

PORTFOLIO CHOICE AND LIFE INSURANCE: THE CRRA CASE

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ABSTRACT

We solve a portfolio choice problem that includes life insurance and labor income under constant relative risk aversion (CRRA) preferences. We focus on the correlation between the dynamics of human capital and financial capital and model the utility of the family as opposed to separating consumption and bequest. We simplify the underlying Hamilton–Jacobi–Bellman equation using a similarity reduction technique that leads to an efficient numerical solution. Households for whom shocks to human capital are negatively correlated with shocks to financial capital should own more life insurance with greater equity/stock exposure. Life insurance hedges human capital and is insensitive to the family’s risk aversion, consistent with practitioner guidance.

INTRODUCTION AND MOTIVATION

There is a glaring disconnect between the way most financial advisers sell life insurance and how they sell or promote investment products such as mutual funds, stocks, or bonds. Aside from the regulatory environment and the different licenses required—namely, securities as opposed to insurance—these two *financial decisions* are presented as if there were a “separation theorem” that justified their relative invariance. Moreover, although the concepts of risk tolerance, risk aversion, and utility are ubiquitous in the lingo and marketing material of the securities industry, the same terminology rarely enters the dialogue in the life insurance arena. This is quite odd since historically the economics of insurance was the breeding ground for much of the development in utility theory.

This observation is more than just anecdotal. The insurance literature’s starting point for the optimal face amount of life insurance is the original work by Solomon Huebner

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in the early 1900s, based on the concept of a human life value (HLV). This idea is also at the heart of much forensic economics and litigation, where the courts must determine the value of a lost life, for the purposes of compensation. See Zietz (2003) or Todd (2004) for a comprehensive review of the academic literature on life insurance. Most financial planning and investment textbooks advocate something called a *needs analysis*, where the family determines how much would be required to meet long-term expenses and other financial goals if the primary source of income were lost. Other authors focus on the present value of lost wages. Regardless of the precise mechanics, rarely is this decision presented as a portfolio allocation problem akin to the investment in stocks and bonds. We believe it should. After all, life insurance is a hedge against the loss of human capital and portfolio hedging decisions should be made jointly, not independently.

Therefore, our objective in this article is to jointly analyze the decision of how much life insurance a *family unit* should have to protect against the loss of its *breadwinner* as well as how the family should allocate its financial resources between risk-free and risky assets vis-à-vis the dynamics of labor income and human capital in our¹ model.

We view this problem within the paradigm of portfolio choice and use the tools of financial economics pioneered by Merton (1969, 1971) to arrive at optimal controls for investment, consumption, and life insurance—but where the family unit is placed front and center.

A number of recent articles have extended the set of decisions included in the portfolio choice problem to highlight the interaction and the risks faced by the household, broadly defined. Thus, for example, Goetzmann (1993), Yao and Zhang (2005), and as Cocco (2005) focus on the role of the housing portfolio; Campbell and Cocco (2003) focus on optimal mortgage choices; Cairns, Blake, and Dowd (2006) examine portfolio choice in defined contribution pension plans; Sundaresan and Zapatero (1997) examine the role of defined benefit pensions; whereas Dybvig and Liu (2004) and Bodie et al. (2004) examine consumption and investment decisions in a life-cycle model with habit formation, stochastic opportunity set, stochastic wages, and labor supply flexibility. They vary retirement dates but do not endogenize it; Jagannathan and Kocherlakota (1996) and Viceira (2001) stress the impact of aging; Faig and Shum (2002) are motivated by the demand for illiquid assets; Koo (1998) and Davis and Willen (2000) model the role of labor income; Dammon, Spatt, and Zhang (2001) focus their attention on capital gains and income taxes; and Heaton and Lucas (2000) focus on the role of entrepreneurial risk. Others go back to basics and extend portfolio choice models to include more sophisticated (and realistic) processes for investment returns, like Chacko and Viceira (2005), or time-varying and mean-reverting risk premiums, like Kim and Omberg (1996) or Detemple, Garcia, and Rindisbacher (2003).

Although the above list is not exhaustive, the unifying theme of the burgeoning portfolio choice literature is that a singular *personal financial* feature is highlighted

¹ And although we refer to this type of protection as life insurance, our framework could also apply to (permanent) disability and other income replacement insurance. We would like to thank one of the *JRI* reviewers for pointing out the applicability of our model beyond traditional life insurance.

and carefully modeled to provide relevant financial economic insights related to that feature. Some of these articles test or calibrate their models against real-world data, whereas others end their contribution with a normative model. Most of them use Merton (1969, 1971) as a starting point.

Our objective is similar. We start with a traditional diffusion model of asset dynamics and constant relative risk aversion (CRRA) utility of consumption, but we focus on modeling labor income and life insurance purchases. A central feature of our model is the correlation between innovations to the labor income process and financial returns. We are specifically interested in the interaction between the utility-maximizing demand for life insurance *and* the optimal consumption pattern and asset allocation when there is a nonzero correlation between these two critical state variables. We deviate somewhat from previous models by focusing on the *family unit* that derives utility from consumption and the risk to this family of losing the wage/income source. The emphasis on the family as a unit, which survives independent of the life status of the breadwinner, alleviates the need for specifying a separate utility of bequest. See Zietz (2003) for a review of the literature and the references therein that analyze life insurance from the perspective of the family as opposed to the survivors.

Life insurance has been previously analyzed in a financial context. Yaari (1965), Hakansson (1969), Fischer (1973), Campbell (1980), Lewis (1989), Hurd (1989), Babbel and Ohtsuka (1989), and more recently, Purcal (2003) develop a number of insights into the demand for life insurance. These articles illustrate how the nature of mortality uncertainty, the utility and strength of bequest desire, risk aversion, increasing household wealth, and intertemporal rates of substitution impact the decision to purchase life insurance and the type of life insurance to buy. But these models do not give us much insight into the role of labor income dynamics and human capital over the life cycle. The same is true of the work by Richard (1975), which extended Merton (1969, 1971) to life insurance but with little numerical insight or analytic solutions, given the complexity of the problem. Along the same lines, Buser and Smith (1983) adopt a portfolio approach to life insurance but from a strictly one-period framework. Our contributions will be emphasized when we present the specifics of our model. Likewise, our work differs from the recent article by Chen et al. (2005), where the analysis is done in discrete time and via simulations. One of the main differences between the current work and most previous research is that we avoid specifying a utility of bequest yet are able to talk about the demand for life insurance. Our results confirm many of the earlier insights.

From a technical point of view, two state-variable problems—i.e., wage income and asset prices—normally lead to highly nonlinear Hamilton–Jacobi–Bellman (HJB) partial differential equations (PDEs) that are difficult to solve. Despite this nonlinearity, closed-form solutions to our model can be obtained in special cases when wage income and asset prices are perfectly (positively or negatively) correlated. Furthermore, when this correlation is between -1 and $+1$, we obtain a variety of efficient approximate solutions. One of the keys to our solution methodology, and indeed one of the contributions of this article, is to locate a *similarity reduction*, which reduces the dimension of the HJB equation.

The method of similarity reductions is a technique for solving PDEs (see, e.g., Ockendon et al., 1999). The basic idea is to find a special combination of the variables so that a new PDE can be obtained, which is simpler than the original PDE. For example, in the classical one-dimensional diffusion equation, spatial (state) and time variables can be collapsed into a new variable by taking the ratio of the spatial variable and the square root of the time variable. This idea (or technique) has been used in the finance literature. For example, Merton (1969, 1971) solved the optimal-consumption and asset allocation problem by scaling the value function with a particular function of wealth. In the option pricing literature, the ratio of an asset and its average (maximum or minimum) value is used as a similarity variable for pricing lookback (and Asian) options (see, e.g., Wilmott, Dewynne, and Howison, 1993). Likewise, Kahl, Liu, and Longstaff (2003) used the ratio of asset and total wealth as a new variable, for solving an asset allocation problem with illiquid stock holdings. Thus, although locating similarity reductions is appealing, the practice is more an art than a science. We will elaborate on this in the “Solution Methodology” section.

