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An Extension of Arrow's Result on Optimal Reinsurance Contract

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Abstract

We consider the problem of finding reinsurance policies that maximize the expected utility, the stability and the survival probability of the cedent for a fixed reinsurance premium calculated according to the maximal possible claims principle. We show that the limited stop loss and the truncated stop loss are the optimal contracts.

Introduction

Let X, a nonnegative random variable on a given probability space $(\Omega, \mathcal{S}, \mathbf{P})$, represent the aggregate claim amount for an insurance portfolio in a certain time period. Suppose the insurer wants to buy a reinsurance policy R arranged on X basis; i.e., $R(X)$ is the part of X covered by the reinsurer and $X - R(X)$ is the cedent's share. The pioneering result of Arrow (1963) states that under the assumptions

$$0 \le R(X) \le X \quad \text{and} \quad \mathbf{E}R(X) = P \quad \text{for fixed} \quad P > 0,$$

the stop loss treaty

$$R^*(X) = (X - a)_+$$

with the retention limit a chosen so that $\mathbf{E}R^*(X) = P$, maximizes the expected utility $\mathbf{E}u(W - X + R(X) - P)$, where u is a concave utility function of the cedent and W stands for the insurer's initial wealth. Here and below, $a_+ = \max\{0, a\}$ and the notation $X = Y$ (resp. $X \le Y$) means $\mathbf{P}(X = Y) = 1$ (resp. $\mathbf{P}(X \le Y) = 1$). Arrow's result has been generalized to many settings: convex costs, nonexpected utility maximizers, moral hazard, background risk, etc. In this article we extend it to the case when the reinsurer's premium is calculated according to the maximal possible claims principle (cf. Gerber, 1979; Straub, 1988).

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$$\pi(R(X)) = \mathbf{E}R(X) + \beta[\sup R(X) - \mathbf{E}R(X)], \tag{1}$$

where $0 < \beta < 1$ and $\sup Z = \sup\{x; \mathbf{P}(Z \le x) < 1\}$. The coefficient β represents weights put by the risk carrier on the expected loss and the maximal loss. Its value depends on how much risk the reinsurer is willing to take. The premium principle (1) has many desirable properties, e.g., independence, risk loading, translation and scale invariance, additivity for independent and comonotonic risks (for details, see the Appendix), and most of them have a natural economic interpretation (see Gerber, 1979; Young, 2004). In particular, the premium belongs to the class of coherent risk measures, which is thoroughly studied in academic journals (see Artzner et al., 1999; Yang and Siu, 2001) and at professional conventions (Meyers, 2001). Another important feature of the maximal loss principle is that it is a limit of zero utility principle and Wang's principle (see the Appendix). The former one is closely related to the economic principle introduced by Bühlmann (1980). Wang's principle also has economic justification: it follows from the use of Yaari's dual theory of risk in place of expected utility theory.

The cedent's optimization problem is nonstandard because the space of reinsurance contracts $R(X)$ constitutes a subset of $L^1 \cap L^\infty$ (i.e., $\mathbf{E}R(X) < \infty$ and $\sup R(X) < \infty$), which is not a Hilbert space. In addition, some important objective functions are not Gâteaux differentiable (e.g., the ruin probability). Therefore the classical optimization methods like the Karush–Kuhn–Tucker theorem cannot be applied here directly. We propose to adapt the Lagrange approach and choose properly the multipliers depending on some parameters.

We show that for fixed premium $P > 0$ and $0 < \beta < 1$, a limited stop loss treaty

$$R^*(X) = \min\{(X - a)_+, b\}$$

with some $a \ge 0$, $b > 0$ is a solution to the problem

$$\sup \mathbf{E}u(W - X + R(X) - P) \quad \text{s.t.} \quad R \in \mathcal{R}, \tag{2}$$

where

$$\mathcal{R} = \{R; 0 \le R(X) \le X,\ (1 - \beta)\mathbf{E}R(X) + \beta \sup R(X) = P\}. \tag{3}$$

This type of reinsurance is commonly used in practice (see, e.g., Mack, 1984; Daykin, Pentikäinen, and Pesonen, 1994; Cummins and Mahul, 2004). Observe that the essential supremum of each feasible policy $R(X)$ is finite although the claim X may be unbounded. The existence of upper bounds for possible losses is a characteristic feature of most reinsurance lines (see Panjer, 1998).

Next we show that a limited stop loss contract is also optimal in the sense that it minimizes some measures of random fluctuations of $X - \mathbf{E}X$. This method of constructing optimal reinsurance treaties relies on the idea of maximizing the stability measured by the variance or other functionals of the cedent's retained risk. See Borch (1975, 1990), Hasselager (1990), Schmitter (2001), Gajek and Zagrodny (2004b), and Kaluszka (2004a,b).

