

HYBRID CAT BONDS

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ABSTRACT

Natural catastrophes attract regularly the media attention and have become a source of public concern. From a financial viewpoint, they represent idiosyncratic risks, diversifiable at the world level. But for various reasons, reinsurance markets are unable to cope with this risk completely. Insurance-linked securities, such as catastrophe (cat) bonds, have been issued to complete the international risk transfer process, but their development is disappointing so far. This article argues that downside risk aversion and ambiguity aversion explain their limited success. Hybrid cat bonds, combining the transfer of cat risk with protection against a stock market crash, are proposed to complete the market. The article shows that replacing simple cat bonds with hybrid cat bonds would lead to an increase in market volume.

INTRODUCTION

Natural catastrophes such as hurricanes, floods, earthquakes, and tsunamis attract regularly the attention of media. Some of them lead to huge human losses, without much impact in terms of economic losses: e.g., 300,000 victims from floods in November 1970 in Bangladesh, or 138,000 victims from cyclone Gorky in the same country in 1991,¹ but insignificant or low insured losses (\$0.3bio for cyclone Gorky). Others lead to huge economic and insurance losses, without many victims: e.g., \$22bio insured losses for Hurricane Andrew in Florida and Bahamas in 1992, but “only” 43 victims. Still others—more recently—emerge in statistics with simultaneously huge economic and insurance losses, and huge death tolls: e.g., the December 2004 tsunami in the

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¹Unless otherwise indicated, statistics are from Sigma (2007). To take inflation into account, all values are adjusted using the U.S. consumer price index to give current (2007) values.

Indian Ocean (220,000 victims, \$14bio economic losses, and \$2bio insured losses), or Hurricane Katrina in August 2005 in Louisiana, Mississippi, and Alabama (1,300 victims, an estimated \$125bio economic losses, and \$45bio insured losses²).

During the last decade, catastrophe-linked securities were very often presented as the appropriate tool to deal with the financial consequences of natural catastrophes, even in the case of poor developing countries (see Freeman, 2001). Catastrophe (cat) risk securitization has occurred using different financial instruments: cat options, such as those proposed by the Chicago Board of Trade (CBOT) for some years (see Cummins and Geman, 1995; Hoyt and McCullough, 1999), cat equity puts (see Doherty, 2000) and cat bonds (see Loubergé, Kellezi, and Gilli, 1999; Cummins, Lalonde, and Phillips, 2004; Nell and Richter, 2004). Cat bonds have met the most success. However, the success so far has not been as high as one could expect when they first appeared in 1996. The market remained stagnant until 2005 with a few issues per year (a maximum of 10 in 1999 and 2005) and a total of \$5bio of capital outstanding at the end of 2005—far below the \$66bio of insured losses for Hurricane Katrina that same year. In 2006 and 2007, the market experienced its takeoff, with 20 issues in 2006 for a total of \$4.69bio and 27 in 2007 for a total of \$7bio. At year-end 2007, outstanding capital had raised to \$13.8bio, nearly three times the 2005 figure (see Guy Carpenter, 2008). This takeoff is welcome, as it tends to show that the securitization of catastrophe risk is well on its way, but it is also fair to say that the years 2006 and 2007 provided favorable conditions for market growth. The year 2006 followed a record year 2005 for natural catastrophe losses: a contraction in supply and more stringent conditions on the reinsurance market provided incentives for cedents to turn to cat bond issues. After low losses in 2006, the market could have slowed down in 2007, but the “subprime crisis” and its impact on the corporate bonds and credit derivatives market led many investors to look for alternative and noncorrelated investment opportunities. As a result, the cat bond market experienced an inflow of funds leading to softer conditions for cat bond sponsors (see Guy Carpenter, 2008). Even if this long-awaited takeoff in the market would be confirmed in the coming years, there is still a long way to go to reach an outstanding capital comparable to the \$66bio in Katrina losses. Note, in addition, that total outstanding capital, measured by the total bond principal, does not represent the actual amount available to the cat bond sponsor for payment of losses: this amount is generally limited to coupons and part of the principal.

Natural catastrophes represent systematic risks at the regional or national level, but their economic impact should be diversifiable at the world level using already available international reinsurance markets, without any need to issue cat-linked securities. However, these markets are not as efficient as could be expected. The recent experience is characterized by chronic shortage of capacity, widely fluctuating prices, and reinsurers’ combined ratios far in excess of 100% (see Sigma, 2002). Once catastrophes have occurred, some participants are pulled out of the market, and risks are not easily transferred until new capital flows in. As a result, major natural catastrophes in developed markets are seriously underinsured (see Sigma, 2007). Different arguments have been provided to explain this situation. Some invoke parameter risk (severity and frequency risk), which would prevent more capital to be invested in the cat reinsurance business. Indeed, the severity of natural catastrophes tends to increase due to

²See Moreau (2005).

concentration of values in exposed areas, such as ocean coasts and earthquake-prone regions,³ and the frequency of storms and hurricanes seems to have increased as well in the recent past, maybe due to global warming (although long-term fluctuations in frequency cannot yet be excluded) (see Sigma, 2005).

In addition to this uncertainty about the severity and frequency parameters, one can observe that the demand for cat insurance coverage is concentrated in a few exposed high-income areas of the world: California, Florida, Texas, Japan, and Western Europe. According to Moreau (2005), 80% of world insured losses due to natural catastrophes are concentrated in the United States. (In 2004, this share was only 68%, due to the South Asia tsunami and other catastrophes in Asia absorbing 25% of losses.) In contrast, the North American cat insurance premium income amounts to 40% of total. An adverse selection process is taking place here, with bad risks crowding good risks out of the market. This questions the solidity of reinsurers operating in this market segment. In the long run, they will have to leave the market or be exposed to bankruptcy when a major event occurs in a high exposure area (see Cummins, Doherty, and Lo, 2002).

Thus, although the risk of losses from natural catastrophes is in principle diversifiable internationally using reinsurance, the uncertainties surrounding the long-term evolution of this risk, its concentration in some regions of the world, and the even greater concentration of demand for coverage in a subset of the latter regions has led to the emergence of catastrophe-linked securities, as possible substitutes for reinsurance. Cat risk securitization presents a number of advantages. First, it relies on the wide international financial market to spread the risk globally (see Kielholz and Durrer, 1997). Second, securitization is a vector of risk disaggregation. Capital is specifically and directly used to support the transfer of cat risk, instead of being allocated to reinsurers involved in a large spectrum of insurance and financial risks. Third, securitization allows optimal risk sharing among market participants, instead of uniform risk spreading if a worldwide mandatory system of catastrophe insurance was eventually organized. If the risk is truly diversifiable globally, i.e., if cat losses are uncorrelated with world wealth, its assumption should not return a risk premium. If, however, part of the risk is undiversifiable, e.g., parameter risk leading to possible mispricing of the cat-linked securities, a risk premium will emerge and the risk will be optimally shared among market participants, in proportion to their risk tolerance.

Given these advantages, why were developments in the markets for cat-linked securities relatively disappointing so far? The modest market increase is all the more surprising that financial returns from cat bonds were better than initially expected.

³As explained in Sigma (2005, p. 13), "The example of Florida is a particularly good illustration of this trend: the number of residents in this state increased by 70% between 1980 and 2001. In the same period, the state's gross domestic product soared by 130%. In 2004, three hurricanes in succession touched land in Florida: Charley, Frances and Jeanne; a fourth, Ivan, also passed over Florida after making landfall in Alabama. Because of the high concentration of assets, the state of Florida alone suffered insured property damage of \$19bn." (Note that the combined total insured losses of these four hurricanes, \$28bio, exceed the record losses from hurricane Andrew in 1992, \$22bio.)

In spite of the uncorrelation with market risk,⁴ and their fully collateralized nature, most of these bonds are rated BB by Standard & Poor's and offer large spreads over LIBOR, in particular those covering multiple perils. Moreover, until Katrina in 2005, losses on principal had never occurred, at least on publicly disclosed cat bond issues.⁵ Nevertheless, cat bonds do not substitute for reinsurance. They are used to complete the market, to fill a gap in market niches, mainly the high-severity, low-frequency layer where traditional reinsurance is unattractive to cedants for two reasons: high premium rates and substantial counterpart credit risk.⁶

Several explanations were already proposed. First, it is often remarked that cat bonds are new exotic securities and that investors need to become familiar with the concept. This argument is unconvincing. Cat bonds have been issued for 10 years now and more exotic securities, such as those used to transfer credit risks, have had a huge success. Products, such as credit default swaps, credit spread options and total-rate-of-return swaps, have allowed firms and financial institutions to more effectively deal with credit risks—even, in some cases, with excessive enthusiasm, as the “subprime crisis” has shown. For the same reason, the argument that financial institutions, such as pension funds, are not allowed to invest in derivative instruments, such as cat bonds, is unconvincing.

Second, parameter uncertainty, as noted before, may be a reason why the market for cat bonds has difficulties to takeoff. But parameter uncertainty is not limited to cat risk. It also affects the financial market and is a potential source of market incompleteness (see Mukerji and Tallon, 2001). In reality, parameter uncertainty and ambiguity aversion did not prevent the stock market volumes to soar over the past two decades. Thus, parameter uncertainty, considered in isolation, is unlikely to explain the stagnation in the annual number and volume of cat bond issues.

