma, the paper develops a simple model for an illiquid capital market, and draws on observations from the Prisoner's Dilemma to examine the conditions under which a capital market may descend into a disorderly collapse. The broader implications for risk management are consequently discussed. The paper assumes a basic familiarity with the axiomatic tenets of game theory as well as the fundamental concept of an equilibrium fixed point, although a review of the Prisoner's Dilemma and a justification for the attainment of the fixed point equilibrium should suffice to allow an intuitive understanding for the non-expert.

Keywords: *non-cooperative game theory, fixed point equilibrium, strategy responses, Nash equilibrium, volatility, illiquidity traps, speculative market*

INTRODUCTION

The propensity for capital markets to exhibit ephemeral bouts of volatility *in extremis* has definitively shifted in recent years and stands in contrast to the secular decline in various measures of cross-asset risk experienced in the first half of this decade. Although one could reasonably believe that the recent injections of volatility perhaps reflect a reversion back to the volatility characteristics exhibited before the decline began, the style of short-term chaos followed by periods of extremely sharp declines in volatility is new (see Figure 1). This summer's credit crunch will, no doubt, join February 2007's globally contagious risk aversion stemming from domestic China's stock market plunge, and the macro inflationary concerns which ignited May 2006's wave of de-leverage, in the annals of risk management war stories.

The predictive inferences in respect of market volatility implied by traditional (value-at-risk) measurement techniques are increasingly belied by the heightened frequency with which such statistical aberrations are occurring (and, in particular, by the relatively short-term nature of the historical data used to calibrate such statistical models). However, these patterns of market disequilibria are entirely rationalised under a game theoretical construct. In the following, the paper presents a pedagogical application of non-cooperative game theory to recent capital markets' behaviour, invoking certain classical results from the theory to demonstrate the conditions which give rise to unsustainable price action and the inevitability, under such circumstances, of market crashes.

THE PRISONER'S DILEMMA

As a motivational introduction, the seminal game theoretical example of the Prisoner's Dilemma can be briefly described as follows. Two suspects are arrested on insufficient evidence to make a successful prosecution of a major crime, although enough evidence exists for a minor offence to be proved. The police separate the two suspects and offer each unilateral plea-bargaining deals with the following possible outcomes:

1. Should neither suspect offer a confession, both prisoners will be prosecuted for the minor offence and serve a short period of incarceration.
2. Should only one of the two suspects confess then the confessor will be released and his partner will be committed to a long sentence in prison.
3. Should both suspects confess, they both will serve a sentence longer than that of the petty crime but shorter than the non-confessor's sentence in outcome 2, above.

It is conventional to represent these outcomes in a canonical form as shown in Figure 2.