Finally, we consider the problem of finding a treaty that minimizes the cedent's ruin probability for fixed reinsurer's premium (1). Such an optimality criterion leads to a truncated stop loss contract

$$R^*(X) = (X - W + P)_+ \mathbf{I}(X - W + P < a)$$

with some $a \geq 0$. Here and subsequently, $\mathbf{I}(c)$ denotes the indicator function, i.e., $\mathbf{I}(c) = 1$ if the condition c is true and $\mathbf{I}(c) = 0$ if it is not. The result is consistent with that of Gajek and Zagrodny (2004a), where the premium of the reinsurer is an increasing function of the expected value of the reinsured part. As shown by Kaluszka (2005), truncated stop loss contracts are also optimal for other pricing rules (like the economic principle, generalized zero-utility principle, Esscher principle and mean-variance principle) and other important objective functions (like the expected utility and the Kahneman–Tversky value function of the cedent, the variance of the cedent's cover and the dividend payment).

MAXIMIZING THE EXPECTED UTILITY

Assume the insurer is risk averse, that is $u'(x) > 0$ and $u''(x) < 0$. Given a reinsurer's premium $P > 0$, the cedent seeks an arrangement that maximizes his expected utility. This leads to problem (2). To keep things simple we assume that $\mathbf{P}(a < X < b) > 0$ for all $0 < a < b < \sup X$. In the sequel, $a \wedge b = \min\{a, b\}$.

Theorem 1: *If* $\mathbf{E}X$ *and* $\mathbf{E}u(W - X - P)$ *are finite, then the limited stop loss contract* $R_{ab}(X) = \min\{(X - a)_+, b\}$ *is a solution to problem (2), where* $a \geq 0$, $b > 0$ *are such that* $(1 - \beta)\mathbf{E}R_{ab}(X) + \beta b = P$ *and*

$$\max_{m \in M} \mathbf{E}u(W - X + R_{am}(X) - P) = \mathbf{E}u(W - X + R_{ab}(X) - P)$$

with

$$M = \{m \geq 0; \beta m < P \leq (1 - \beta)\mathbf{E}(X \wedge m) + \beta m\}. \tag{4}$$

Proof: To shorten notation, we write $w(x)$ instead of $-u(W - P - x)$. Of course, $w' > 0$ and $w'' > 0$; i.e., w is increasing and convex. Observe that problem (2) is equivalent to

$$\inf \mathbf{E}w(X - R(X)) \quad \text{s.t.} \quad R \in \mathcal{R}.$$

Put $\mathcal{R}_m = \{R \in \mathcal{R}; \sup R(X) = m\}$. Clearly, $\mathcal{R} = \cup_{m \in M} \mathcal{R}_m$. Hence

$$\inf_{R \in \mathcal{R}} \mathbf{E}w(X - R(X)) = \inf_{m \in M} \inf_{R \in \mathcal{R}_m} \mathbf{E}w(X - R(X)).$$

Define

$$\mathcal{R}^*_m = \left\{ R; 0 \leq R(x) \leq x \ \forall x \geq 0, \sup_{x \geq 0} R(x) = m, (1 - \beta)\mathbf{E}R(X) + \beta m = P \right\}. \tag{5}$$

Since for each $R \in \mathcal{R}_m$ there exists $R^* \in \mathcal{R}_m^*$ such that $\mathbf{P}(R(X) = R^*(X)) = 1$, we have

$$\begin{aligned}\inf_{R\in\mathcal{R}} \mathbf{E}w(X - R(X)) &\geq \inf_{m\in M} \inf_{R\in\mathcal{R}_m^*} \mathbf{E}w(X - R(X))\\ &\geq \inf_{m\in M} \inf_{R\in\mathcal{R}_m^*} [\mathbf{E}w(X - R(X)) + c((1-\beta)\mathbf{E}R(X) + \beta m) - cP]\\ &\geq \inf_{m\in M} \int \left(\inf_{R\in\mathcal{R}_m^*} [w(x - R(x)) + c(1-\beta)R(x)] + c(\beta m - P)\right) dF(x)\\ &\geq \inf_{m\in M} \left[\int \min_{0\leq y\leq x\wedge m} [w(x-y) + c(1-\beta)y]\, dF(x) + c(\beta m - P)\right], \qquad (6)\end{aligned}$$