Moral hazard and basis risk are other possible reasons for the low volume of cat bonds, but both must be relevant to explain the lack of real takeoff in the market. Moral hazard may be relevant when the activation of the cat bond is triggered by the insurance

⁴According to calculations made by Bank Leu, the 5-year (2000–2005) correlations of a cat bond portfolio with stock market indices are low but not zero: 0.09 with the S&P 500, 0.15 with the DJ EuroStoxx 50, and 0.12 with the Swiss Market Index (see Graemiger Theler, 2005). In contrast, Hoyt and McCullough (1999) found zero correlations. But they investigated the correlation between the stock market and the PCS index used as a basis for cat options. As noted by Loubergé, Kellezi, and Gilli (1999), the call feature imbedded in cat bonds makes these securities sensitive to interest rate risk. For this reason, their model predicts a nonzero correlation between cat bond returns and stock market returns, even if cat losses are uncorrelated with stock market returns.

⁵See MMC Securities (2005). In spite of the high losses due to Florida hurricanes and Japanese typhoons in 2004, no cat bond was triggered by these events. However, a 2004 Japanese earthquake did activate an \$85 million second event tranche part of a 2003 issue sponsored by Swiss Re. According to MMC securities, “The activation of a second event tranche indicates that the occurrence of the next qualifying event may trigger a loss to the tranche’s principal” (p. 25).

⁶The model presented by Nell and Richter (2004), combining reinsurance and cat bonds in an expected utility framework, obtains precisely this result: the cedant uses reinsurance for small losses and index-linked cat bonds for the higher layers of losses.

sponsor's record of claims due to a natural catastrophe. In this case, investors may be worried by the sponsor's incentive to inflate claims reported once a catastrophe has occurred. If moral hazard binds, the obvious response is to issue cat bonds based on an industry index of losses (such as a PCS index⁷), or on a parametric index, the mere occurrence of a catastrophic event with defined and observable parameter characteristics (wind speed and location of a hurricane, magnitude and location of an earthquake). This was the more common form of cat bond issues over the past years. In this case, moral hazard disappears, but basis risk emerges. The payout from the cat bond does not fully hedge the sponsor's actual cat losses (see Cummins, Lewis, and Phillips, 2002; Nell and Richter, 2004). Therefore, to explain the low volume of issues, both moral hazard and basis risk should be a severe problem. This is not excluded, but unlikely.

Another possible reason motivates this article. It is the aversion to downside risk among investors, combined with parameter uncertainty. In particular, investors can show an aversion against the ambiguity on the effective dependency between the occurrence of a natural catastrophe and that of a market crash. Catastrophes convey a notion of dread. Their financial impact on insurance companies returns is easily associated with the impact of a market crash on stock portfolio returns. Although losses from natural catastrophes have been historically uncorrelated with financial market returns, it is not guaranteed that this absence of interaction will still hold in the future. A potential causality effect between an important catastrophe and a market crash is not so unrealistic. The terrorist attack on the World Trade Center on September 11, 2001, had simultaneous impact on the market for cat reinsurance and the stock market. More recently, the stock market reacted to the Katrina event. Not only insurance company stocks, but also oil stocks, dropped significantly—although no crash occurred. Similarly, potential rare disasters such as world wars, epidemics of diseases, and large-scale natural catastrophes are often invoked in the asset pricing literature to argue that a significant probability of a deep financial market downturn helps to solve the equity premium puzzle (see Barro, 2006; Rietz, 1988).

This article argues that the volume of cat bond issues would likely increase if intermediaries issued hybrid cat bonds, financial instruments combining a simple cat bond and a protection against a simultaneous drop in stock market prices. The latter could be conceived as a down-and-in digital put option on a market index, with the occurrence of the catastrophe as activating barrier. In this case, investors would be protected against downside risk. If a relevant catastrophe or a stock market crash occurred, the diversified investor would lose only part of her investment. If both happened, the downside risk would be hedged by the activation of the put protection. Using the concept of zero-premium options, this put protection could be financed by the simultaneous issue of a up-and-in digital call option. This latter option would be activated if no natural catastrophe was recorded and the financial market experienced at the same time a sufficient increase in stock prices. In such a case, investors would gain on both sides of their investment and would be more prepared to pay a known amount, as determined in the call component of the hybrid transaction. One could argue that such hybrid structures can be replicated by the investors themselves, and

⁷The claims indices calculated by Property Claims Services (PCS) were formerly used as underlyings for the CBOT cat options.

therefore the current absence of protection against combined events should not have prevented investors from buying cat bonds. But several decades of developments in the financial market have shown that financial intermediation is useful in providing sophisticated structures to investors—even if the latter could have replicated these structures by constructing portfolios of primary assets and other available instruments, such as derivative securities. Hybrid cat bonds would simply represent an additional structured product in a wide population of such products.

The model presented in the article uses the concept of convex risk measures to present a game involving three players: an insurer/reinsurer exposed to a risk of catastrophic loss, an investor already holding a diversified portfolio of investment, and a financial intermediary seeking to maximize the economic surplus from the cat bond transaction, under the risk measure constraints of the two other agents. The convex risk measures introduced by Föllmer and Schied (2002b) are particularly well suited to a context of downside risk, as they provide an improvement with respect to the much used value at risk (denoted by $V@R$) concept—a nonconvex risk measure. More specifically, our model is based on the modified risk measures introduced by Barrieu and El Karoui (2003, 2005). These modified measures increase the tractability of convex risk measures when dealing with risk transfer issues for investors already holding a diversified portfolio of securities. In addition, a special class of convex risk measure, the entropic risk measure, allows a convenient link with expected utility theory, a familiar concept for financial economists.

The “Framework and Notations” section presents the model and the risk measure concepts used throughout the article. In the “Analysis of the Simple Transaction” section, the base case of a simple cat bond transaction is developed. The “Analysis of the Hybrid Transaction” section introduces hybrid cat bonds and derives the optimal trading conditions for the three players concerned by the transactions: the insurer/reinsurer, the financial intermediary, and the investor. Finally, the “Impact of the Hybrid Component on the Volume of the Transaction” section analyzes the effect of introducing hybrid cat bonds on the volume of cat bond issues. Our main result shows that hybrid cat bonds provide more volume than simple cat bonds. They enlarge the class of insurance-linked securities and allow better diversification of cat risk in the financial market.

FRAMEWORK AND NOTATIONS

General Framework

In a universe described by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}$ the prior probability measure, a given economic agent, typically an insurer or a reinsurer—agent A —is exposed to the occurrence of a natural catastrophe. If a catastrophe occurs, the amount of losses is denoted by X . This amount is supposed to be known in advance. The randomness comes from the occurrence of the catastrophe itself. The time horizon is $T > 0$ and the random event “occurrence of a catastrophe” is denoted by τ . Hence, $\{\tau \leq T\}$ denotes the occurrence of a natural catastrophe before time T , while $\{\tau > T\}$ characterizes the situation where no catastrophe occurs.

Agent A wants to issue a cat bond to hedge his exposure. In this study, we consider successively two different types of products for the potential investors and look at their impact on the volume of the possible transactions. In each case, different agents

are involved: agent *A*, the “issuer” or “sponsor”; agent *C*, the investor (possibly a hedge fund); and agent *B*, an intermediary between *A* and *C*, typically a special purpose vehicle (SPV) sponsored by an investment bank acting on behalf of agent *A* to issue the cat bond and playing an advisory role. Agent *B* is particular in the sense that his role is just to make the overall transaction feasible. He cannot retain any risk and acts as a pure intermediary. Such an agent may appear redundant, if not to help with legal constraint related to the issue. For his intermediation role, agent *B* may collect some servicing fees for the design and issue of the transaction.

Remark 1: Note that, in actual practice, the SPV is also an intermediary between cat bond investors and the financial markets. In this capacity, an SPV usually performs a dual-risk transfer function. First, by investing the proceed from the cat bond issue (as well as the reinsurance premium paid by the insurer) in Treasury securities, it frees the whole arrangement from default risk. Second, by contracting an interest rate swap with a financial counterpart, it cleans the cat bond from interest rate risk. These two risk transfer functions are important in practice. They enhance the attractiveness of the cat bond. In our model, they do not play a role, given our simplifying assumptions. The insurer faces no default risk, as he receives the proceeds from the issue at time $t = 0$, and a possible default on the insurer’s side is simply not considered. Interest rate risk is removed by assuming a zero-coupon cat bond and holding of the position until maturity T . We are thus able to concentrate on a different aspect related to cat bonds, namely, the downside risk.

The First Model Is Based on a Simple Cat Bond. In this case, agent *B* buys the entire cat bond issue from agent *A* and transfers it to agent *C*. Two parameters characterize the cat bond: its price, representing the volume of the transaction,⁸ and its nominal amount, N . The latter is fully transferred by agent *B*. As far as the price is concerned, agent *B* pays a price π_A to agent *A* to purchase the security and then receives a price π_C from agent *C*. A very simple structure is considered: no coupons are paid, and the nominal amount N is paid at maturity T to the buyer (first agent *B*, then agent *C*) only if no catastrophe occurs between 0 and T . If a catastrophe occurs before T , nothing is paid and the cat bond return is zero.

The Second Model Introduces a Hybrid Transaction. As for the simple transaction, agent *B* buys the entire cat bond from agent *A* with the previous characteristics. Agent *B* will then issue a hybrid product, which is bought by agent *C*. This structure has some optional features. Against a price π_C^h , agent *C* receives the prospect of nominal amount N^h if no catastrophe occurs before maturity T , and the prospect of a fixed amount H when both a financial market crash and a natural catastrophe occur before T . If, however, there is a market boom and no natural catastrophe before the maturity, agent *C* will pay this amount H to agent *B*, or equivalently will only receive $N^h - H$. More precisely, the hybrid structure received by agent *C* includes the purchase of a digital put, paying H if there is a natural catastrophe and a market crash, and the sale of a digital call paying H if there is no natural catastrophe and a market boom. The

⁸The price represents the volume of capital flowing into the cat bond market.

initial price of the put and that of the call coincide, and hence it simply remains the exchange of the contingent payoff at maturity. Since agent B cannot retain any risk in his book, he will transfer the payoff of the call option if exercised to the financial market.