Our main qualitative results are as follows. First, we find that the optimal amount of (face value) life insurance a family should have depends on volatility of the wage/income process as well as its correlation to investment returns. Higher volatility and higher correlations lead to lower, optimal, life insurance levels. *Ceteris paribus*, a tenured university professor needs more life insurance than a Wall Street investment banker, even if they both expect to earn the same wages over time. Furthermore, as one would expect, the risk profile of the wage/income process has a strong impact on the optimal mix between risky and risk-free assets for the family. This result should come as no surprise. The original work by Bodie, Merton, and Samuelson (1992), all the way to the recent work by Cocco, Gomez, and Maenhout (2005), have stressed the financial characteristics of human capital and how it serves as a substitute for financial capital. Our model easily reproduces this result and emphasizes the role of life insurance, which is a hedge for human capital. Finally, our model confirms the intuition shared by many financial advisors that the optimal amount of life insurance a family should hold is not sensitive to the risk aversion or risk tolerance of the family. With CRRA, the optimal demand for life insurance is insensitive to the degree of relative risk aversion, even though the family’s financial portfolio is obviously quite sensitive to this parameter. This opens up the possibility of testing whether the actual holding of life insurance is, in fact, correlated with other proxies for a family’s risk aversion. Our model provides justification for finding no such relationship.

The rest of the article is organized as follows. The setup of the problem is described in the “Model of the Family Life Cycle” section, followed by the setup of the HJB equation in the “The Hamilton–Jacobi–Bellman” section. Solution methodologies are discussed in the “Solution Methodology” section. In the “Special Case: $\rho = \pm 1$ ” section we focus on special cases when the wage and risky asset are perfectly correlated. The general case is discussed in the “General Case: $-1 < 1$ ” section. We further simplify the HJB equation using the similarity reduction technique, which is followed by the presentation of the numerical method for solving the reduced HJB equations, where the proper boundary conditions are also discussed. In the “Closed-Form Approximation Solution” section we present a closed-form approximate solution, which

proves to be quite accurate in practice. In the “Numerical Results and Discussion” section, some results and the financial implications of these results are presented. We summarize the article in the “Conclusion.” The derivations of the HJB equation and other nonessential derivations are placed in the appendices.

MODEL OF THE FAMILY LIFE CYCLE

The Input Variables

We will work with three dates of interest. The first date is the time horizon of the family T_H , which is exogenous and deterministic. The second date is the time of retirement $T_R < T_H$. The third date of interest is the death of the breadwinner, which is the end of the wage/income process taking place at a random stopping time, denoted by τ . The wage/income process jumps to zero at the minimum of T_R and τ .

The family purchases short-term insurance on the life of the breadwinner, which is renegotiated and guaranteed renewable on an ongoing basis at a predetermined schedule that is driven by an *instantaneous force of mortality (IFM) curve* denoted by λ_{y+t} , where y is the age of the primary breadwinner at time zero. Let I_t denote the insurance premium (in dollars) payable per unit time; a variable that is under the direct control of the family. One can think² of I_t as a budget for insurance that induces a certain face value or death benefit I_t/λ_{y+t} .

We let M_t denote the market value of the family’s assets including the value of all (risky) stocks and (risk-free) bonds on a mark-to-market basis. We assume that M_0 denotes the initial marketable wealth at time $t = 0$. The breadwinner works and converts labor and time into wages and income. A portion of this income is consumed and the remainder is saved in a diversified portfolio, consisting of risky stocks and safe bonds.

The variable α_t denotes the fractional allocation of the family’s marketable wealth M_t to the risky asset at time t . Thus, if $\alpha_t = 0$, the family allocates all its marketable wealth—but no more—to risk-free bonds, and if $\alpha_t = 1$, the family allocates all its marketable wealth—but no more—to risky stocks. The model also allows for $\alpha_t > 1$ which would imply leverage. Thus, for example, $\alpha_t = 2$ implies that 200 percent of net worth is invested in the risky stock. This is financed by borrowing 100 percent of wealth at some (constant) rate of interest, denoted by r . Recall that α_t is under direct control of the family and is one of the *three* choice variables in our model. We do not address liquidity constraints faced by younger families when borrowing. Many of our numerical results lead to large amount of leverage.

Let c_t denote the instantaneousness consumption rate of the family per unit time. In general, our model is specified in real terms, and all parameters and choice variables will reflect this. The consumption rate is our third choice variable and c_t is chosen to maximize the family’s *utility* of consumption, which includes the breadwinner as well. We refer readers to the review article by Zietz (2003), and the various references

² For example, if the IFM curve at age $y = 35$ is $\lambda_{35} = 0.001$, and the family spends $I = \$50$ on life insurance, this entitles the family to a death benefit of $50/0.001 = \$50,000$.

therein, in which the goals of a family versus an individual is presented as a possible objective.

Even though a more realistic state-dependent instantaneous utility function can be used under our PDE-based methodology, we adopt the simpler CRRA utility function in this article. The precise functional form is

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma} \quad (1)$$

for some positive coefficient of relative risk aversion $\gamma > 0$.

Let Z_t denote the market value of the risky asset (stock index, market portfolio) at time t , which follows

$$dZ_t = \mu_m Z_t dt + \sigma_m Z_t dB_t^m, \quad (2)$$

where μ_m denotes the constant drift and σ_m the volatility. The risk-free rate r , which is the rate at which the family can borrow money, must be lower than μ_m for economic equilibrium purposes.

Let W_t denote the real wage/income rate of the family's breadwinner per unit time. This stochastic process is expected to increase over time and might be correlated with the risky asset. The wage process satisfies

$$dW_t = \begin{cases} \mu_w W_t dt + \sigma_w W_t dB_t^W & t < \tau, \\ 0 & t \geq \tau, \end{cases} \quad (3)$$

where τ is the random time of death, μ_w denotes the constant drift, σ_w denotes the constant volatility, and B_t^W denotes the Brownian motion driving the wages process W_t . The Brownian motion B_t^m driving the risky asset is instantaneously correlated with the B_t^W driving the wage process via the relationship $d\langle B_t^m, B_t^W \rangle = \rho \sigma_m \sigma_w dt$. This correlation variable ρ is the primary focus of our numerical case study and results.

Note that if two individuals start working at time 0 and face the same (μ_w, σ_w) wage process, the expected present value of their income, i.e., the discounted value of their human capital, is identical. In other words, if both individuals start out earning \$100,000 per year, and they expect it to grow by an inflation adjusted 5 percent each year, with a volatility of 3 percent, for example, then the discounted value of their human capital is identical. However, of course, the *economic value of their human capital* can be quite different. That will depend on the correlation between their wages and the market portfolio—and that is exactly our point. The amount of life insurance they will purchase should depend on the economic value of their human capital, *not* the discounted value of their human capital. Indeed, practitioners in the insurance industry tend to focus on the discounted value (i.e., seven times annual salary) in their insurance rules of thumb, whereas in truth, this might be an over/underestimate, depending on the correlation coefficient.

As mentioned earlier, the IFM (or hazard rate) is denoted by λ_{y+t} , where y is the initial age. The conditional probability of survival, from age y to age $y + t$, under the

law of mortality defined by λ_{y+t} can be computed via

$${}_t p_y = e^{-\int_0^t \lambda_{y+s} ds}, \quad (4)$$

where the notation on the left-hand side of the equation is standard in the actuarial literature. If the future-lifetime random variable is exponentially distributed, we have that $\Pr[\tau \geq s] = e^{-\lambda s}$, and the hazard rate is constant at all ages $\lambda_{y+t} = \lambda$. In the general case, the function λ_{y+t} is expected to increase with time (age). In our model, the family unit knows exactly how much they will have to pay for insurance—regardless of how much and when they want to purchase it—from the current time t , to the time of retirement T_R . Thus, in addition to precluding whole life and other more complicated forms of insurance, we do not allow for stochastic mortality rates or antiselection effects that might complicate the insurance purchase problem. For now, we ignore insurance loadings, commissions, and profits, which can easily be handled by working with a loaded hazard rate $\widehat{\lambda}_{y+t}$ instead of a biological mortality rate λ_{y+t} .