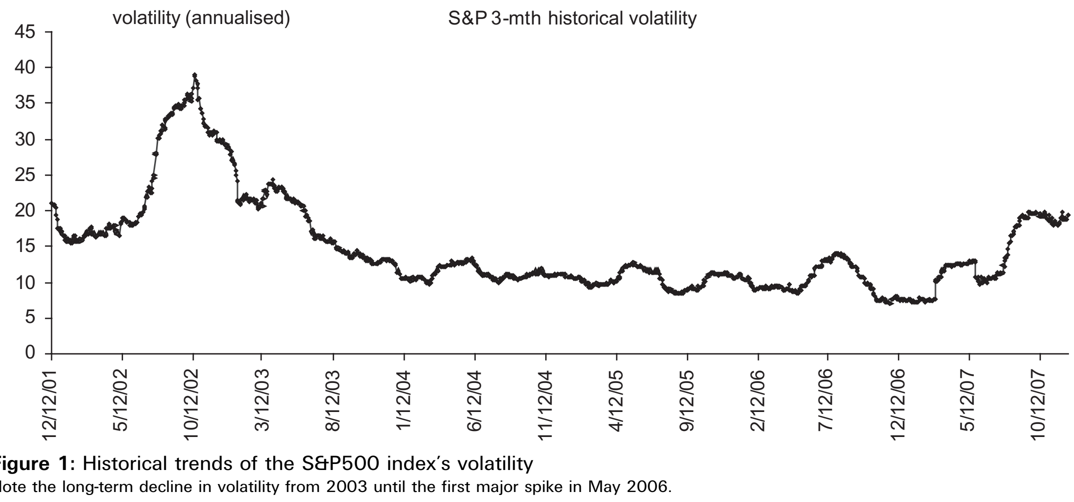

Figure 1: Historical trends of the S&P500 index's volatility
Note the long-term decline in volatility from 2003 until the first major spike in May 2006.

What is a rational strategy to play? Consider player A's choices *given a confession from player B*, as displayed in the highlighted column in Figure 2: player A can confess, resulting in, say, a five-year prison term, or not confess, resulting in a 10-year sentence. He thus rationally confesses, as five years is less than 10 years. Similarly, consider player A's choices *given no confession from player B*: if he confesses then he is released, whereas if he does not then he joins B with a five-year term. He thus confesses hoping for release.

Therefore, player A should confess irrespective of player B's strategy. Rational players expect rational players to play rationally: player B, understanding that player A's best, rational choice is to confess thus, rationally, pursues the best possible response to A's strategy: he confesses too. Players' choices thus attain an equilibrium state of mutual best response to each other. This is the eponymous Nash equilibrium, which proves the existence of a fixed point equilibrium to such non-cooperative games.[1,2]

The Nash equilibrium represents a set of strategies deployed by players whereby any unilateral deviation from the strategy produces a suboptimal outcome for that deviating player. Note this solution is a local equilibrium: it may thus be non-unique and it may be suboptimal to each player's global potential outcomes.

Suboptimality of the solution

Paradoxically, this rational choice of mutual best response represents a suboptimal outcome for each player who would both otherwise receive a minimum sentence in the event they both refuse to confess. Neither player has knowledge of how the other will play. Together with the distribution of players' incentives, this lack of perfect information results in concurrent behaviour which, while rational in the sense of being a best response to the various potential strategies deployed by the other player, avoids the best of all possible outcomes.

In the technical nomenclature of game theory this paradoxical solution is termed *Pareto-inefficient*.

APPLICATION TO RISK MANAGEMENT: THE LIQUIDITY TRAP GAME

This theory has immediate application to financial risk management. Consider, for example, a liquidity trap where investors

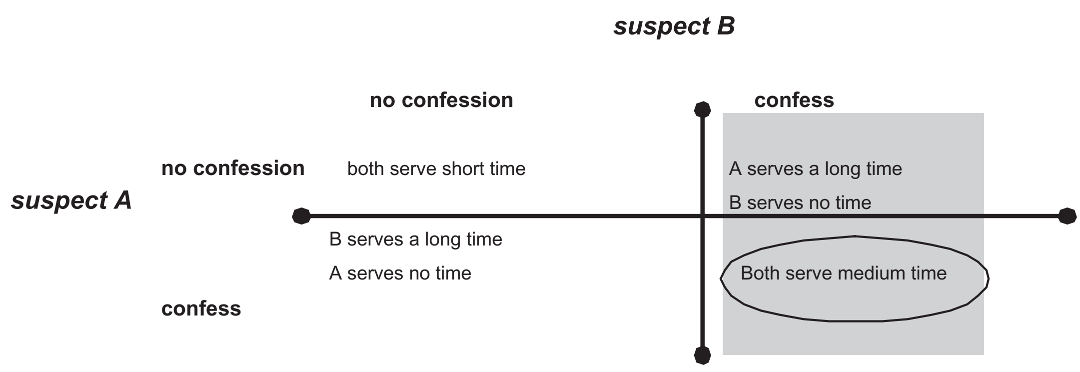

Figure 2: Schematic representation of the choices and outcomes of the Prisoner's Dilemma
Choices for suspect A are on the vertical axis and the choices for B on the horizontal axis. The ringed choice represents the equilibrium point of this game: any unilateral deviation from this strategy results in a worse outcome for the deviant.