Note that the hybrid structure depends on the possibility for agent B to find such hybrid options on the market. This is an assumption we will make in this article. It can be easily relaxed by considering a call and a put written simply on the market index, without any reference to the occurrence of a natural catastrophe. Moreover, in this structure, the two options should have the same initial price. This puts a constraint on the reference market events. More precisely, what is meant by “crash” and “boom” is characterized by this condition on the market prices of the related options, as we will see in the “Analysis of the Hybrid Transaction” section. Note also that, for the sake of simplicity, we assume that all cash flows are capitalized up to T .

Convex Risk Measures as a Choice Criterion

In order to determine optimally the characteristics of both transactions, each agent needs to refer to a choice criterion. In this article, we consider convex risk measures. Such a framework is motivated by the fact that risk measures are becoming increasingly popular in both the financial and actuarial industry but also they enable a concise writing, facilitating interpretation as we will see in the following. Moreover, such a framework enables to take into account the aversion toward ambiguity of the different agents. Indeed, risk measures do take into account the model risk (or the parameter uncertainty) by assessing the risk over a whole family of scenarios, weighted differently according to their likelihood, as underlined by the dual representation.

Definition and General Comments. When assessing the risk related to a given position, a first natural approach is based on the distribution of the risky position itself. In this framework, the most classical measure of risk is simply the *variance*. However, it does not take into account the whole distribution’s features (as asymmetry or skewness) and especially it does not focus on the “real” financial risk which is the downside risk. Therefore different methods have been developed to focus on the risk of losses: the most widely used (as it is recommended to bankers by many supervisory authorities) is the so-called $V@R$, based on quantiles for the lower tail of the distribution. More precisely, the $V@R$ associated with the position Ψ at a level ε is defined as

$$V@R_{\varepsilon}(\Psi) = \inf\{k : \mathbb{P}(\Psi + k < 0) \leq \varepsilon\}.$$

The $V@R$ corresponds to the minimal amount to be added to a given position to make it acceptable. Such a criterion has several key properties: it is decreasing in Ψ ; it satisfies the monetary property in the sense that it is translation invariant: $\forall m \in \mathbb{R}, V@R_{\varepsilon}(\Psi + m) = V@R_{\varepsilon}(\Psi) - m$; and finally, it is positive homogeneous as $\forall \lambda \geq 0, V@R_{\varepsilon}(\lambda \Psi) = \lambda V@R_{\varepsilon}(\Psi)$. This last property reflects the linear impact of the size of the position on the risk measure.

However, as noticed by Artzner et al. (1999), this criterion fails to meet a natural consistency requirement: it is not a convex risk measure while the convexity property translates the natural fact that diversification should not increase risk. In particular,

any convex combination of “admissible” risks should be “admissible.” The absence of convexity of the $V@R$ may lead to arbitrage opportunities inside the financial institution using such criterion as risk measure.

Based on this logic, Artzner et al. (1999) have adopted a more general approach to risk measurement. Their article is seminal as it has initiated a systematic axiomatic approach to risk measurement. A *coherent measure of risk* should be convex and satisfy the three key properties of the $V@R$.

More recently, the axiom of positive homogeneity has been discussed. Indeed, such a condition does not seem to be compatible with the notion of liquidity risk existing on the market as it implies that the size of the risky position has simply a linear impact on the risk measure. To tackle this shortcoming, Föllmer and Schied (2002a, b) consider instead *convex risk measures* defined on $\mathcal{X}$ a linear space of bounded functions including constant functions, as follows:

Definition 1: *The functional ρ is a convex risk measure if it satisfies the following properties:*

- a) *Convexity;*
- b) *Monotonicity;*
- c) *Translation invariance:* $\forall m \in \mathbb{R} \quad \rho(X + m) = \rho(X) - m.$

The Agents’ Risk Assessment. The two main agents of this study assess their risk using a convex risk measure, respectively, denoted by ρ_A and ρ_C . For the sake of simplicity, after the presentation of the general problem, we will consider *entropic risk measures*⁹ in order to derive explicit formulae for the different quantities involved. Denoting by γ_i (>0), the risk tolerance coefficient of agent i ($i = A, C$), his entropic risk measure associated with the terminal investment payoff Ψ is expressed as:

$$\rho_i(\Psi) = \gamma_i \ln \mathbb{E} \left[\exp \left(-\frac{1}{\gamma_i} \Psi \right) \right]. \quad (1)$$

Note that due to the cash translation invariance property of the different risk measures, we do not need to introduce the agents’ initial wealth as their impact will disappear in the different computations. Therefore, without any loss of generality, we fix them equal to 0.

The entropic risk measure has the following dual representation:

$$\rho_i(\Psi) = \sup_{\mathbb{Q}} \left(\mathbb{E}_{\mathbb{Q}}(-\Psi) - \gamma_i h(\mathbb{Q}/\mathbb{P}) \right),$$

⁹Note that the entropic risk measure corresponds to the certainty equivalent associated with the exponential utility. In particular, the risk-tolerance coefficient in both frameworks coincide.

where $h(\mathbb{Q}/\mathbb{P})$ is the relative entropy of $\mathbb{Q}$ with respect to the prior probability $\mathbb{P}$, defined by

$$h(\mathbb{Q}/\mathbb{P}) = \begin{cases} \mathbb{E}_{\mathbb{P}} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right), & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise.} \end{cases}$$

The model risk is fully taken into consideration here since the risk assessment is made over a whole class of possible probability measures or scenarios.

The problem of agent B is different in the sense that he cannot retain any risk and therefore his role is the structuring and issuing of the securities (simple cat bonds and hybrid cat bonds). Therefore, as he is not really exposed to risk, he has no particular measure of risk. His problem (just as that of the investment bank sponsoring the whole transaction) is to generate enough servicing fees. For this reason, we will assume that agent B is risk neutral.

Financial Investments on the Market and Market-Modified Risk Measures

In this article, we also assume that the two agents have the possibility to invest on the financial markets to hedge and diversify their risk. We first present the possible hedging strategies and then look at the impact of such hedging instruments on the risk measures of the different agents.

Hedging Portfolios and Investment Strategies. The financial market is represented by a set $\mathcal{V}_T$ of bounded terminal gains¹⁰ at time T , resulting from self-financing investment strategies with a null initial value. The key point is that all agents in the market agree on the initial value of every strategies. In other words, the market value at time 0 of any strategy is null. In particular, an admissible strategy is associated with a derivative contract with bounded terminal payoff Φ only if its forward market price at time T , $q^m(\Phi)$, is a transaction price for all agents in the market. Then, $\Phi - q^m(\Phi)$ is the bounded terminal gain at time T and is an element of $\mathcal{V}_T$. Typical example of admissible terminal gains are the terminal wealth associated with transactions based on options.

Moreover, in order to have coherent transaction prices, we assume in the following that the market is arbitrage free. In our framework, this can be expressed by:

$$\exists \mathbb{Q} \sim \mathbb{P} \quad \forall \xi_T \in \mathcal{V}_T \quad \mathbb{E}_{\mathbb{Q}}(\xi_T) \leq 0.$$

In particular, considering the financial assets, with a terminal payoff Φ that can be sold and bought, such a condition is written as

$$q^m(\Phi) = \mathbb{E}_{\mathbb{Q}}(\Phi).$$

¹⁰That is, the net potential gain corresponds to the spread between the terminal wealth resulting from the adopted strategy and the capitalized initial wealth.

The probability measure $\mathbb{Q}$ may be viewed as a static version of the classical $\mathcal{V}_T$ -martingale measures in a dynamic framework.

The set $\mathcal{V}_T$, previously defined, has also to satisfy some properties to be consistent with fundamental investment principles. For example, to comply with the diversification principle, any convex combination of admissible gains should also be an admissible gain, and therefore, the set $\mathcal{V}_T$ is always taken as a convex set.

Market-Modified Risk Measures. Each agent A and C will have to determine his optimal financial investment by solving the following hedging/investment problem:

$$\min_{\xi_T \in \mathcal{V}_T} \rho_i(\Psi - \xi_T),$$

where Ψ is agent i 's exposure, for $i = A, C$. The value functional of this optimization problem characterizes a new convex risk measure, which corresponds to the risk measure agent i will have after having optimally chosen her financial investment or hedge on the market. It is called the market-modified risk measure of agent i and denoted by ρ_i^m . In the entropic framework, we get:

$$\rho_i^m(\Psi) = \gamma_i \ln \mathbb{E}_{\hat{\mathbb{Q}}} \left[\exp \left(-\frac{1}{\gamma_i} \Psi \right) \right]. \quad (2)$$

where $\hat{\mathbb{Q}}$ is the minimal entropy probability measure.¹¹

Therefore, in the following, the risk measure to consider when dealing with any agent is directly his market-modified risk measure instead of the original one. This allows taking into account simultaneous optimal investment decisions in the financial market, when the agent also trades in the cat bond market. In terms of probability measures, this means that we will not work directly with the historical probability measure $\mathbb{P}$, but instead with the minimal entropy probability measure $\hat{\mathbb{Q}}$, which is common to all agents as they have the same access to the financial market.