The Financial Wealth Dynamics

Based on the wage/income process W_t and the evolution of the risky asset price Z_t , the family budget constraint for the wealth process M_t follows

$$\begin{aligned} dM_t &= W_t dt - c_t dt - I_t dt + \alpha_t M_t (\mu_m dt + \sigma_m dB_t^m) + (1 - \alpha_t) r M_t dt \\ &= [(\mu_m - r) M_t \alpha_t + r M_t + W_t - c_t - I_t] dt + \sigma_m \alpha_t M_t dB_t^m, \end{aligned} \quad (5)$$

for $t < \min[\tau, T_R]$, and

$$\begin{aligned} dM_t &= -c_t dt + \alpha_t M_t (\mu_m dt + \sigma_m dB_t^m) + (1 - \alpha_t) M_t r dt \\ &= [(\mu_m - r) M_t \alpha_t + r M_t - c_t] dt + \sigma_m \alpha_t M_t dB_t^m \end{aligned} \quad (6)$$

for $t > \min[\tau, T_R]$.

At the instant of death $t = \tau$, we add the death benefit by distinguishing between the value of marketable wealth one instant prior to death and one instant after. More precisely,

$$M_{\tau+} = M_{\tau-} + \frac{I_\tau}{\lambda_{y+\tau}}, \quad (7)$$

where $I_\tau/\lambda_{y+\tau}$ is the death benefit or the face value of the insurance policy at time τ and $\lambda_{y+\tau}$ is the IFM, or hazard rate.

Finally, the family-unit or household objective function

$$\max_{\{\alpha_s, I_s, c_s\}} E_t \left[\int_t^{T_H} e^{-\delta s} u(c_s) ds \right], \quad (8)$$

where δ is the subjective discount rate and $u(c)$ denotes the instantaneous utility of consumption at time t . The family is searching for an asset allocation strategy α_s , an insurance buying strategy I_s , and a consumption strategy c_s that maximize the discounted value of utility of consumption between time t and the terminal horizon of the family unit.

As explained and justified earlier, we have “eliminated” the need for a utility of bequest by treating the family as one unit that is dependent on the breadwinner for their source of income. Thus, in contrast to the insurance models of Yaari (1965), Fischer (1973), or Campbell (1980), the primary breadwinner is not forced to decide how much “they love their family,” relative to how much they “love themselves.” Rather, the family allocates consumption across the entire horizon of the family, while protecting the wage/income flow using insurance. This assumption alleviates the need to measure the strength of bequest but assumes that the family derives the exact same level of utility, whether the primary breadwinner is alive or not.

THE HJB EQUATION

With the model’s foundations and formulation in place, let

$$J(M_t, W_t, t) = \max_{\{\alpha_s, I_s, c_s\}} E \left[\int_t^{T_H} e^{-\delta s} u(c_s) ds | \mathcal{F}_t \right] \quad (9)$$

be the indirect utility function. Assume we can find the optimal control that satisfies Equation (9), then J satisfies the following HJB equation (see Appendix A for the detailed derivation):

$$\begin{aligned} \lambda_{y+t} J &= J_t + \max_{c_t} (e^{-\delta t} u(c_t) - c_t J_M) \\ &+ \max_{I_t} \left(\Phi \left(M_t + \frac{I_t}{\lambda_{y+t}}, t \right) \lambda_{y+t} - I_t J_M \right) \\ &+ \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M_t J_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M_t^2 J_{MM} + \alpha_t \rho \sigma_W \sigma_m W_t M_t J_{WM} \right] \\ &+ r M_t J_M + W_t J_W + \mu_W W_t J_W + \frac{1}{2} \sigma_W^2 W_t^2 J_{WW}, \end{aligned} \quad (10)$$

where the Φ satisfies a simpler (fundamental) HJB equation but without wage income and life insurance:

$$\begin{aligned} 0 &= \Phi_t + \max_{c_t} (e^{-\delta t} u(c_t) - c_t \Phi_M) \\ &+ \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M_t \Phi_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M_t^2 \Phi_{MM} \right] + r M_t \Phi_M. \end{aligned} \quad (11)$$

For CRRA utility, the analytic form of $\Phi(M, t; T)$ can be obtained as

$$\Phi(M, t; T) = h(t; T) \frac{M^{1-\gamma}}{1-\gamma}, \quad (12)$$

where $h(t; T)$ satisfies the following ordinary differential equation

$$\frac{1}{1-\gamma} \frac{h'}{h} + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} h^{-\frac{1}{\gamma}} + r + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2} = 0. \quad (13)$$

Hereafter, the prime symbol is used to denote the derivative with respect to time. The closed-form solution of Equation (13), with zero utility at time T , can be obtained as

$$h(t; T) = e^{-\delta t} \left(\frac{e^{\xi(T-t)} - 1}{\xi} \right)^{\gamma}, \quad (14)$$

where

$$\xi = -\frac{\delta}{\gamma} + \frac{1-\gamma}{\gamma} \left(r + \frac{(\mu_m - r)^2}{2\gamma\sigma_m^2} \right). \quad (15)$$

After applying the first-order conditions, the HJB equation for J can be rewritten as

$$\begin{aligned} \lambda_{y+t} J = & J_t + \frac{\gamma}{1-\gamma} c_t^* J_M + \frac{\lambda_{y+t}\gamma}{1-\gamma} (M + \lambda_{y+t}^{-1} I_t^*) J_M + \lambda M J_M \\ & + \alpha_t^* (\mu_m - r) M J_M + \frac{1}{2} (\alpha_t^*)^2 \sigma_m^2 M^2 J_{MM} + \alpha_t^* \rho \sigma_W \sigma_m W M J_{WM} \\ & + r M J_M + W J_M + \mu_W W J_W + \frac{1}{2} \sigma_W^2 W^2 J_{WW}. \end{aligned} \quad (16)$$

The terminal condition is

$$J(M, W, T_R) = \Phi(M, T_R; T_H). \quad (17)$$

Finally, the optimal control c_t^* (consumption), α_t^* (allocation to risky equity), and I_t^* (amount spent on life insurance purchases) are given by

$$c_t^* = (e^{\delta t} J_M)^{-\frac{1}{\gamma}}, \quad (18)$$

$$\alpha_t^* = -\frac{(\mu_m - r) J_M + \rho \sigma_m \sigma_W W J_{WM}}{\sigma_m^2 M J_{MM}}, \quad (19)$$

$$I_t^* = \left[\left(\frac{J_M}{h(t; T_H)} \right)^{-\frac{1}{\gamma}} - M \right] \lambda_{y+t}. \quad (20)$$

SOLUTION METHODOLOGY

From here our article proceeds in the following manner. Remember that at this point, all we have is a two-dimensional PDE, representing the two uncertain variables—wages and investment returns—for the solution we have denoted by J ; the indirect

utility function. The optimal consumption, asset allocation, and insurance are all functions of J and its various derivatives.

To our knowledge, it is impossible to obtain a closed-form expression for J under the general conditions. However, there are some special and unique cases in which J can be obtained explicitly. For example, the well-known Merton (1969, 1971) analytic solution is obtained when the wage parameters are all set to zero. Another of those special cases is when the correlation between shocks to wages and investment returns are perfect. This effectively creates only one source of economic uncertainty in the model, and in this case too we derive explicit answers for the optimal consumption, asset allocation, and insurance.

In the general case, however, when the correlation between the random variables is not perfect, which means that we truly have two financial sources of uncertainty, the situation is more complicated. Nevertheless, we are able to employ a so-called similarity reduction technique, which is explained later on, that helps us to reduce the dimensionality of the problem, leaving us with only a one-dimensional PDE.

Of course, even though we are left with a one-dimensional PDE for J , we are still not “out of the woods” because even this (simpler) PDE does not have an analytic solution. At that point our narrative proceeds along two distinct paths.

First, we employ numerical methods to solve the new (albeit reduced dimension) PDE by brute force. More specifically, we use the method of lines, which converts the PDE into a system of ODEs that are then solved using numerical integration techniques. Although this provides us with reasonable solutions for any given set of parameters, it is time consuming and hence prone to the usual stability problems. It is not ideal for full-scale real-time implementation unless a more efficient numerical scheme can be found and implemented.