hold large quantities of the same highly illiquid asset. Each investor's holding is larger than the secondary market's capacity to absorb the liquidation of the asset. Liquidation would thus provoke a wholesale collapse in market confidence and result in losses to all investors as clearing prices established at fire-sale levels would crystallise losses through the mark-to-market process. At the same time, investors have an incentive in not liquidating to continue to earn carry and maintain holding levels of the risk inventory at nominal (untested) pricing levels. Under this modelling construct there are three possible outcomes:

1. *Neither investor liquidates:* carry is earned and nominal holding price levels are maintained.
2. *Unilateral liquidation:* partial exit, mark-to-market losses on balance of position.
3. *Bilateral liquidation:* small amount of position exited (as the availability of market liquidity is shared across liquidating investors), collapse in prices for residual positions held.

Scenario 1 is the best outcome for all parties.

Under scenario 2, the liquidating investor's outcome is better than that of his counterpart as he exits a portion of his risk (not all due to secondary market liquidity constraints), while his counterpart faces large mark-to-market losses in being left carrying his entire risk inventory at the distressed market levels established as a consequence of his fellow investor's act of liquidation.

In scenario 3, investors exit smaller proportions of the risk than they would have otherwise achieved through unilateral exit in scenario 2 (the same finite secondary market liquidity is now shared between the two investors), although having cleared some of their risk they are both better off than the non-liquidating investor's outcome in scenario 2.

Should either investor decide to refrain from liquidating then the best response from the other investor is to do likewise. This mutual best response outcome is thus, by definition, a Nash equilibrium and remains so as long as the benefits of maintaining the investment exceed the benefits to the seller in scenario 2. In contrast to the Prisoner's Dilemma, there exists a second Nash equilibrium: the bilateral liquidation scenario. The best response to liquidation is to liquidate oneself rather than be the last one fully invested in a falling market whose collapse was triggered by the liquidation of one's fellow investor. In this game, two bimodal equilibria are thus possible.

Parameterisation of the liquidity trap game

The liquidity trap game can be summarised as follows:

- Two investors have holdings of illiquid assets. They have two choices: hold or sell.
- If either chooses to sell they can only sell out 50 per cent of their holding given market illiquidity constraints.
- Selling will clear that portion of the risk for zero cost (assume reserves were maintained to insulate against this loss). However, selling establishes a new market level significantly lower than the pre-sell nominal level. Selling will result in mark-to-market losses for the balance of all holdings.

Figure 3 represents this game in canonical form.

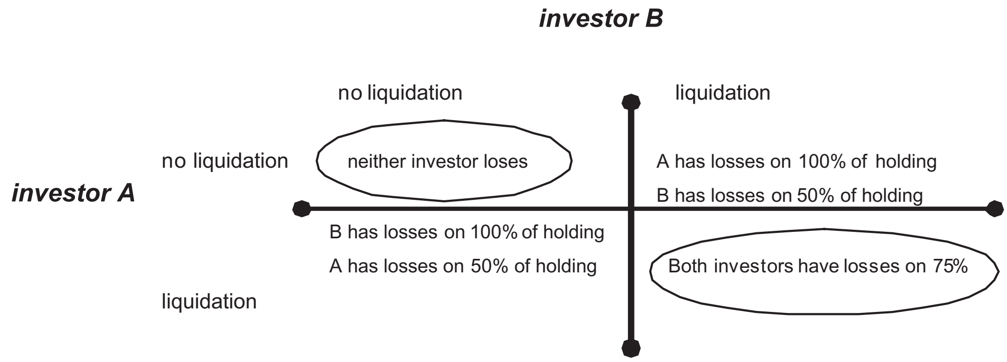

Figure 3: Canonical representation of the 'liquidity trap' game
Suppose investors hold equal amounts of an asset and the market is only capable of absorbing, for example, half the size of each holding. In this game investors have a choice: hold or sell. Liquidation causes a market collapse. Unilateral liquidation will leave the liquidator facing mark-to-market losses on only his residual balance of 50 per cent whereas the non-liquidator will face mark-to-market losses on 100 per cent of his holding. It is assumed, in this idealised model, that investors hold illiquidity reserves which mitigate losses on the first piece of their liquidation. Note the two highlighted equilibria.