Remark 2: Even if there exists a unique large underlying financial market, the different agents may not have the same access to it. Indeed, the various agents may be of very different natures *a priori* and the set of hedging products to which they have access may be completely different because of specific regulations, of usual strategies, and so on. This will not modify the general results of this article but will simply introduce some compensation terms that correspond to the likelihood ratio of the different probability measures obtained (for more details, please refer to Barrieu and El Karoui, 2003).

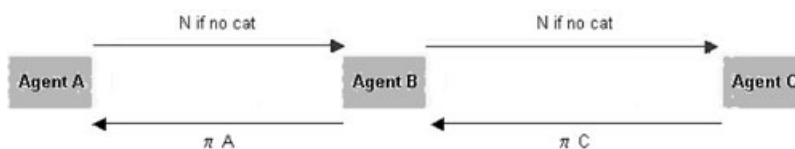
ANALYSIS OF THE SIMPLE TRANSACTION

This section is devoted to the study and analysis of the simple transaction, which has the structure shown in Figure 1.

¹¹For more details, please refer to Barrieu and El Karoui (2003, 2005).

FIGURE 1

Simple Transaction



Note that two different prices are involved in this transaction: π_A is the price for the simple cat bond issued by agent *A* and purchased by agent *B*, while π_C is the price paid by agent *C* in his transaction with agent *B*. Note also that agent *B* is assumed to fully transfer the risk of the cat bond, embedded in the contingent payment of the nominal N . He acts as a pure intermediary. We want to determine the optimal structure of the cat bond, i.e., the price(s) and the nominal amount. To do so, we proceed in several steps. We first characterize the different payoffs of the three agents involved in the transaction, as well as their respective role in characterizing the optimal transaction.

Payoffs and the Structure of Cash Flows

In this simple framework, there are only two relevant states of nature depending on whether a natural catastrophe occurs before T ($\tau < T$) or not ($\tau > T$). The probability of each event has to be considered under the reference probability measure for the study; i.e., the minimal entropy probability measure $\hat{\mathbb{Q}}$ and $\hat{p}$ denotes $\hat{\mathbb{Q}}(\tau \leq T)$. The subsequent table summarizes the payoff of the three agents in the different possible situations. Note that agent *B* is indifferent between the occurrence and the nonoccurrence of a natural catastrophe as he transfers all the contingent cash flows from agent *A* to agent *C*.

State of Nature	Probability Under $\hat{\mathbb{Q}}$	Agent <i>A</i>	Agent <i>B</i>	Agent <i>C</i>
$T < \tau$	$1 - \hat{p}$	$\pi_A - N$	$\pi_C - \pi_A$	$-\pi_C + N$
$\tau < T$	$\hat{p}$	$\pi_A - X$	$\pi_C - \pi_A$	$-\pi_C$

Transaction Feasibility and Pricing Rule

Looking more closely at the structure of the simple transaction and at the related cash flows, the different characteristics of the transaction are determined in the following way:

- Agent *A* determines the volume of the transaction with agent *B*; in another words, he determines the “reservation price,” π_A , of the first part of the transaction.
- Agent *C*, on the other hand, determines the volume of the other transaction, i.e., the “reservation price” π_C .
- Finally, as advisor of the whole transaction, agent *B* chooses the nominal amount of both transactions, N , so as to maximize the fees he can generate under the participation constraint imposed by both agents *A* and *C*. These fees are usually called servicing fees. They cover the expenses related to the design, setup, and issue of the transaction.

Therefore, the optimization program of the transaction can be written as follows:

$$\begin{aligned} & \max_{N, \pi_A, \pi_C} (\pi_C - \pi_A) \\ \text{s.t. } & \begin{cases} \rho_A^m(\pi_A - N\mathbf{1}_{\tau > T} - X\mathbf{1}_{\tau \leq T}) \leq \rho_A^m(-X\mathbf{1}_{\tau \leq T}) \\ \rho_C^m(-\pi_C + N\mathbf{1}_{\tau > T}) \leq \rho_C^m(0) \end{cases}. \end{aligned} \quad (3)$$

Such an optimization problem is rather standard in risk theory. The existence of a solution relies upon the convexity of the various functionals involved, and does not depend on any specific relationship between the various risk tolerance coefficients of the different agents. In particular, as underlined in Proposition 4, without any additional assumption, the optimal transaction, obtained as solution of the optimization problem, will always take place as the three agents improve their situation by doing it.

The indifference pricing rules are obtained by binding each constraint at the optimum and using the cash translation invariance property of the market modified risk measure of agent A and agent C :

$$\begin{aligned} \pi_A &= \rho_A^m(-N\mathbf{1}_{\tau > T} - X\mathbf{1}_{\tau \leq T}) - \rho_A^m(-X\mathbf{1}_{\tau \leq T}), \\ \pi_C &= \rho_C^m(0) - \rho_C^m(N\mathbf{1}_{\tau > T}). \end{aligned}$$

The different characteristics will be studied in details in the subsequent subsections, considering entropic risk measures to obtain explicit formulation of the relevant quantities.

Volume of the Transaction

We first consider the volume of the simple transaction. As previously noticed, the price of each part of the transaction represents the volume of capital flowing into the cat bond market. Both pricing rules can be obtained explicitly in the entropic framework as shown subsequently.

Cat Bond Transaction Between Agent A and Agent B .

The lower bound to the price agent A is ready to accept for the cat-bond, giving also a lower bound to the volume of the transaction, is

$$\pi_A = \rho_A^m(-N\mathbf{1}_{\tau > T} - X\mathbf{1}_{\tau \leq T}) - \rho_A^m(-X\mathbf{1}_{\tau \leq T}).$$

From there, using the entropic risk measure, we get:

Proposition 1: *The indifference volume for the cat-bond issue by agent A is given by:*

$$\pi_A = \gamma_A \ln \left\{ \frac{(1 - \hat{p}) \exp\left(\frac{1}{\gamma_A} N\right) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)}{(1 - \hat{p}) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)} \right\}. \quad (4)$$

Note that $\pi_A > 0$ iff $N > 0$.

Cat Bond Transaction Between Agent B and Agent C.

On the other hand, the upper bound to the price agent C is ready to pay for the cat bond, giving also an upper bound to the volume of the transaction, is

$$\pi_C = \rho_C^m(0) - \rho_C^m(N\mathbf{1}_{\tau > T}).$$

From there, using the entropic risk measure, we get:

Proposition 2: *The indifference volume for the cat bond issue by agent B is given by:*

$$\pi_C = -\gamma_C \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_C} N \right) + \hat{p} \right]. \quad (5)$$

Note that $\pi_C > 0$ iff $N > 0$, and $\partial \pi_C / \partial N > 0$ for any N .

Problem of Agent B: Determination of the Nominal Amount of the Transaction

It now remains to optimally determine the nominal amount of both parts of the transaction. Agent B has to solve Program (3) under the pricing constraints imposed by agent A and agent C given by Equations (4) and (5). The following result prevails:

Proposition 3: *The optimal nominal amount for the simple cat bond is given as:*

$$N = \frac{\gamma_C}{\gamma_A + \gamma_C} X. \quad (6)$$

The optimality condition given by Equation (6) is the well-known Borch's condition for Pareto-optimal risk sharing. We get $N = \alpha X$, with $\alpha = \frac{\gamma_C}{\gamma_A + \gamma_C} < 1$. In equilibrium, agent A retains part of the risk X and this part increases with his risk tolerance—for a given risk tolerance of agent C. Note that the role of agent B is to make the transaction feasible. He perceives the difference between π_C and π_A , which would not exist if agent A could contract with agent C directly (only π_C would appear in this case, with the same value).

Proof: Using the optimal pricing rule together with the cash-translation invariance property of the risk measure ρ_A^m , program (3) becomes

$$\min_N \left\{ \rho_A^m(-X\mathbf{1}_{\tau \leq T} - N\mathbf{1}_{\tau > T}) - \rho_A^m(-X\mathbf{1}_{\tau \leq T}) + \rho_C^m(N\mathbf{1}_{\tau > T}) - \rho_C^m(0) \right\},$$

or, as $\rho_C^m(0) = 0$ in the entropic framework,

$$\begin{aligned} & \min_N \{ \rho_A^m(-X \mathbf{1}_{\tau \leq T} - N \mathbf{1}_{\tau > T}) - \rho_A^m(-X \mathbf{1}_{\tau \leq T}) + \rho_C^m(N \mathbf{1}_{\tau > T}) \} \\ &= \min_N \left\{ \gamma_A \ln \left\{ \frac{(1 - \hat{p}) \exp\left(\frac{1}{\gamma_A} N\right) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)}{(1 - \hat{p}) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)} \right\} \right. \\ & \quad \left. + \gamma_C \ln \left[(1 - \hat{p}) \exp\left(-\frac{1}{\gamma_C} N\right) + \hat{p} \right] \right\}. \end{aligned}$$

Equivalently,

$$\begin{aligned} &= \min_N \left\{ \gamma_A \ln \left\{ (1 - \hat{p}) \exp\left(\frac{1}{\gamma_A} N\right) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right) \right\} \right. \\ & \quad \left. + \gamma_C \ln \left[(1 - \hat{p}) \exp\left(-\frac{1}{\gamma_C} N\right) + \hat{p} \right] \right\}. \end{aligned}$$

We write the first-order condition:¹²

$$\gamma_A \frac{\frac{1}{\gamma_A} (1 - \hat{p}) \exp\left(\frac{1}{\gamma_A} N\right)}{(1 - \hat{p}) \exp\left(\frac{1}{\gamma_A} N\right) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)} + \gamma_C \frac{-\frac{1}{\gamma_C} (1 - \hat{p}) \exp\left(-\frac{1}{\gamma_C} N\right)}{(1 - \hat{p}) \exp\left(-\frac{1}{\gamma_C} N\right) + \hat{p}} = 0$$

or

$$\frac{\exp\left(\frac{1}{\gamma_A} N\right)}{(1 - \hat{p}) \exp\left(\frac{1}{\gamma_A} N\right) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)} = \frac{\exp\left(-\frac{1}{\gamma_C} N\right)}{(1 - \hat{p}) \exp\left(-\frac{1}{\gamma_C} N\right) + \hat{p}}.$$

Finally, after some simplifications, the result immediately follows. Q.E.D.