Then, as an alternative path to the above computationally intensive approach, we formulate an analytic approximation to the implied utility. This approximating function for J —which we can solve more easily—is quite accurate when compared with numerical results for nonperfect wage/returns correlations. It is confirmed against the case of perfect correlation. This analytic approximation functions at its best when the underlying economic parameters fall in a certain region, which we describe later.

Finally, we provide some case studies using our analytic approximation, since the parameter values we are interested in fall within the required range. We believe that one of the contributions of this article is our methodological approach to solving the main PDE, which is a problem that vexes many other articles in the literature. We now describe the highlights of the solution and refer the reader to the appendices for more details.

Special Case: $\rho = \pm 1$

As shown in Appendix A, the solution of the HJB Equation (16) takes the form

$$J(M, W, t) = \frac{h(t)(M + k(t)W)^{1-\gamma}}{1-\gamma}, \quad (21)$$

where h is given by (14) and k satisfies

$$\frac{k'}{k} + \frac{1}{k} + \eta = 0, \quad (22)$$

with

$$\eta = \mu_w - r - \lambda_{y+t} - \beta(\mu_m - r) \quad (23)$$

and

$$\beta = \rho\sigma_w/\sigma_m. \quad (24)$$

Using the terminal condition $k(T_R) = 0$, k can be solved as

$$k = e^{-\int_{T_R}^t \eta(s)ds} \int_{T_R}^t e^{\int_{T_R}^s \eta(n)dn} ds. \quad (25)$$

When $\lambda_{y+t} = \lambda$, i.e., a constant mortality rate, we have

$$k = \frac{1}{\eta} (e^{\eta(T_R-t)} - 1). \quad (26)$$

Remark: Since $\beta = \pm \sigma_w/\sigma_m$ for $\rho = \pm 1$, η is the only parameter affected by the value of ρ . For given values of W/M and t , the solution is solely determined by the value of β or η , while other parameter values are fixed. This relationship illustrates how risk aversion does not impact the optimal face value of life insurance.

General Case: $-1 < \rho < 1$

For the general case of correlated wage and risky asset, a closed-form solution cannot be obtained. However, we can define a new variable $x = W/M$ and use this as our “similarity variable” without the restriction of a simple analytical form. Even though our model itself is valid when M is negative, for technical reasons we will restrict our discussion (and examples) to the case of $M > 0$ for the rest of this article. We note that this is not a serious restriction since total household wealth is usually positive in reality. In the rare event that M is negative, we can either use the closed-form approximate solution (see the “Closed-Form Approximate Solution” section) or solve the two-dimensional Equation (16) using numerical methods described in Wang (2006).

Similarity Reduction. We now show that the two-dimensional HJB equation can be reduced by one dimension via the similarity reduction technique. We start by letting

$$J(M, W, t) = \frac{M^{1-\gamma}}{1-\gamma} F(x, t).$$

This leads to a new PDE,

$$F_t + c_0(1 - \gamma)F + c_1xF_x + c_2x^2F_{xx} = 0, \quad (27)$$

where

$$\begin{aligned} c_0 &= \frac{c_t^*}{M} \frac{\gamma}{1 - \gamma} + \frac{I_t^*}{M} \frac{\gamma}{1 - \gamma} + \alpha_t^*(\mu_m - r) + x + r - \frac{1}{2}\gamma(\alpha_t^*\sigma_m)^2, \\ c_1 &= \mu_w - \frac{c_t^*}{M} \frac{\gamma}{1 - \gamma} - \frac{I_t^*}{M} \frac{\gamma}{1 - \gamma} - \frac{\lambda_{y+t}}{1 - \gamma} \\ &\quad - x - r - \alpha_t^*(\mu_m - r) - \alpha_t^*\rho\sigma_m\sigma_w\gamma + (\alpha_t^*\sigma_m)^2\gamma, \\ c_2 &= \frac{1}{2}(\alpha_t^*\sigma_m - \rho\sigma_w)^2 + \frac{1}{2}\sigma_w^2(1 - \rho^2), \\ \frac{c_t^*}{M} &= \exp\left(\frac{\delta t}{\gamma}\right) \left(F - \frac{x F_x}{1 - \gamma}\right)^{-\frac{1}{\gamma}}, \\ \frac{I_t^*}{M} &= \left[h^{\frac{1}{\gamma}} \left(F - \frac{x F_x}{1 - \gamma}\right)^{-\frac{1}{\gamma}} - 1\right] \lambda_{y+t}, \\ \alpha_t^* &= \frac{(\mu_m - r)[(1 - \gamma)F - x F_x] - \rho\sigma_m\sigma_w(\gamma x F_x + x^2 F_{xx})}{\sigma_m^2[\gamma(1 - \gamma)F - 2\gamma x F_x - x^2 F_{xx}]}. \end{aligned} \quad (28)$$

The terminal condition is now

$$F(x, T_R) = h(T_R, T_H). \quad (29)$$

The coefficients c_0, c_1, c_2 , and the terminal conditions are functions of x only, so that the HJB equation permits the similarity solution, which is a function of x and t .

Numerical Method. First, we truncate the semi-infinite domain ($x > 0$) into a finite one ($0 < x < X$) and set up a uniform mesh with size $\delta x = X/N$ and grid points $x_j = j\delta x$ for $j = 0, 1, \dots, N$. Then, we use the method of lines to solve (27); i.e., we approximate F by F_j on the grid points and its spatial derivatives by finite differences, which leads to a system of ordinary differential equations with respect to time on grid point x_j :

$$\begin{aligned} F_j' + c_0 F_j + \frac{|c_1| + c_1}{2\delta x} (F_{j+1} - F_j) + \frac{|c_1| - c_1}{2\delta x} (F_j - F_{j-1}) \\ + \frac{c_2}{\delta x^2} (F_{j+1} + F_{j-1} - 2F_j) = 0 \end{aligned} \quad (30)$$

for $j = 1, 2, \dots, N - 1$. Here c_0, c_1 , and c_2 are evaluated on the grid points, based on the formula in (28). Note that the first upwind method is used for the first-order derivative term in x .

Boundary Conditions. We observe that $x_0 = 0$ corresponds to $W = 0$, i.e., no wage income. Therefore, the solution is of Merton type (with the addition of insurance),

with F not a function of x . In such cases, F satisfies the ordinary differential equation

$$F_t + c_0(1 - \gamma)F = 0$$

or

$$F'_0 + c_0(1 - \gamma)F_0 = 0. \quad (31)$$

At $x_N = X$ (the truncation of the domain), the situation is more complicated. Motivated by the earlier closed-form solution, we postulate

$$F^{\frac{1}{1-\gamma}} \sim 1 + kx,$$

which is a linear function of x , therefore $(F^{\frac{1}{1-\gamma}})_{xx} \sim 0$. We approximate this by

$$F^{\frac{1}{1-\gamma}}_{N+1} = 2F^{\frac{1}{1-\gamma}}_N - F^{\frac{1}{1-\gamma}}_{N-1}. \quad (32)$$

The system of ordinary differential Equation (30) with boundary conditions (31) and (32) is then solved by an appropriate numerical integration technique. For example, we can use a semi-implicit method, where the coefficients c_0 , c_1 , and c_2 are evaluated at the previous time level and the spatial derivatives are treated implicitly. In this article, we use a built-in Matlab solver `ode15s.m`.³

Closed-Form Approximate Solution

Motivated by the solution for the special case $\rho = \pm 1$, we now seek a closed-form approximate solution for the general case $-1 < \rho < 1$. To proceed formally, we define a new parameter

$$\epsilon = (1 - \rho^2) \left(\frac{\gamma \sigma_m \sigma_w}{\mu_m - r} \right)^2 \quad (33)$$

and consider the case when $\epsilon \ll 1$. Note that the special cases considered earlier correspond to $\epsilon = 0$.