The model migrates to the classical Prisoner's Dilemma with a unique equilibrium when the benefits of unilateral liquidation in scenario 2 begin to exceed those of holding the assets in scenario 1. For example, suppose the market begins to clear at lower pricing levels, in small size, exogenous to investors A and B. Observable pricing will necessitate the mark-to-market of investors' positions.

Should either investor formulate an expectation that, given this price discovery, further falls are likely, then this may prompt a 'dash to the exit' under the following constraint:

Modest losses on 100 per cent of holding $>$ Big losses on 50 per cent of holding.[1]

Note that modest losses on 100 per cent of the holding will be equal to the expected losses given exogenous market trading less the carry earned from holding the position (the risk premium of the position). Modest losses may also include an increase in the cost of funding the position and/or increases in the regulatory cost of capital.

In this situation the problem reduces to a single Nash equilibrium: simultaneous liquidation (Figure 4).

This migration explains how a market can exist for some time in a state of benign equilibrium of no liquidation until exogenous factors weaken the benefits of the hold strategy. In this sense, this local equilibrium is unstable. When such factors come into play, players all turn to a liquidation strategy which produces a market collapse: this explains the recent collateralised debt obligation (CDO) subprime crisis where, for years, investment banks carried inventories of essentially the same illiquid structured CDO risk. The exogenous trigger for the model migration came in June 2007 with the public display of brinkmanship between Bear Stearns & Co and the bank creditors of two of its structured credit strategies funds. The threat of fire-sale liquidation of the CDO collateral, thereby establishing market-clearing prices of retained CDO risk, was enough to prompt the rush to the exit for CDO investors.

A more interesting situation arises when equality holds in constraint.[1] For example, the investors may hold