Remark on the Transaction Feasibility

The following result ensures the feasibility of the simple transaction. Agent B will get a positive fee from organizing the simple cat bond arrangement.

¹²The second-order is satisfied as the second derivatives with respect to N is positive.

Proposition 4: *The indifference buyer's price of agent C is larger than the indifference seller's price of agent A.*

The result reflects the economic surplus provided by Pareto-optimal risk sharing between agents A and C. The surplus goes primarily to agent B, who makes the overall transaction feasible. Note, however, that if the difference between π_C and π_A is high enough, and exceeds the usual servicing fee required by agent B, agent A may negotiate with the financial intermediary to get part of this surplus.

Proof: Let us consider $\pi_A - \pi_C$ and use the fact that at the optimum $N = \alpha X$, with $\alpha = \frac{\gamma_C}{\gamma_A + \gamma_C}$. Therefore,

$$\begin{aligned} \pi_A - \pi_C &= \gamma_A \ln \left\{ \frac{(1 - \hat{p}) \exp \left(\frac{1}{\gamma_A} \alpha X \right) + \hat{p} \exp \left(\frac{1}{\gamma_A} X \right)}{(1 - \hat{p}) + \hat{p} \exp \left(\frac{1}{\gamma_A} X \right)} \right\} \\ &\quad + \gamma_C \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_C} \alpha X \right) + \hat{p} \right], \\ &= \gamma_A \ln \left[(1 - \hat{p}) \exp \left(\frac{1}{\gamma_A} (\alpha - 1) X \right) + \hat{p} \right] \\ &\quad - \gamma_A \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_A} X \right) + \hat{p} \right] \\ &\quad + \gamma_C \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_C} \alpha X \right) + \hat{p} \right]. \end{aligned}$$

Replacing α by its value, we get:

$$\begin{aligned} \pi_A - \pi_C &= \gamma_A \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_A + \gamma_C} X \right) + \hat{p} \right] \\ &\quad - \gamma_A \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_A} X \right) + \hat{p} \right] \\ &\quad + \gamma_C \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_A + \gamma_C} X \right) + \hat{p} \right] \end{aligned}$$

or

$$\begin{aligned} \pi_A - \pi_C &= (\gamma_A + \gamma_C) \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_A + \gamma_C} X \right) + \hat{p} \right] \\ &\quad - \gamma_A \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_A} X \right) + \hat{p} \right]. \end{aligned}$$

But the function $\gamma \rightarrow \gamma \ln[(1 - \hat{p}) \exp(-\frac{1}{\gamma} X) + \hat{p}]$ is a strictly decreasing function. Since $\gamma_A + \gamma_C > \gamma_A$, we immediately obtain:

$$\pi_A - \pi_C < 0.$$

Hence the result.

Q.E.D.

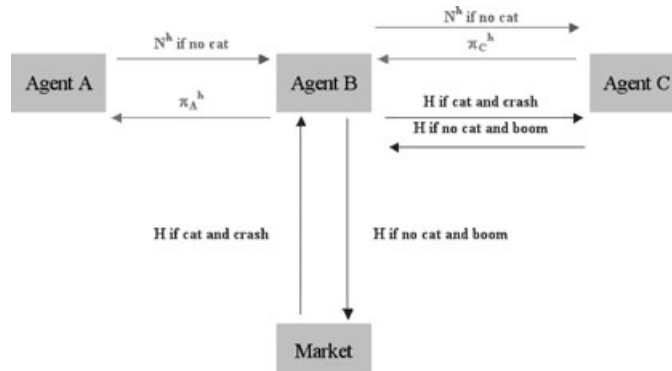
ANALYSIS OF THE HYBRID TRANSACTION

This section is devoted to the study and analysis of the hybrid transaction, which has the structure shown in Figure 2.

Note that two different prices are involved in this transaction: π_A^h is the price for the simple cat bond issued by agent *A* and purchased by agent *B*; π_C^h is the price for the hybrid cat bond issued by agent *B* and purchased by agent *C*. Moreover, the nominal amount N^h of the cat bond embedded in the hybrid transaction will be in general different from the optimal nominal amount $N = \alpha X$ obtained in the case of a simple cat bond.

Note in addition that, in contrast to the simple transaction, the intermediation role of agent *B* becomes more active and necessary in the hybrid transaction. The necessity arises from risk management considerations. Agent *A* cannot issue directly the hybrid cat bond himself because of the additional hybrid payment H . In contrast, agent *B* is able to issue the hybrid product, involving some zero-premium options. More precisely, the hybrid structure received by agent *C* includes the purchase of a put, paying H if there is a natural catastrophe and a market crash, and the sale of a call paying H if there is no natural catastrophe and a market boom. The initial price of the put and that of the call coincide, and hence it simply remains the exchange of the contingent payoff at maturity. Such a condition on the initial prices of both options imposes some constraint on the market events under consideration, as we will see in the subsection below when looking at the various cash flows. Once the transaction has been completed, no agent is simultaneously exposed to both risks—natural catastrophe and market crash. Agent *A* retains some exposure from the cat

FIGURE 2
Hybrid Transaction



risk if $N^h < X$. Agent C assumes the remaining exposure to the cat risk against a random suitable compensation (the payoff prospect from the hybrid cat bond) and agent B does not retain any risk, as he is a pure intermediary and transfers all the cash flows to either agent C or to the market.

We want to determine the optimal structure of this hybrid transaction. To do so, we proceed in several steps. We first characterize the different payoffs for the three agents involved in the transaction, as well as their respective roles in the transaction process and then we determine the transaction volume and the optimal conditional payoff amounts N^h and H .

Payoffs and the Structure of Cash Flows

Since the payoff structure of the hybrid cat bond depends on the joint occurrence of a natural catastrophe and a crash, and that of no natural catastrophe and a market boom, it is essential to introduce the joint distribution, as the timing of the various events also matters. We introduce the following notation: as previously, τ denotes the random time of occurrence of a natural catastrophe, $\underline{\tau}$ will now represent the random time of occurrence of a market crash and $\bar{\tau}$ that of a market boom.

To simplify, we make the two following assumptions:

- There is no dependence between the occurrence of a natural catastrophe and that of market-based events. For this reason, the order of occurrence has no impact on the probability of realization. In particular, the amount H will be paid as soon as a catastrophe and a market crash have both occurred before T .
- The events “market crash” and “market boom” are mutually disjoint in the sense that they cannot both occur before T . Such an assumption can be justified if the maturity of the transaction T is sufficiently close.

Under these assumptions, the following states of nature are relevant and the joint distribution of both events can be written under the reference probability measures of this study, i.e., the minimal entropy probability measure $\hat{\mathbb{Q}}$, as:

Cat Event	Probability Under $\hat{\mathbb{Q}}$	Relevant States of Nature	Probability Under $\hat{\mathbb{Q}}$
<i>no cat</i>		$\underline{\tau} < T < \tau$	$(1 - \hat{p})q_U$
$\tau > T$	$1 - \hat{p}$	$T < \bar{\tau} < \tau$ and $T < \tau < \bar{\tau}$	$(1 - \hat{p})(1 - q_U)$
<i>cat</i>		$\underline{\tau} < \tau < T$ and $\tau < \underline{\tau} < T$	$\hat{p}q_L$
$\tau < T$	$\hat{p}$	$\tau < T < \underline{\tau}$	$\hat{p}(1 - q_L)$

with $q_U = \hat{\mathbb{Q}}(\bar{\tau} < T)$ and $q_L = \hat{\mathbb{Q}}(\underline{\tau} < T)$.

The table on the next page summarizes the payoff of the three agents in the different possible situations.