From an economic point of view, one can think of ϵ as a measure of “incompleteness” of the market. Clearly, when either $\rho = 1$ or -1 or $\sigma_m = 0$ or $\sigma_w = 0$, the value of $\epsilon = 0$, and we are in a world in which there is only one source of economic risk. Likewise, when $\gamma = 0$, the individual is risk neutral, i.e., various risk-states of nature are treated the same and $\epsilon = 0$ as well. This can be viewed (with a bit of creativity and imagination) as yet another form of market completeness.

³ We have experimented with different solvers and the solutions are almost identical.

Based on the procedure outlined in Appendix B, we can write our solution in the form $J = J_0 + O(\epsilon)$. For J_0 , we have the following explicit form:

$$J_0(M, W, t) = \frac{M^{1-\gamma}}{1-\gamma} h(1+kx)^{1-\gamma}, \quad (34)$$

where h and k are given earlier. For a constant mortality rate, they can be solved explicitly as

$$h(t) = e^{-\delta t} \left[\frac{e^{\xi(T_H-t)} - 1}{\xi} \right]^\gamma, \quad (35)$$

$$k(t) = \frac{1}{\eta} (e^{\eta(T_R-t)} - 1), \quad (36)$$

with ξ and η defined earlier by Equations (15) and (23). Recall that a constant mortality rate $\lambda_t = \lambda$ implies that future lifetime is exponentially distributed.

NUMERICAL RESULTS AND DISCUSSION

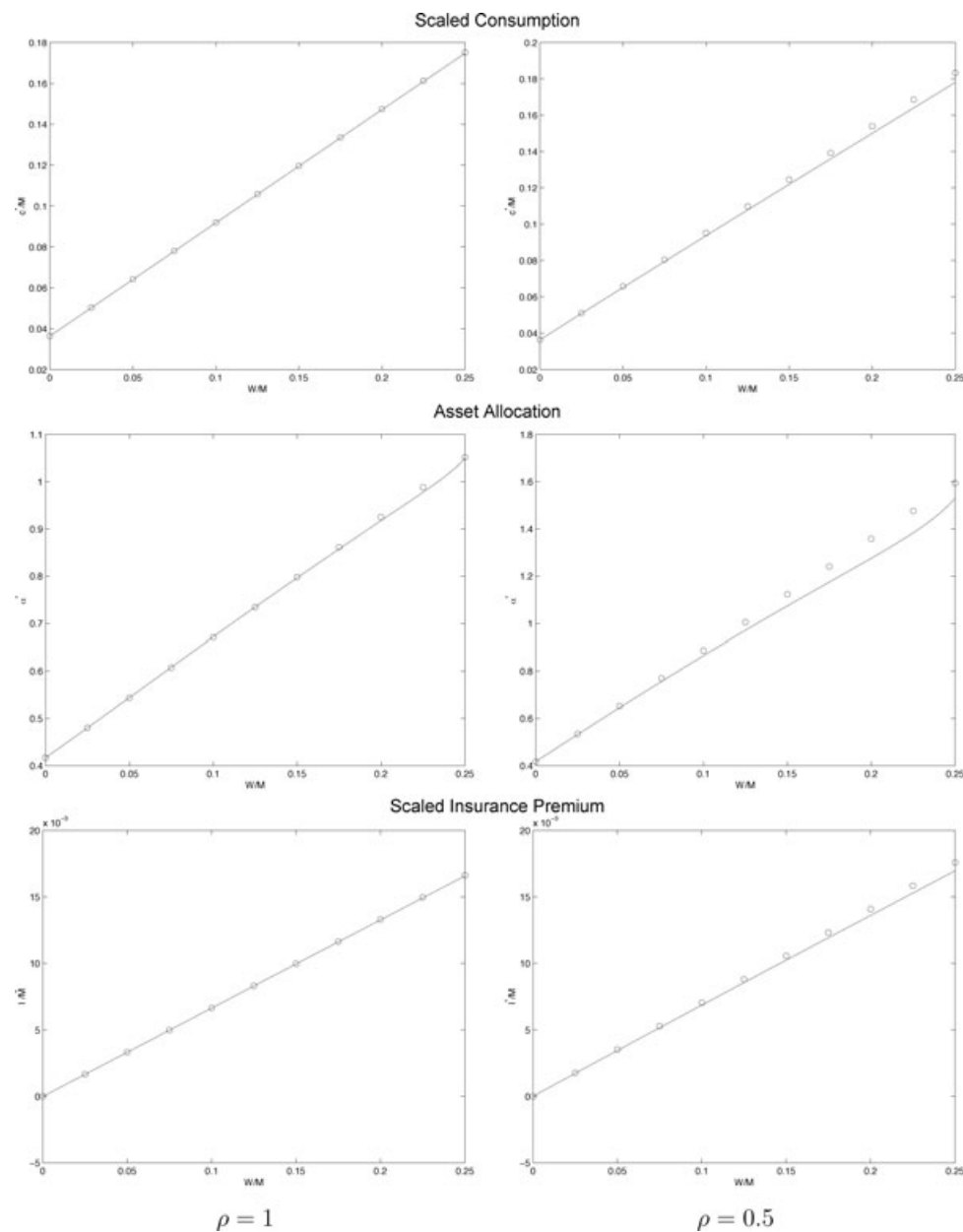
We now present a case study for a realistic set of economic parameters. The parameter values we have chosen for our results are as follows: $\mu_m = 7$ percent; $\sigma_m = 20$ percent; $r = 2$ percent; $\mu_w = 1$ percent; $\sigma_w = 5$ percent; $M_0 = 1, 200$; $W_0 = 50$; $T_R = 10, 20, 30$; $T_H = 40, 50, 60$; $\delta = 2$ percent; $y = 35, 45, 55$.

The financial justification for these numbers are as follows. We assume that the family can invest their financial wealth in a broadly diversified index-fund that is expected to earn an inflation-adjusted arithmetic mean of $\mu_m = 7$ percent per annum, with a volatility of $\sigma_m = 20$ percent. These numbers are consistent with other asset allocation research in the literature (e.g., Campbell and Viciera, 2002; Chen et al., 2005). The risk-free rate of $r = 2$ percent after inflation is also consistent with current conditions in the inflation-adjusted bond market. Wages grow in real terms by $\mu_w = 1$ percent per annum approximately, with a standard deviation⁴ of $\sigma_w = 5$ percent. This is consistent with wage inflation being higher than price inflation but also allows for shocks to wages, such as unemployment or business cycle effects. Finally, we assume the family's discount rate—or subjective rate of time preference—is $\delta = r = 2$ percent.

In terms of other embedded parameters, we assume that insurance prices (and mortality rates) are driven by a Gompertz law of mortality, which, at age 45, starts at $\lambda_{45+t} = \exp\{(45 + t - 86.3)/9.5\}/9.5$. These numbers are consistent with survival rates implicit in pension-based mortality tables, and the analytic law of mortality has

⁴ We use 5 percent for the standard deviation of wages even though there is a lack of consensus in the literature regarding the precise magnitude of this variable. For example, MaCurdy (1982) estimates the error structure for wages and earnings of prime-age males, using data from the Michigan Panel of Income Dynamics, and estimates numbers much higher than 5 percent.

FIGURE 1
Comparison Between Numerical and Approximate Solutions



been used in a variety of actuarial and insurance articles (e.g., Frees, Carriere, and Valdez, 1996).

Our first order of business is to confirm that for these parameters values, our analytic approximation for $\rho \neq 1$ is valid and accurate. Figure 1 provides a visual confirmation

TABLE 1

Life Insurance (in Thousands) and Allocation to Stocks 20 Years Away From Retirement, at Age 45

Risk Aversion		Correlation Between Wages and Stock Market				
		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0.0$	$\rho = +0.5$	$\rho = +1.0$
$\gamma = 1.5$	α_t^*	6.06	5.18	4.40	3.69	3.06
Low	I_t^*/λ_{y+t}	\$964.7	\$907.7	\$855.2	\$806.8	\$762.1
$\gamma = 3$	α_t^*	3.63	2.88	2.20	1.59	1.05
Medium	I_t^*/λ_{y+t}	\$964.7	\$907.7	\$855.2	\$806.8	\$762.1
$\gamma = 4.5$	α_t^*	2.82	2.11	1.47	0.894	0.384
High	I_t^*/λ_{y+t}	\$964.7	\$907.7	\$855.2	\$806.8	\$762.1

Note: Parameter assumptions: $\mu_m = 7\%$, $\sigma_m = 20\%$, $r = 2\%$, $\mu_w = 1\%$, $\sigma_w = 5\%$, $M_0 = 200$, $W_0 = 50$, $T_R = 20$, $T_H = 50$, $\delta = 2\%$, $y = 45$, $m = 86.3$, $b = 9.5$.

of this fact by comparing the optimal insurance purchase I_{y+t} as a function of the ratio between wages W_0 and marketable wealth M_0 , when the correlation parameter is $\rho = 0.5$.