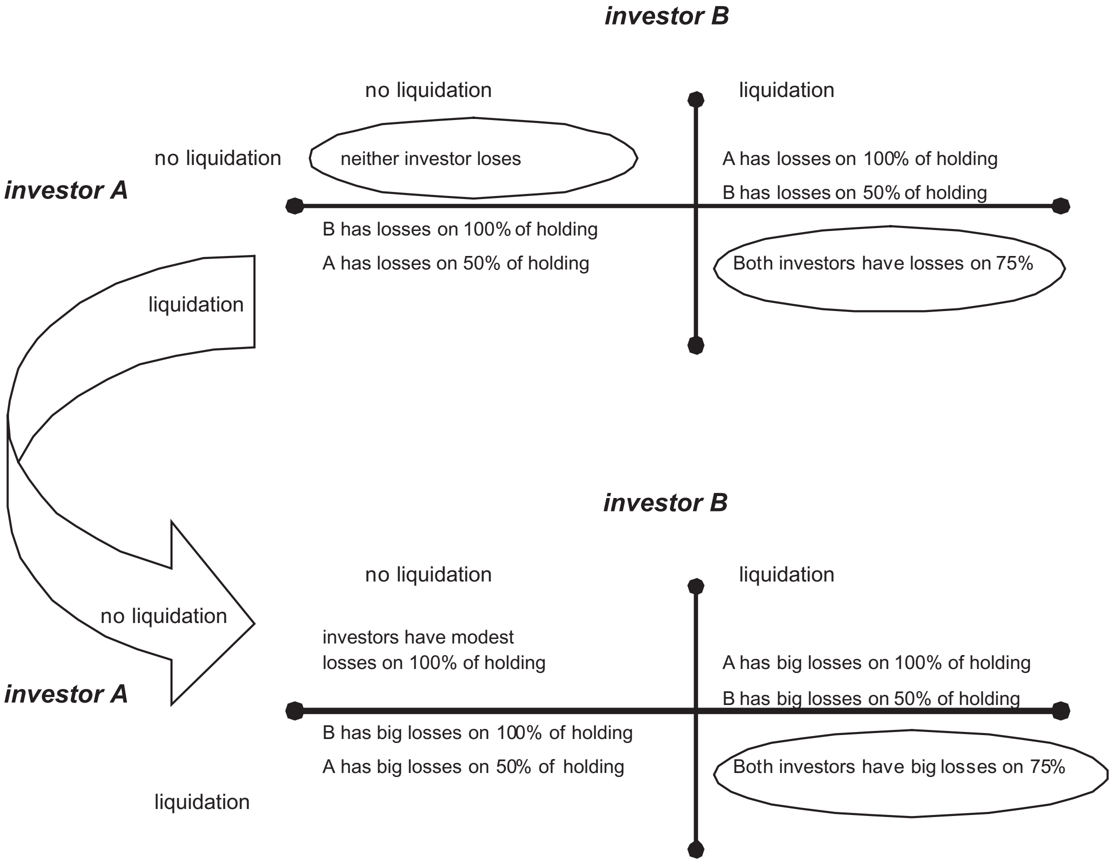

Figure 4: Migration of 'liquidity trap' game to the Prisoner's Dilemma
When the expectation of future losses exceeds the loss involved in unilateral liquidation then the best response is to liquidate irrespective of the other investor's strategy. This problem then reduces to the Prisoner's Dilemma.

additional accounting reserves to insulate the loss from mark-to-market draw-downs on their residual holdings such that they are each indifferent to:

- *their own* unilateral liquidation; or
- their continued holding of the assets.

Specifically, the following constraint is established:

Modest losses on 100 per cent of holding
= Big losses on 50 per cent of holding.

In this case, the mutual hold solution is not a strict equilibrium. This introduces an ambiguity into each investor's behaviour and, more relevantly, to each investor formulating a rational strategy response to the perceived ambiguity in the behaviour of the other. This ambiguity results in an unstable equilibrium: should investor B decide on the hold strategy then A is indifferent from holding or selling under the above equality: if A holds then B is no better off than selling, although if A decides, just as likely, to sell, then the hold strategy is disastrous for B.

One way to recast this problem is to assume player B, aware of the ambiguity in the strategy response brought about by A's indifference, will graft a probability measure on the space of player A's strategy choices. Given A's indifference when B chooses not to liquidate, then it is reasonable to assume he thinks A is equally likely to hold as sell in this situation. Assuming A has a 50 per cent probability to sell and 50 per cent probability to hold when B holds, then B's expected outcome under this probability measure is:

50 per cent × large loss on 100 per cent of holding + 50 per cent × modest losses on 100 per cent of holding.

This is a strictly worse outcome than either of B's alternative outcomes should he choose to sell when this loss exceeds losses under the mutual liquidation scenario:

50 per cent × large loss on 100 per cent of holding + 50 per cent × modest losses on 100 per cent of holding
> large loss on 75 per cent of holding
ie
modest losses on 50 per cent of holding
> big losses on 25 per cent of holding.

That is, whenever the market move under a modest loss event is more than half that under a big loss event. When this inequality holds, B chooses to sell, in response to which A sells and the market meltdown solution of mutual liquidation is unavoidable.

APPLICATION TO SPECULATIVE MARKET DISEQUILIBRIA

An asset bubble is sustained by the continued net inflow of capital to a market. This can be modelled by augmenting the liquidity trap game with the introduction of a third player — a potential new investor. This new player has a choice: invest or sit on the sidelines. Without an essential loss of generality, it is assumed that this investor has capital equal to that of each of the two existing investors:

- An investment strategy of fresh capital will sustain the bubble and both existing investors are better off as a consequence of this fresh investment lifting the market higher, unless both current investors simultaneously deploy the sell strategy, in which case selling will outweigh the influx of fresh capital.
- Should only one investor sell, then the inflow of fresh capital offsets the outflow of capital, maintaining the market equilibrium.