As previously mentioned, both options should have the same initial price, computed under the pricing measure $\hat{\mathbb{Q}}$. As both options are traded in the market, we could

State of Nature	Probability under $\hat{\mathbb{Q}}$	Agent A	Agent B	Agent C
$\bar{\tau} < T < \tau$	$(1 - \hat{p})q_U$	$\pi_A^h - N^h$	$\pi_C^h - \pi_A^h$	$-\pi_C^h + N^h - H$
$T < \bar{\tau} < \tau$ and $T < \tau < \bar{\tau}$	$(1 - \hat{p})(1 - q_U)$	$\pi_A^h - N^h$	$\pi_C^h - \pi_A^h$	$-\pi_C^h + N^h$
$\bar{\tau} < \tau < T$ and $\tau < \bar{\tau} < T$	$\hat{p}q_L$	$\pi_A^h - X$	$\pi_C^h - \pi_A^h$	$-\pi_C^h + H$
$\tau < T < \bar{\tau}$	$\hat{p}(1 - q_L)$	$\pi_A^h - X$	$\pi_C^h - \pi_A^h$	$-\pi_C^h$

assume that they are priced using the risk-neutral pricing rule with respect to any equivalent martingale measure (the market is *a priori* not complete, so there is an infinity of such measures), in particular with respect to the minimal entropy probability measure, which is our reference probability measure in this study. More precisely, we should have

$$\mathbb{E}_{\hat{\mathbb{Q}}}(H\mathbf{1}_{\tau, \bar{\tau} \leq T}) = \mathbb{E}_{\hat{\mathbb{Q}}}(H\mathbf{1}_{\bar{\tau} \leq T \leq \tau})$$

or equivalently

$$\hat{\mathbb{Q}}(\tau, \bar{\tau} \leq T) = \hat{\mathbb{Q}}(\bar{\tau} \leq T \leq \tau),$$

which can also be written as the following condition:

$$\hat{p}q_L = (1 - \hat{p})q_U. \quad (7)$$

Such a condition on the probability of occurrence (under $\hat{\mathbb{Q}}$) of the events “market boom” and “market crash” has an impact on the definition of these events themselves. More precisely, if a crash is defined as the fall of the stock market index below a certain level $\underline{L}$, this condition characterizes another level $\bar{L}$ for the market index such that a stock market boom is defined to occur exactly when the index rises beyond this level. From condition (7), to every definition of a market crash corresponds a particular definition of a market boom.

Transaction Feasibility and Pricing Rule

Looking more closely at the structure of the hybrid transaction and at the related cash flows, the different characteristics of the hybrid transaction are determined in the following way:

- Agent A determines the volume of the transaction with agent B, in another words, the “price” π_A^h of the first part of the transaction.
- Agent C, on the other hand, determines the volume of the other transaction, i.e., the “price” π_C^h .
- As advisor of the whole transaction, agent B chooses optimally the nominal amount N^h of both transactions and the payoff H of digital puts and calls involved in the hybrid structure. As for the simple structure, agent B wants to maximize the fees he can generate from this transaction, under the participation constraint of both

agents A and C . These servicing fees will cover the expenses related to the design, setup, and issue of the hybrid transaction.

Therefore, the optimization program is now written as follows:

$$\begin{aligned} & \max_{N^h, H, \pi_A, \pi_C} (\pi_C^h - \pi_A^h), \\ & \text{s.t.} \begin{cases} \rho_A^m(-N^h \mathbf{1}_{\tau > T} + \pi_A^h - X \mathbf{1}_{\tau \leq T}) \leq \rho_A^m(-X \mathbf{1}_{\tau \leq T}) \\ \rho_C^m(N^h \mathbf{1}_{\tau > T} - \pi_C^h + H \mathbf{1}_{\tau, \mathcal{I} \leq T} - H \mathbf{1}_{\bar{\tau} \leq T \leq \tau}) \leq \rho_C^m(0). \end{cases} \end{aligned} \quad (8)$$

Volume of the Transactions

We first consider the volume of the simple transaction. As previously noticed, the price of each part of the transaction represents the volume of capital flowing into the cat bond market. Both pricing rules can be obtained explicitly in the entropic framework as shown subsequently.

Cat Bond Transaction Between Agent A and Agent B

The lower bound to the price agent A is ready to accept for the cat bond, giving also a lower bound to the volume of the transaction, is

$$\pi_A^h = \rho_A^m(-N^h \mathbf{1}_{\tau > T} - X \mathbf{1}_{\tau \leq T}) - \rho_A^m(-X \mathbf{1}_{\tau \leq T}).$$

From there, using the entropic risk measure, we get:

Proposition 5: *The indifference volume for the cat bond issue by agent A is given by:*

$$\pi_A^h = \gamma_A \ln \left\{ \frac{(1 - \hat{p}) \exp\left(\frac{1}{\gamma_A} N^h\right) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)}{(1 - \hat{p}) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)} \right\}. \quad (9)$$

Note that $\pi_A^h > 0$ if $N^h > 0$.

Cat Bond Transaction Between Agent B and Agent C

On the other hand, the upper bound to the price agent C is ready to pay for the cat bond, giving also an upper bound to the volume of the transaction, is

$$\pi_C^h = \rho_C^m(0) - \rho_C^m(N^h \mathbf{1}_{\tau > T} + H \mathbf{1}_{\tau, \mathcal{I} \leq T} - H \mathbf{1}_{\bar{\tau} \leq T \leq \tau}).$$

From there, using the entropic risk measure, we get:

Proposition 6: *The indifference volume for the hybrid cat bond issue by agent B is given by:*

$$\pi_C^h = -\gamma_C \ln \left((1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p} (1 - q_L) \right). \quad (10)$$

Problem of Agent B: Determination of the Nominal Amount of the Transaction and of the Hybrid Amount

It now remains to optimally determine the nominal amount N^h of both parts of the transaction and the hybrid amount H . Agent B has to solve program (8) by considering the pricing rules given by Equations (9) and (10).

The problem of agent B can be expressed as:

$$\max_{H, N^h} (\pi_C^h - \pi_A^h),$$

subject to

$$\begin{cases} \pi_C^h = -\gamma_C \ln \left((1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p} (1 - q_L) \right) \\ \pi_A^h = \gamma_A \ln \left\{ \frac{(1 - \hat{p}) \left(\frac{1}{\gamma_A} N^h \right) + \hat{p} \exp \left(\frac{1}{\gamma_A} X \right)}{(1 - \hat{p}) + \hat{p} \exp \left(\frac{1}{\gamma_A} X \right)} \right\}. \end{cases}$$

Optimization in N^h . We aim at solving the following program for any given H :

$$\max_{N^h} \{ \pi_C^h(N^h, H) - \pi_A^h(N^h, H) \}.$$

Proposition 7: *The optimal nominal amount of the transaction is given by:*

$$N^h = \frac{\gamma_C}{\gamma_A + \gamma_C} X + \frac{\gamma_A}{\gamma_A + \gamma_C} \mathcal{L}_C(q_U, q_L, H),$$

where

$$\mathcal{L}_C(q_U, q_L, H) = \rho_C^m(-H \mathbf{1}_{\bar{\tau} \leq T}) - \rho_C^m(H \mathbf{1}_{\bar{\tau} \leq T}).$$

Note that N^h is composed of two terms: the first term is N (the Borch risk-sharing condition from the simple cat bond); the second term introduces a link with the financial market conditions and the contingent payment H , but in a nonlinear way. As soon as H is positive, this second term is positive (using the decreasing monotonicity of the risk measure ρ_C^m) and the nominal amount N^h is larger than that of the simple transaction N . Note also that this second term is increasing in the share of agent A in the aggregate risk tolerance $\gamma_A + \gamma_C$. Therefore, the difference between both cat bond transactions will be all the more important; therefore, because agent A is more risk tolerant, compared to agent C , everything else remaining constant. This will be further commented in the "Remarks on the Hybrid Transaction" section.

Proof: In the entropic framework, the program becomes

$$\min_{N^h} \left\{ \begin{aligned} & \gamma_A \ln \left\{ \frac{(1 - \hat{p}) \exp \left(\frac{1}{\gamma_A} N^h \right) + \hat{p} \exp \left(\frac{1}{\gamma_A} X \right)}{(1 - \hat{p}) + \hat{p} \exp \left(\frac{1}{\gamma_A} X \right)} \right\} \\ & + \gamma_C \ln \left((1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) \right. \\ & \left. + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p}(1 - q_L) \right) \end{aligned} \right\}$$

or equivalently:

$$\min_{N^h} \left\{ \begin{aligned} & \gamma_A \ln \left((1 - \hat{p}) \exp \left(\frac{1}{\gamma_A} N^h \right) + \hat{p} \exp \left(\frac{1}{\gamma_A} X \right) \right) \\ & + \gamma_C \ln \left((1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) \right. \\ & \left. + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p}(1 - q_L) \right) \end{aligned} \right\}.$$

We write the first-order condition:

$$\begin{aligned} & \frac{\exp \left(\frac{1}{\gamma_A} N^h \right)}{(1 - \hat{p}) \left(\frac{1}{\gamma_A} N^h \right) + \hat{p} \exp \left(\frac{1}{\gamma_A} X \right)} \\ & = \frac{q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right)}{(1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p}(1 - q_L)} \end{aligned}$$

or equivalently, after some simplifications:

$$\begin{aligned} & \exp\left(\frac{1}{\gamma_A} N^h\right) \left(q_L \exp\left(-\frac{1}{\gamma_C} H\right) + (1 - q_L)\right) \\ &= \exp\left(-\frac{1}{\gamma_C} N^h\right) \exp\left(\frac{1}{\gamma_A} X\right) \left(q_U \exp\left(\frac{1}{\gamma_C} H\right) + (1 - q_U)\right). \end{aligned}$$

Finally, taking the logarithm on both sides leads to

$$\left(\frac{1}{\gamma_A} + \frac{1}{\gamma_C}\right) N^h = \frac{1}{\gamma_A} X + \ln \left(\frac{q_U \exp\left(\frac{1}{\gamma_C} H\right) + (1 - q_U)}{q_L \exp\left(-\frac{1}{\gamma_C} H\right) + (1 - q_L)} \right).$$

Hence the result. Q.E.D.