Indeed, based on the above-mentioned parameter values, we compute the value of $\epsilon = 0.35(1 - \rho^2)$, which is small compared with unity. The numerical solution was obtained using 100 grid points on a spatial interval $x = [0, 0.25]$ and a time step size of 10^{-5} on the interval of $t = [0, 20]$. Our numerical experiments reveal that the computation becomes unstable at a larger time step due to oscillations in the optimal value of α_t^* . This severe stability constraint may be caused by the nonlinear nature of the boundary conditions. Despite the fact that small time step sizes have to be used, our numerical method is still competitive, since we have reduced the dimension of the HJB, compared with the method in Purcal (2003), where a two-dimensional HJB was solved directly.

Given the accuracy of the closed-form approximate solution for the particular set of parameter values used, we turn our attention to the financial implications of the solution. In particular, we compute a number of cases that demonstrate the impact of risk aversion and wage correlation on the optimal asset allocation and demand for term life insurance.

Table 1 displays the optimal allocation to stocks and the optimal face amount (death benefit) for life insurance, assuming the primary breadwinner has $T_R = 20$ years to retirement and the family horizon is $T_H = 50$ years. This individual is $y = 45$ years old, earns $W_0 = 50,000$ per year in real terms, and currently has $M_0 = 200,000$ saved.

To be consistent with how insurance is discussed in practice, we display the optimal face value of life insurance I_t^*/λ_{y+t} instead of the amount spent on life insurance I_t^* or the ratio of wealth insurance I_t^*/M_t .

The main qualitative insights from our model are as follows. First, the optimal amount of life insurance and allocation to stocks depend on the correlation between wage

shocks and the investment return on stocks. As we mentioned in the earlier part of the article, by varying only the correlation between wage shocks and market shocks, we are essentially comparing the demand for insurance and asset allocation for two individuals that have identical discounted value of wages (since the wage process parameters are the same). The economic value of their human capital is obviously different, as evidenced by the fact that the amount of life insurance is lower when the correlation is greater than zero. In other words, the lower amount of life insurance is a proxy for a reduced economic value of human capital. Indeed, if the economic value of human capital was the same, the amount of insurance would be the same as well.

For example, a 45-year-old who earns \$50,000 per annum and who already has \$200,000 in accumulated savings will optimally purchase between \$762,000 and \$965,000 of life insurance, depending on the correlation between wages (i.e., human capital) and markets (financial capital). If this individual is working in a job/career whose wages profile is countercyclical to the financial market ($\rho = -1$), then the optimal amount of insurance is \$964,000. On the other hand, if the wage profile is perfectly correlated with financial markets, the optimal amount of life insurance is a lower \$762,000. The economic intuition for the impact of wage–market correlations on the optimal demand, for insurance, as in Chen et al. (2005), is that when human capital is highly correlated with financial capital, the *utility-adjusted value* (i.e., economic value) of human capital is much lower, and hence the family requires less life insurance. Also, when the wage process is highly correlated with the returns from the risky asset, the allocation to financial risky asset is reduced to counteract the existing (and implicit) allocation to risky assets within the wage process. A number of models developed in the cited articles (e.g., Bodie, Merton, and Samuelson, 1992) confirm this. Notice, however, that the optimal face value life insurance, I_t^*/λ_{y+t} , does not depend on the risk aversion of the family. For example, at zero wage-to-market correlation ($\rho = 0$), the optimal face value is approximately \$855,000, which is more than 16 times the breadwinner's annual wage. This number is somewhat higher than the often-heard rule of thumb that people should have 5–8 times their annual income in life insurance.

Some might find it puzzling that our utility-based model does not predict even higher levels of life insurance for families that are more risk averse. On a theoretical level, it is important to remember that we did utilize fair actuarial pricing. In practice though, this invariance is fairly consistent with practitioner intuition, when it comes to life insurance. Namely, a family should be protected with a minimal level of insurance, regardless of how risk tolerant the family (or breadwinner) consider themselves. Of course, risk tolerance and risk aversion have a very strong impact on the demand for the risky asset α_t , as one would expect. Families that are more risk tolerant will allocate more financial and marketable wealth to risky equities, in highly leveraged amounts, as evidenced by our numerical values of $\alpha_t \gg 1$. It is not clear how more general HARA utility, highly loaded insurance premiums as well as liquidity and borrowing constraints would impact these results. Preliminary and ongoing research reported in Huang and Milevsky (in press) indicates that risk aversion does impact the demand for insurance in these more general cases, yet the practical numerical impact is relatively small.

TABLE 2

Life Insurance (in Thousands) and Allocation to Stocks 10 Years Away From Retirement, at Age 55

Risk Aversion		Correlation Between Wages and Stock Market				
		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0.0$	$\rho = +0.5$	$\rho = +1.0$
$\gamma = 1.5$	α_t^*	3.46	3.09	2.73	2.40	2.09
= Low	I_t^*/λ_{y+t}	\$484.8	\$470.2	\$456.1	\$442.6	\$429.6
$\gamma = 3$	α_t^*	2.03	1.69	1.37	1.06	0.775
= Medium	I_t^*/λ_{y+t}	\$484.8	\$470.2	\$456.1	\$442.6	\$429.6
$\gamma = 4.5$	α_t^*	1.56	1.22	0.911	0.616	0.337
= High	I_t^*/λ_{y+t}	\$484.8	\$470.2	\$456.1	\$442.6	\$429.6

Notes: Parameter assumptions: $\mu_m = 7\%$, $\sigma_m = 20\%$, $r = 2\%$, $\mu_w = 1\%$, $\sigma_w = 5\%$, $M_0 = 200$, $W_0 = 50$, $T_R = 10$, $T_H = 40$, $\delta = 2\%$, $y = 55$, $m = 86.3$, $b = 9.5$.

Table 2 provides similar results but with the key difference that the individual is now $y = 55$ years old, and therefore $T_R = 10$ years from retirement and $T_H = 40$ years from the terminal horizon of the family unit. All other parameters are exactly the same so that we can focus on the impact of “aging” on the optimal face value (i.e., death benefit) of life insurance and the optimal asset allocation.

Table 2 reveals substantially lower values for the optimal asset allocation proportions α_t , as well as the face amount of life insurance I_t^*/λ_{y+t} . Intuitively, the individual is closer to retirement, with only 10 more remaining years of work and wages. In this case, the optimal amount of life insurance lies between \$430,000 and \$480,000, depending on the wage-market correlation ρ . This is only eight times the annual wage of $W_0 = 50,000$. The optimal amount of life insurance coverage closely tracks the expected discounted value of future wages. The breadwinner has 10 more years of labor income and buys eight times annual income of life insurance. In Table 1 the breadwinner had 20 more years of labor income and he/she purchased 16 times annual income. In addition, once the family unit is 10 years away from retirement, the amount of financial wealth invested in the risky asset is greatly reduced. In fact, when the wage-market correlation $\rho > 0$, the family eliminates leverage and stops borrowing, as evidenced by $\alpha_t < 1$ values.

Finally, Table 3 takes us back to age 35, which is $T_R = 30$ years prior to retirement and $T_H = 60$ years prior to the end of the family’s planning horizon. One other change in Table 3 is that we have reduced the initial (current) wealth $M_0 = 1$ in contrast to the $M_0 = 4$, $W_0 = 100$ of the previous two tables. This particular parameter assumption is motivated by the low levels of wealth one would expect to observe among (young) 35-year-old individuals.