The various scenarios of this game can be represented in two canonical forms (Figure 5).

In this game, there is a proliferation of non-strict equilibria. For example, under the hold strategy for both existing investors, investor C is indifferent with respect to buying or waiting (price gains due to his purchase are shared among existing investors). Given a non-investment strategy by C, then the game reduces to the liquidity trap above and the mutual hold strategy by existing investors is a strict, although non-unique, equilibrium. Should C decide to invest, then again the hold strategy by existing investors is a strict equilibrium (this time unique) since either of their unilateral liquidations or their bilateral liquidations leaves the deviant(s) missing out on the positive price action due to C's investment. From the vantage point of A and B's hold strategy, as in the case of the liquidity trap example, one may assume existing investors A and B graft a probability measure over potential investor C's then indifferent behaviour.

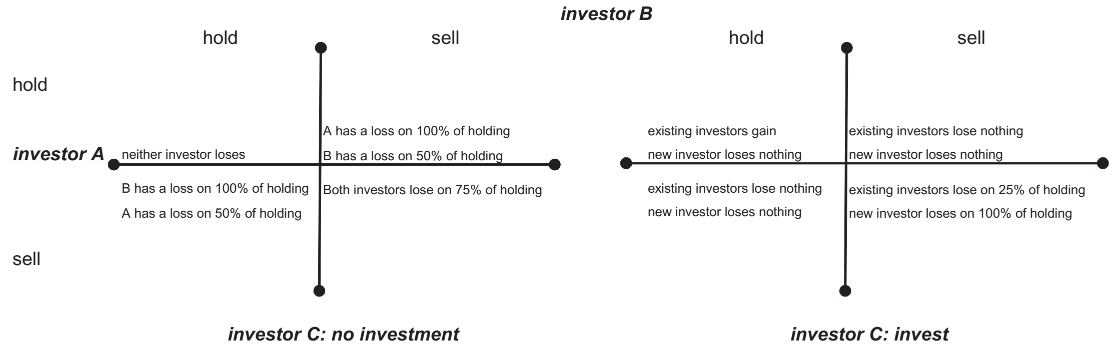

Figure 5: Schematic representation of three-person investment bubble game
Investors A and B are existing holders and have two choices: hold or sell. While investor C, a potential new investor, sits on the sidelines, the game reduces to that of the liquidity trap: selling into a speculative market with no fresh capital to support prices will lead to an illiquid collapse in prices. Should investor C decide to invest then existing investors benefit from the price appreciation brought about by the influx of fresh capital. If either one of the existing investors decides to sell when the new investor decides to buy then the market retains its equilibrium state with a rotation of new and old investors. Should both existing investors decide to sell while investor C buys then the market collapses.

A 50 per cent chance of C's investment yields expected outcomes for investors A and B strictly better than outcomes resulting from deviation from the hold strategy, resulting in a 'quasi' stable equilibrium.

When the conditions of the liquidity trap game migrate to the Prisoner's Dilemma, there becomes one strict equilibrium: A and B exit with investor C sitting on the sidelines. As described earlier, this migration occurs when, irrespective of C's actions, the benefits of unilateral deviation of existing investors exceed the benefits of remaining invested, in which case the solution collapses to a single equilibrium fixed point: mutual liquidation giving rise to a systemic market collapse. As exogenous pressure gradually builds for existing investors (capital requirements, higher value-at-risk costs, tighter monetary policy, higher funding costs, administrative controls etc), thus undermining the benefits of remaining fully invested in a speculative market, non-cooperative game theory predicts, under these conditions, the inevitability of an ultimate disorderly correction to such asset markets.