Optimization in H . Let us now look at the optimization with respect to H . We now have to solve the following program for any given N^h :

$$\max_H \{\pi_C^h(N^h, H) - \pi_A^h(N^h, H)\}.$$

Proposition 8: *The optimal hybrid amount to be paid to the investor when both a crash and a catastrophe occur before the maturity is given by:*

$$H = \frac{1}{2} N^h. \tag{11}$$

Proof: In the entropic framework, the program becomes

$$\min_H \left\{ \begin{aligned} & \gamma_A \ln \left\{ \frac{(1 - \hat{p}) \exp\left(\frac{1}{\gamma_A} N^h\right) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)}{(1 - \hat{p}) + \hat{p} \exp\left(\frac{1}{\gamma_A} X\right)} \right\} \\ & + \gamma_C \ln \left((1 - \hat{p}) \left(q_U \exp\left(\frac{1}{\gamma_C} (H - N^h)\right) + (1 - q_U) \exp\left(-\frac{1}{\gamma_C} N^h\right) \right) \right. \\ & \quad \left. + \hat{p} q_L \exp\left(-\frac{1}{\gamma_C} H\right) + \hat{p} (1 - q_L) \right) \end{aligned} \right\},$$

or equivalently,

$$\min_H \gamma_C \ln \left((1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) \right. \\ \left. + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p} (1 - q_L) \right).$$

We write the first-order condition:

$$\frac{(1 - \hat{p}) q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) - \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right)}{(1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p} (1 - q_L)} = 0$$

or

$$(1 - \hat{p}) q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) = \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right).$$

Hence, after some simplifications, we have:

$$H = \frac{1}{2} N^h + \gamma_C \ln \left(\frac{\hat{p} q_L}{(1 - \hat{p}) q_U} \right).$$

From the condition 7, we have the desired result.

Q.E.D.

Remarks on the Hybrid Transaction

Let us now make a few comments on the hybrid transaction and in particular compare it with the simple transaction in terms of nominal amounts.

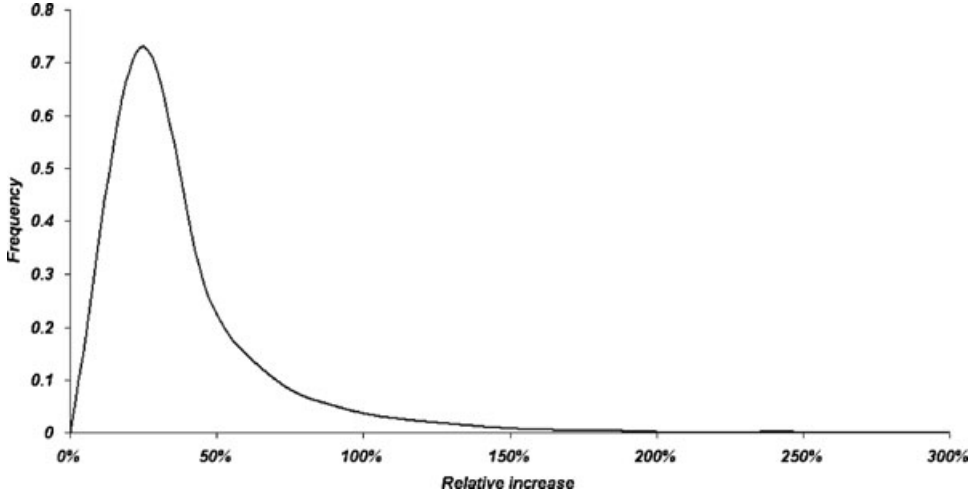
First, the nominal N^h is an increasing function of H and for $H > 0$, $N^h > N$. On the other hand, the amount H is a simple increasing function of N^h . The situation $H = 0$ is impossible and therefore the nominal amount of the hybrid transaction will always be larger than that of the simple cat bond. This result is illustrated in Figure 3 representing the distribution of the relative increase in the nominal amount $\frac{N^h - N}{N}$ obtained when performing random experiments. More precisely, we considered X , γ_A , γ_C , p , and q_U as independent uniform random variables taking values in

$$\begin{cases} X \in [10, 210] \\ \gamma_A \in [0, 1000] \\ \gamma_C \in [0, 1000] \\ \hat{p} \in [0, 1] \\ q_U \in [0, 1] \end{cases}$$

and then did 5,000 independent drawings.

FIGURE 3

Relative Increase in the Nominal Between the Hybrid Transaction and the Simple Cat Bond



The average relative increase in the nominal amount is 27%.

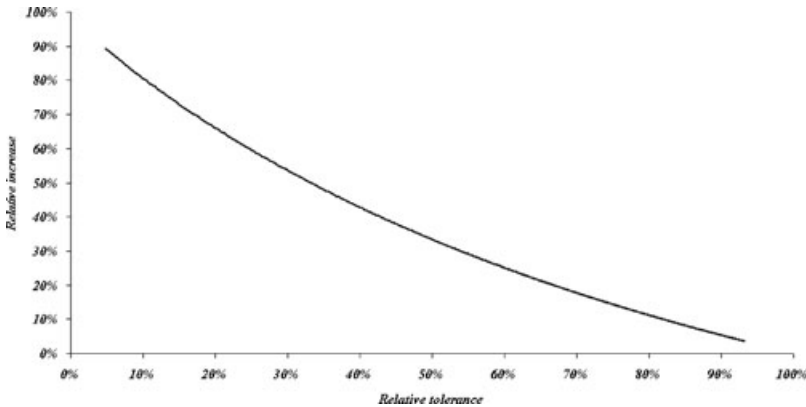
Moreover, the relative increase in the nominal amount $\frac{N^h - N}{N}$ is a decreasing function of the relative tolerance of agent C, as illustrated in the following simple example where

$$\begin{cases} X = 150 \\ \gamma_A = 10 \\ \hat{p} = 0.2 \\ q_U = 0.2 \end{cases}.$$

In other words, when the investor is relatively more risk tolerant, the hybrid nominal amount is relatively not so important. The investor is not so eager in terms of nominal amount to receive when there is no natural catastrophe. As previously noticed, the difference between both cat bond transactions will be all the more important; therefore, because agent A is more risk tolerant, compared to agent C, everything else remaining constant. Moreover the simple nominal N is increasing in the the share of agent C in the aggregate risk tolerance. Therefore, when agent C is more risk tolerant, compared to agent A, N will be larger and the difference between both cat bond structures will be small, everything else remaining constant. In contrast, if agent C is less risk tolerant, compared to agent A, he will take less of the cat risk (N will be smaller) and the difference between both cat bonds will be more important. In other words, when investors are relatively risk tolerant, compared to insurers, the nominal amounts involved in simple cat bond transactions will be large, and hybrid cat bonds will not make much a difference. In contrast, if investors are relatively risk averse, compared to insurers, simple cat bonds will attract less capital, and introducing

FIGURE 4

Relative Increase in Nominal With Respect to the Relative Tolerance of Agent C When $X = 150$, $p = q_u = 0.2$, and $y_A = 10$



hybrid cat bonds will have a significant impact on the volume of capital flowing into the cat bond market (Figure 4).

IMPACT OF THE HYBRID COMPONENT ON THE VOLUME OF THE TRANSACTION

As previously mentioned, the volume of the transaction (simple or hybrid) is measured by the bound imposed on the price. The price corresponds to the amount of money involved at the beginning of the transaction and therefore can be naturally interpreted as the volume.

Point of View of Agent A

First note that agent A is indifferent from his risk measure point of view between doing the hybrid transaction and the simple transaction. This is a direct consequence of the indifference price calculation. Using the same type of arguments, to have an idea of the impact the hybrid component has on the volume of the transaction for agent A , we look at the difference between π_A^h and π_A . The following result holds true:

Proposition 9: *The volume of the hybrid transaction is more important for agent A :*

$$\pi_A^h > \pi_A.$$

Therefore, the volume of the cat bond market for the insurance industry is larger when a hybrid component is added to the transaction. In other words, the amount of capital received by the insurance sector is more important when a hybrid structure is issued.

Proof: The result is a straightforward consequence of the characterization of the indifference prices itself:

$$\pi_A^h = \rho_A^m(-N^h \mathbf{1}_{\tau > T} - X \mathbf{1}_{\tau \leq T}) - \rho_A^m(-X \mathbf{1}_{\tau \leq T})$$

$$\text{and } \pi_A = \rho_A^m(-N \mathbf{1}_{\tau > T} - X \mathbf{1}_{\tau \leq T}) - \rho_A^m(-X \mathbf{1}_{\tau \leq T}),$$

and the fact that $N^h > N$, using the decreasing monotonicity of the risk measure ρ_A^m . Q.E.D.

Point of View of Agent C

First note that agent C is indifferent from his risk measure point of view between doing the hybrid transaction and the simple transaction. This is a direct consequence of the indifference price calculation. To have an idea of the impact the hybrid component has on the volume of the transaction for agent C, we look at the difference between π_C^h and π_C . The following result holds true:

Proposition 10: *The volume of the hybrid transaction is more important for agent C:*

$$\pi_C^h > \pi_C.$$

Therefore, the volume of the cat bond market for the investors is larger when a hybrid component is added to the transaction. In other words, the investors are ready to put more capital in this type of hybrid structure than in traditional cat bonds.

Proof: From the characterization of the indifference prices, we know that

$$\begin{aligned} \pi_C^h - \pi_C &= -\gamma_C \ln \left((1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p} (1 - q_L) \right) \\ &\quad + \gamma_C \ln \left[(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_C} N \right) + \hat{p} \right], \\ &= \gamma_C \ln \left[\frac{(1 - \hat{p}) \exp \left(-\frac{1}{\gamma_C} N \right) + \hat{p}}{(1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p} (1 - q_L)} \right], \end{aligned}$$

where N_h is the nominal amount of the hybrid transaction.