In this third example, the amount of life insurance purchased by the family falls in the millions of dollars, ranging from \$1.03 million to \$1.45 million, depending on the correlation variable ρ . With 30 years to retirement, the family purchases almost 20 times annual wages in life insurance. These numbers are consistent with a human capital perspective that protects the family against the loss of the income source.

TABLE 3

Life Insurance (in Thousands) and Allocation to Stocks 30 Years Away From Retirement, at Age 35

Risk Aversion		Correlation Between Wages and Stock Market				
		$\rho = -1.0$	$\rho = -0.5$	$\rho = 0.0$	$\rho = +0.5$	$\rho = +1.0$
$\gamma = 1.5$	α_t^*	1,570	1,270	1,010	789	599
= Low	I_t^*/λ_{y+t}	\$1,450	\$1,320	\$1,210	\$1,110	\$1,030
$\gamma = 3$	α_t^*	966	717	505	325	171
= Medium	I_t^*/λ_{y+t}	\$1,450	\$1,320	\$1,210	\$1,110	\$1,030
$\gamma = 4.5$	α_t	765	533	337	170	28.8
= High	I_t^*/λ_{y+t}	\$1,450	\$1,320	\$1,210	\$1,110	\$1,030

Notes: Parameter assumptions: $\mu_m = 7\%$, $\sigma_m = 20\%$, $r = 2\%$, $\mu_w = 1\%$, $\sigma_w = 5\%$, $M_0 = 1$, $W_0 = 50$, $T_R = 30$, $T_H = 60$, $\delta = 2\%$, $y = 35$, $m = 86.3$, $b = 9.5$.

Another important insight from Table 3 is the (abnormally) high amounts allocated to the risky investment asset α_t . Regardless of the level of risk aversion, whether it is a high γ or a low γ , the family places thousands of a percent in risky stocks. This is financed by borrowing thousands of a percent, a.k.a. shorting the risky free bond in our model. Clearly, these very high numbers are unrealistic (in practice) and at the very least are unfeasible, given the various institutional restrictions on borrowing with very little collateral. However, we do not believe this (odd) result represents a flaw or problem with our model, since this tends to plague most portfolio choice and asset allocation models in which risk aversion is low, borrowing is unlimited and time horizons are long.

CONCLUSION

This article focused on a subset of *portfolio choice* problems, where the emphasis was placed on the demand for life insurance, as a function of labor income. Our primary research question was to investigate the interaction between the demand for life insurance, the demand for risky assets, and the optimal level of consumption, as a function of one's human capital, all within a CRRA framework.

From a technical point of view, we show that the underlying HJB equation can be simplified using a similarity reduction technique, which then allows for the implementation of an efficient numerical method. Furthermore, for realistic financial economic parameter values, a closed-form approximation is derived, which greatly simplifies the calculations. Various tests confirm the robustness and accuracy of the analytic approximation compared with the full numerical solutions. Ongoing research by Huang and Milevsky (in press) is extending the application of this similarity reduction technique to more general HARA utility functions and a broader universe of mortality-contingent claims, such as retirement pensions and longevity insurance.

A variety of financial case studies illustrating our algorithm are also provided within the article. Our main qualitative and practical result is that households, whose primary breadwinner's wages are negatively correlated with financial market returns should

optimally purchase more life insurance and can afford to take more risky positions with their financial portfolio. However, we also find that the optimal face value of life insurance is insensitive to the family's risk aversion.

From a practical perspective, our model validates a number of rules of thumb, used by financial planners when making portfolio and investment recommendations for their wealth management clients. First, we find that younger investors, i.e., people who are farther away from the date at which their employment wage process will hit zero, are more able to tolerate financial risk and will therefore invest more in the risky asset compared with the risk-free bond. In fact, we find a very high (optimal) leverage ratio that can range in the hundreds of a percent, for families whose risk aversion is low. Second, we find that households should insure against the loss of their primary breadwinner's wages relatively independent of how risk tolerant or risk averse the family defines itself. This is consistent with the HLV concept, originally introduced by Solomon Huebner in the early 1900s. According to this approach to financial planning and insurance, there is a universal multiple of wages that a family should insure against losing, regardless of whether this particular family wants to gamble (i.e., is risk averse.).

Finally, we find that the demand for life insurance should depend on the riskiness of the human capital (wage process) as well as its correlation with financial returns. High anticipated levels of wage volatility and/or high levels of correlation induce a reduction in the demand for life insurance. In other words, the discounted HLV alone is not enough to determine the optimal amount of life insurance. The HLV alone might over/underestimate the demand for life insurance when the remainder of the lifecycle portfolio is taken into account. Indeed, it is the *economic* value of human capital that is the quantity that must be insured.

APPENDIX A: DERIVATION OF THE HJB EQUATION

We assume in the time period h , the survival probability of the individual is ${}_h p_{y+t}$, the death probability is ${}_h q_{y+t} = 1 - {}_h p_{y+t}$, the current age of the individual is y , and the current time is t .

Let h be an arbitrarily small time increment; we have

$$\begin{aligned} J(M_t, W_t, t) \geq & {}_h p_{y+t} E \left[\int_t^{t+h} e^{-\delta s} u(c_s) ds + J(M_{t+h}, W_{t+h}, t+h) \right] \\ & + {}_h q_{y+t} E \left[\int_t^{t+h} e^{-\delta s} u(c_s) ds + \Phi(M_{t+h} + I_{t+h} \lambda_{y+t+h}^{-1}, t+h) \right], \end{aligned} \quad (A1)$$

where $\Phi(\cdot, \cdot)$ is the optimal objective function when there is no wage income and no need for paying insurance premium. By Ito's formula, we have

$$dJ(M_t, W_t, t) = (J_t + A_t^I) dt + \sigma_m M_t J_M d B_t^m + \sigma_W W_t J_W d B_t^W, \quad (A2)$$

where

$$A_t^J = (\alpha_t(\mu_m - r)M_t + rM_t + W_t - c_t - I_t)J_M + \mu_W W_t J_W + \frac{1}{2}\alpha_t^2 \sigma_m^2 M_t^2 J_{MM} + \frac{1}{2}\sigma_W^2 W_t^2 J_{WW} + \alpha_t \rho \sigma_W \sigma_m M_t W_t J_{WM}. \quad (A3)$$

Integrating Equation (A2), we obtain

$$J(M_{t+h}, W_{t+h}, t+h) = J(M_t, W_t, t) + \int_t^{t+h} (J_s + A_s^J)ds + \int_t^{t+h} \sigma_m M_s J_M dB_s^m + \int_t^{t+h} \sigma_W W_s J_W dB_s^W. \quad (A4)$$

Similarly, for Φ , we have

$$d\Phi(M_t, t) = (\Phi_t + B_t^\Phi)dt + \sigma_m \alpha_t M_t \Phi_M dB_t^m, \quad (A5)$$

where

$$B_t^\Phi = ((\mu_m - r)M_t \alpha_t + rM_t - c_t)J_M + \frac{1}{2}\sigma_m^2 \alpha_t^2 M_t^2 J_{MM}.$$

Integrating Equation (A4) yields

$$\Phi(M_{t+h}, t+h) = \Phi(M_t, t) + \int_t^{t+h} (\Phi_s + B_s^\Phi)ds + \int_t^{t+h} \sigma_m \alpha_s M_s \Phi_M dB_s^m. \quad (A6)$$

Combining (A1), (A5), and (A3), we obtain

$$h q_{y+t} J(M_t, W_t, t) \geq h p_{y+t} E \left[\int_t^{t+h} e^{-\delta s} u(c_s) ds + \int_t^{t+h} (J_s + A_s^J) ds \right] + h q_{y+t} E \left[\int_t^{t+h} e^{-\delta s} u(c_s) ds + \Phi(M_t, t) + \int_t^{t+h} (\Phi_s + B_s^\Phi) ds \right]. \quad (A7)$$

Dividing by h and rearranging Equation (A6), as $h \rightarrow 0$, we obtain

$$J(M_t, W_t, t) \lambda_{y+t} \geq J_t + A_t^J + \Phi \left(M_t + \frac{I_t}{\lambda_{y+t}}, t \right) \lambda_{y+t} + e^{-\delta t} u(c_t). \quad (A8)$$