JUSTIFICATION FOR ATTAINMENT OF THE NASH EQUILIBRIUM

This paper has so far invoked elements from the theory of non-cooperative games to suggest that, given rational responses to the market risks, investors necessarily arrive at a fixed-point equilibrium of behaviour. The bimodal equilibria described above represent the classic investment interplay of 'greed and fear'. When greed wins over fear, investors rationally arrive at the hold strategy. When fear abounds, the rational play is to sell. As soon as the benefits of unilateral selling outweigh holding then there is only one logical outcome: a systemic market collapse triggered by mutual liquidation.

Nothing has yet justified *ex ante* investors' coordinated responses to arrive at a Nash equilibrium. In fact, the Federal Reserve's monetary policy approach in not targeting speculative asset prices has, according to former Fed chairman Alan Greenspan,[3] been based on a belief in the perceived difficulty in preemptively identifying an asset market experiencing a bubble. Montet and Serra (2003) provide an excellent taxonomy of explanations for the inevitability of the Nash equilibrium.[4] The most relevant explanation in the present case is that of the so-called 'Nashian regulator', where an external agent enforces a compatibility of game players' strategies to arrive at a Nash equilibrium.

Ironically, it is central banks and financial authorities themselves, through their capital ratio requirements, which assume this 'Nashian regulator' role in the markets and thus prompt investors to necessarily arrive at the previously discussed equilibrium point of mutual self-destruction. When markets become volatile, capital ratios become impaired as mark-to-market losses accrue, at a time when the cost of regulatory risk capital, as measured by value-at-risk models, increases. This self-enforcing mechanism leads regulated investors towards the liquidation fixed point because liquidation exacerbates volatility, further reducing capital ratios. There has been some discussion of this issue in the academic literature.[5,6]

In the above model, the benefits of unilateral liquidation begin to exceed the hold strategy as the cost of maintaining regulatory capital against retained risk exceeds the carry benefits of maintaining the risk position. As banks are required to uphold capital ratio standards at all times, this volatility feedback mechanism inherent in a capital framework based on value-at-risk inevitably results in the migration of the bimodal equilibrium model into the Prisoner's Dilemma with attainment of the mutual liquidation Nash equilibrium (the systemic collapse) described above. A recent example of this situation was during the volatility in the equity market in 2003 when the UK Financial Services Authority's value-at-risk approach to capital regulation required life insurers to reduce portfolios of risk as their value-at-risk measures rose, which fuelled further price declines. This vicious circle was broken only when the 'rules of the regulatory game' were changed and a moratorium was placed on those capital requirements.

IMPLICATIONS FOR THE MANAGEMENT OF RISK

At a practical level, the results demonstrated above provide useful lessons for risk management. In particular, given the prevalence of like-minded trading strategies which give rise to the aforementioned 'liquidity trap' and the 'speculative trap' game-theoretical models, risk managers need to remain vigilant for commonly-held, crowded trades. This can be achieved by tracking inventories of such consensual strategies, and stress testing those strategies for:

- significant liquidity impairment predicated on the liquidation scenario suggested by game theory above;
- sharp increases in correlation across consensual strategies as the liquidation scenario transmits a latent correlation on the de-leverage mutual liquidation event.

References

1 Nash, J. F. (1950) 'Equilibrium points in n-person games', Proceedings of the National Academy of Sciences, 36, No. 1, pp. 48–49.
2 Nash, J. F. (1951) 'Non-cooperative games', *Annals of Mathematics*, 54, No. 2, pp. 286–295.
3 Greenspan, A. (2007) 'The Age of Turbulence: Adventures in a New World', Penguin, USA.
4 Montet, C. and Serra, D. (2003) 'Game Theory & Economics', Palgrave, New York City, pp. 75–77.

5 Danielsson, J. (2000) 'The emperor has no clothes: Limits to risk modelling', *Journal of Banking & Finance*, Vol. 26, No. 7, pp. 1273–1296.
6 Danielsson, J. and Shin, H. (2002) 'Endogenous risk', working paper of the Financial Markets Group at the London School of Economics, available at: www.riskresearch.org. and at www.nuff.ox.uk/users/shin/working.htm