Let us first work on the denominator:

$$\begin{aligned} \mathcal{D} \triangleq & (1 - \hat{p}) \left(q_U \exp \left(\frac{1}{\gamma_C} (H - N^h) \right) + (1 - q_U) \exp \left(-\frac{1}{\gamma_C} N^h \right) \right) \\ & + \hat{p} q_L \exp \left(-\frac{1}{\gamma_C} H \right) + \hat{p} (1 - q_L). \end{aligned}$$

From Proposition 8, we know that the optimal hybrid amount $H = \frac{1}{2} N^h$. Moreover, condition (7) imposes that $(1 - \hat{p}) q_U = \hat{p} q_L$. Using these two identities, the

denominator becomes

$$\mathcal{D} = (1 - \hat{p}) \exp\left(-\frac{1}{\gamma_C} N^h\right) + \hat{p} + 2\hat{p}q_L \exp\left(-\frac{1}{\gamma_C} \frac{1}{2} N^h\right) - \hat{p}q_L \exp\left(-\frac{1}{\gamma_C} N^h\right) - \hat{p}q_L.$$

Since

$$\begin{aligned} & 2\hat{p}q_L \exp\left(-\frac{1}{\gamma_C} \frac{1}{2} N^h\right) - \hat{p}q_L \exp\left(-\frac{1}{\gamma_C} N^h\right) - \hat{p}q_L \\ &= -\hat{p}q_L \left(\exp\left(-\frac{1}{\gamma_C} \frac{1}{2} N^h\right) - 1 \right)^2 < 0, \end{aligned}$$

the denominator is always

$$\mathcal{D} < (1 - \hat{p}) \exp\left(-\frac{1}{\gamma_C} N^h\right) + \hat{p},$$

and we deduce the desired result.

Q.E.D.

Remark on the Hybrid Transaction Feasibility

We finally have to consider the point of view of the intermediate agent B . His role is essential as he ensures the feasibility of the global transaction. In the simple case, we have previously seen that he always has an interest in doing the transaction. We now have to look at his interest in doing the hybrid transaction, compared to the simple case. To do so, we compare the fees he can get when doing the hybrid transaction with his fees when doing the simple transaction:

$$\begin{cases} \text{Fees when hybrid transaction: } \pi_C^h - \pi_A^h \\ \text{Fees when simple transaction: } \pi_C - \pi_A \end{cases}.$$

The following result ensures the feasibility of the hybrid transaction, since agent B will get a positive fee from organizing the hybrid arrangement. It also ensures the dominance of the hybrid cat bonds over the simple structure.

Proposition 11: *The indifference buyer's price of agent C for the hybrid structure is always larger than the indifference seller's price of agent A :*

$$\pi_C^h > \pi_A^h.$$

Moreover, the hybrid economic surplus is larger than the simple economic surplus:

$$\pi_C^h - \pi_A^h > \pi_C - \pi_A.$$

This result provides a necessary condition for the feasibility of the hybrid cat bond transaction. As in the case of simple cat bonds, the difference $\pi_C^h - \pi_A^h \geq 0$ reflects the economic surplus derived from Pareto-optimal risk sharing. It will go to agent B or will be shared between agents A and B if the surplus is high enough. Moreover, as the economic surplus derived from hybrid cat bonds, $\pi_C^h - \pi_A^h$, is larger than the one derived from simple cat bonds, $\pi_C - \pi_A$, we get a final strong argument in favor of hybrid cat bonds. The hybrid surplus is indeed high enough to pay for the service provided by agent B , who plays a more active role when hybrid cat bonds replace simple cat bonds.

Proof: By construction, the optimal hybrid nominal, denoted here by $N^{h,*}$, and the hybrid amount, denoted here by H^* , have been characterized as to maximize the fees collected by agent B

$$(N^{h,*}, H^*) = \arg \max \{ \pi_C^h(N^h, H) - \pi_A^h(N^h, H) \}.$$

Hence, for any other amounts (N^h, H) , we have

$$\pi_C^h(N^{h,*}, H^*) - \pi_A^h(N^{h,*}, H^*) > \pi_C^h(N^h, H) - \pi_A^h(N^h, H).$$

Considering for instance $H = 0$ and $N^h = N = \frac{\gamma_C}{\gamma_A + \gamma_C} X$ (which is clearly suboptimal since $H^* = \frac{1}{2}N^{h,*}$ by 8), we have:

$$\pi_C^h(N^{h,*}, H^*) - \pi_A^h(N^{h,*}, H^*) > \pi_C^h(N, 0) - \pi_A^h(N, 0).$$

But, from the characterization of the various indifference pricing rules, we have:

$$\pi_C^h(N, 0) = \pi_C(N) \quad \text{and} \quad \pi_A^h(N, 0) = \pi_A(N).$$

Hence, from Proposition 4, we get the desired result:

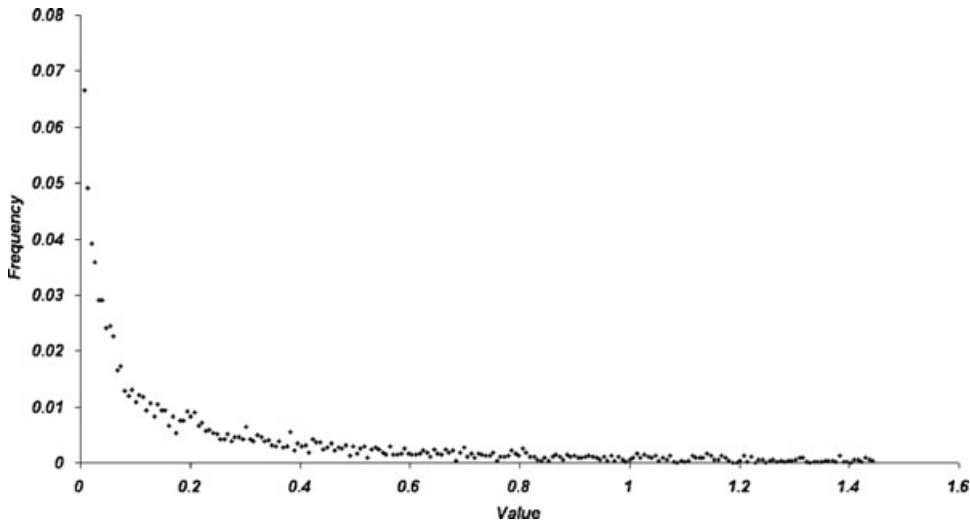
$$\pi_C^h - \pi_A^h > \pi_C - \pi_A > 0. \quad \text{Q.E.D.}$$

The following random experiments give an illustration of the previous proposition. More precisely, we considered X , γ_A , γ_C , $\hat{p}$, and q_U as independent uniform random variables taking values in

$$\left\{ \begin{array}{l} X \in [10, 210] \\ \gamma_A \in [0, 1000] \\ \gamma_C \in [0, 1000] \\ \hat{p} \in [0, 1] \\ q_U \in [0, 1] \end{array} \right.$$

FIGURE 5

Random Values of Delta_B



and then did 5,000 independent drawings. A distribution for the smallest values of

$$\text{Delta}_B = \pi_C^h - \pi_A^h - (\pi_C - \pi_A)$$

is presented in Figure 5.

CONCLUSION

In this article, we analyze the effect of introducing hybrid cat bonds on the volume of capital flowing into the cat bond market. We use the concept of risk measure as analytical tool and proceed in two steps. In the first model, agent *A* is endowed with a catastrophic risk and uses a simple cat bond to transfer part of this risk to a nonexposed agent *C*. The transaction is performed with the intermediation of an SPV, agent *B*, who does not retain any risk. In the entropic framework, the optimal risk transfer depends linearly on the absolute risk tolerance coefficients of agents *A* and *C*, as expected from the classical theory of risk sharing.

In the second model, the downside risk faced by agent *C* is taken into account. This risk materializes when both a natural catastrophe and a financial market crash occur. In this situation, agent *C* loses on both sides of her investment: the cat bond is triggered and returns nothing; at the same time, her financial portfolio value drops severely. To take both risks into account, a hybrid cat bond is set up by agent *B*, who takes care of transferring the conditional stock market risk to the index options market using zero-premium digital calls and puts. The created hybrid cat bond protects agent *C* against the downside risk. In the optimal arrangement, the risk is shared again by agents *A* and *C* according to a more elaborate nonlinear sharing rule.

In both models, we take into account the fact that both agents *A* and *C* are intrinsically exposed to the market risk, as they already hold optimal financial portfolios. This exposure is taken into account in their risk assessment via a modification of their initial risk measure. This has a direct impact on the volume of each transaction, determined in terms of the modified risk measure.

The main result of the article is that introducing hybrid cat bonds would increase the volume of capital flowing into the cat bond market, in particular when investors are strongly risk averse, compared to issuers of cat bonds (insurers, reinsurers, and large corporations). Such a development is much needed given the somewhat disappointing experience recorded with simple cat bonds since they were first introduced in 1997. Hybrid cat bonds would not only enlarge the class of insurance-linked securities. They would also allow better diversification of catastrophe risk in the financial market.

This article represents a first step in the analysis of hybrid cat bonds. For this reason, it is necessarily exploratory. The two models presented are based on some simplifying assumptions, such as a binomial probability distribution for the catastrophe risk and zero-coupon bonds. Future research on this topic will be helpful to check that the main result of this article still holds when some of the simplifying assumptions are relaxed. The ultimate test of our arguments will obtain, however, when hybrid cat bonds appear on the market.

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