The equality holds when we take the optimal control:

$$J(M_t, W_t, t) \lambda_{y+t} = \max_{\alpha_t, I_t, c_t} \left(J_t + A_t^J + \Phi \left(M_t + \frac{I_t}{\lambda_{y+t}}, t \right) \lambda_{y+t} + e^{-\delta t} u(c_t) \right). \quad (A9)$$

Using Equations (A3) and (A7), after rearrangement, we obtain

$$\begin{aligned}
 \lambda_{y+t} J &= J_t + \max_{c_t} (e^{-\delta t} u(c_t) - c_t J_M) \\
 &+ \max_{I_t} \left(\Phi \left(M_t + \frac{I_t}{\lambda_{y+t}}, t \right) \lambda_{y+t} - I_t J_M \right) \\
 &+ \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M_t J_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M_t^2 J_{MM} + \alpha_t \rho \sigma_W \sigma_m W_t M_t J_{WM} \right] \\
 &+ r M_t J_M + W_t J_M + \mu_W W_t J_W + \frac{1}{2} \sigma_W^2 W_t^2 J_{WW}.
 \end{aligned} \tag{A10}$$

Here Φ satisfies the following HJB (without wage and insurance):

$$\begin{aligned}
 0 &= \Phi_t + \max_{c_t} (e^{-\delta t} u(c_t) - c_t \Phi_M) \\
 &+ \max_{\alpha_t} \left[\alpha_t (\mu_m - r) M_t \Phi_M + \frac{1}{2} \alpha_t^2 \sigma_m^2 M_t^2 \Phi_{MM} \right] + r M_t \Phi_M.
 \end{aligned} \tag{A11}$$

In order to simplify the presentation, we now drop the subscript t from M_t and W_t .

APPENDIX B: DERIVATION OF CLOSED-FORM SOLUTION

Motivated by the closed-form solution for constant wage Merton (1971), we seek the solution of the HJB Equation (16) in the form

$$J(M, W, t) = \frac{h(t)(M + k(t)W)^{1-\gamma}}{1-\gamma}. \tag{B1}$$

Note that when $W = 0$ (no wage income), this solution reduces to the Merton formula (12). Simple calculation and algebraic manipulation lead to the following expressions

$$\frac{I_t^*}{\lambda M} = kx, \tag{B2}$$

$$\alpha_t^* = \frac{\mu_m - r}{\gamma \sigma_m^2} (1 + kx) - \beta kx, \tag{B3}$$

$$\frac{c_t^*}{M} = e^{\frac{\delta}{\gamma} t} h^{-\frac{1}{\gamma}} (1 + kx), \tag{B4}$$

where $\beta = \rho \sigma_w / \sigma_m$ and $x = W/M$. From (16), we obtain

$$\begin{aligned}
& -\frac{\lambda_{y+t}}{1-\gamma}(1+kx)^2 + \frac{h'}{h}\frac{(1+kx)^2}{1-\gamma} + \frac{k'}{k}(1+kx)kx \\
& + \left[\frac{\gamma}{1-\gamma}e^{\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}}(1+kx) + \frac{\lambda_{y+t}}{1-\gamma}(1+\gamma kx) \right. \\
& \left. + r + x + \frac{(\mu_m-r)^2}{2\gamma\sigma_m^2}(1+kx) - \frac{\beta(\mu_m-r)}{2}kx \right](1+kx) \\
& - \frac{\mu_m-r}{2}\beta(1+kx)kx + \mu_w(1+kx)kx + \frac{\gamma}{2}\sigma_w^2(kx)^2(\rho^2-1) = 0. \tag{B5}
\end{aligned}$$

When $\rho = \pm 1$, the last term in the equation above drops out, and when $1+kx \neq 0$, we have

$$\begin{aligned}
& \frac{h'}{h}\frac{1+kx}{1-\gamma} + \frac{k'}{k}kx + \frac{\gamma}{1-\gamma}e^{\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}}(1+kx) - \lambda_{y+t}kx \\
& + r + \frac{1}{k}kx + \frac{(\mu_m-r)^2}{2\gamma\sigma_m^2}(1+kx) - (\mu_m-r)\beta kx + \mu_w kx = 0. \tag{B6}
\end{aligned}$$

This can be viewed as a linear algebraic equation for x and for the solution to exist, we must have the coefficients vanish for all x , which leads to the following two first-order ordinary differential equations for h and k :

$$\frac{1}{1-\gamma}\frac{h'}{h} + \frac{\gamma}{1-\gamma}e^{\frac{\delta}{\gamma}t}h^{-\frac{1}{\gamma}} + r + \frac{(\mu_m-r)^2}{2\gamma\sigma_m^2} = 0, \tag{B7}$$

$$\frac{k'}{k} + \frac{1}{k} + \mu_w - r - \lambda_{y+t} - \beta(\mu_m-r) = 0. \tag{B8}$$

Note that (B7) is the same as (13), and h is simply the Merton solution given by (14). Furthermore, our solution reduces to (12) at $t = T_R$, which yields the terminal conditions for k ,

$$k(T_R) = 0. \tag{B9}$$

Remark: When (B5) is viewed as a quadratic equation of x , the solution (B1) exists when this quadratic equation is of a special form (with a common factor $1+kx$) since we only have two degrees of freedom (h and k). Another special case when the quadratic equation has a common factor $1+kx$ is when $\sigma_w = 0$, in which case we can also eliminate a common factor $1+kx$. Merton's solution of constant wage (Merton, 1971) belongs to this case when μ_w is also zero.

The Case of $-1 < \rho < 1$

We assume that there exists an asymptotic expansion of the value function in the form $J = J_0 + \epsilon J_1$, where $\epsilon = (1-\rho^2)(\frac{\gamma\sigma_m\sigma_w}{\mu_m-r})^2$,

$$J_0(M, W, t) = \frac{M^{1-\gamma}}{1-\gamma}h(t)(1+k(t)x)^{1-\gamma},$$

$$J_1(M, W, t) = \frac{M^{1-\gamma}}{1-\gamma}F_1(x, t).$$

In the following, we show that h and k are the same functions obtained in the “Special Case: $\rho = \pm 1$ ” section valid for $-1 \leq \rho \leq 1$ as long as ϵ is small.

We proceed as before, by substituting J into the HJB Equation (16) and collect the terms of the same order of ϵ . At the zero order, we have

$$\begin{aligned} & -\frac{\lambda_{y+t}}{1-\gamma}(1+kx)^2 + \frac{h'}{h} \frac{(1+kx)^2}{1-\gamma} + \frac{k'}{k}(1+kx)kx \\ & + \left[\frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} h^{-\frac{1}{\gamma}}(1+kx) + \frac{\lambda_{y+t}}{1-\gamma}(1+\gamma kx) \right. \\ & \left. + r + x + \frac{(\mu_m-r)^2}{2\gamma\sigma_m^2}(1+kx) - \frac{\beta(\mu_m-r)}{2}kx \right](1+kx) \\ & - \frac{\mu_m-r}{2}\beta(1+kx)kx + \mu_w(1+kx)kx = 0, \end{aligned} \quad (\text{B10})$$

which is (B6) multiplied by $1+kx$. At the first order of ϵ , we have

$$\frac{\partial F_1}{\partial t} + \hat{c}_0(1-\gamma)F_1 + \hat{c}_1x \frac{\partial F_1}{\partial x} + \hat{c}_2x^2 \frac{\partial^2 F_1}{\partial x^2} = G \quad (\text{B11})$$

where $\hat{c}_0$, $\hat{c}_1$, and $\hat{c}_2$ are similarly defined as c_0 , c_1 , and c_2 in the “General Case: $-1 < 1$ ” section, with F replaced by $h(1+kx)^{1-\gamma}$. G is a function of F_1 , as well as h and k .

For practically relevant parameter values, as shown by the numerical examples, the zero-order approximation gives sufficiently accurate results. In principle, if we are interested in a more accurate approximation, we could always use the numerical method outlined earlier.

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