where $w'(0)/(1-\beta) \leq c < w'(\infty)/(1-\beta)$ and $F(x) = \mathbf{P}(X \leq x)$ for $x \in \mathbf{R}$. Let $R_{cm}(x)$ denote an element at which the function $[0, x \wedge m] \ni y \to w(x-y) + c(1-\beta)y$ attains the minimum. It is easy to check that

$$R_{cm}(x) = \min\left\{[x - w^*(c(1-\beta))]_+, m\right\}, \quad x \geq 0,$$

where w^* is the inverse function to w'. From (6) we obtain

$$\begin{aligned}&\inf_{R\in\mathcal{R}} \mathbf{E}w(X - R(X))\\ &\quad\geq \inf_{m\in M}[\mathbf{E}w(X - R_{c(m)m}(X)) + c(m)((1-\beta)\mathbf{E}R_{c(m)m}(X) + \beta m - P)] \qquad (7)\end{aligned}$$

for any $w'(0)/(1-\beta) \leq c(m) < w'(\infty)/(1-\beta)$.

We now show that there exists a function $c(m)$ continuous on M and such that $R_{c(m)m} \in \mathcal{R}_m^*$ for every $m \in M$. Define $\Psi(c) = (1-\beta)\mathbf{E}R_{cm}(X) + \beta m$. Obviously,

$$\Psi\left(\frac{w'(0)}{1-\beta}\right) = (1-\beta)\mathbf{E}(X \wedge m) + \beta m \geq P, \quad \Psi\left(\frac{w'(\infty)}{1-\beta}\right) = \beta m < P$$

and Ψ is strictly decreasing. By the dominated convergence theorem and the continuity of w', the function $(m, c) \to (1-\beta)\mathbf{E}R_{cm}(X) + \beta m$ is continuous. Hence there exists a continuous and nondecreasing function $c(m)$ such that $\Psi(c(m)) = P$ for every $m \in M$. From (7) we have

$$\inf_{R\in\mathcal{R}} \mathbf{E}w(X - R(X)) \geq \inf_{m\in M} \mathbf{E}w(X - R_{c(m)m}(X)). \qquad (8)$$

Since $0 \leq R_{cm}(X) \leq X$, $\mathbf{E}X < \infty$ and $\lim_{m\to P/\beta} [(1-\beta)\, \mathbf{E}R_{c(m)m}(X) + \beta m] = P$, by the dominated convergence theorem,

$$\lim_{m\to P/\beta} \mathbf{E}R_{c(m)m}(X) = \mathbf{E} \lim_{m\to P/\beta} R_{c(m)m}(X) = 0,$$

and consequently $\lim_{m\to P/\beta} R_{c(m)m}(x) = 0$ for each $x \geq 0$. An application of the dominated convergence theorem enables us to show that the function $m \to \mathbf{E}w(X - R_{c(m)m}(X))$ is continuous on M. As w is increasing,

$$\lim_{m \to P/\beta} \mathbf{E}w(X - R_{c(m)m}(X)) = \mathbf{E}w\left(X - \lim_{m \to P/\beta} R_{c(m)m}(X)\right)$$
$$= \mathbf{E}w(X) \geq \mathbf{E}w(X - R_{c(m)m}(X)) \tag{9}$$

for each $m \in M$. Observe that the closure of M is of the form $cl\,M = \{m \geq 0;\, \beta m \leq P \leq (1-\beta)\mathbf{E}(X \wedge m) + \beta m\}$, so the set clM is an interval. From (8), (9), the continuity of $m \to \mathbf{E}w(X - R_{c(m)m}(X))$, and the compactness of clM, it follows that the function $m \to \mathbf{E}w(X - R_{c(m)m}(X))$ attains minimum at a point $b \in M$. In consequence,

$$\inf_{R \in \mathcal{R}} \mathbf{E}w(X - R(X)) \geq \mathbf{E}w(X - R_{c(b)b}(X)). \tag{10}$$

By its construction $R_{c(b)b}$ belongs to $\mathcal{R}$. The proof is complete. □

Remark 1: Arguing similarly as in the proof of Theorem 1 one can extend the result to the case $P = f(\mathbf{E}R(X), \sup R(X))$, where f is nonnegative, continuous, and increasing in each variable (cf. Kaluszka, 2004a,b).

Remark 2: Suppose the reinsurer calculates the reinsurance premium on the basis of expected loss and maximal loss, namely, uses a convex combination of these two criterions for valuating reinsurance risk. Suppose the cedent maximizes expected utility from terminal wealth and does not know either the weight β or the reinsurer's utility function. Suppose also the reinsurer does not know the cedent's utility function. It follows from Theorem 1 that, from the cedent's point of view, a limited stop loss contract is optimal for a fixed premium P. The reinsurer can offer different limited stop loss treaties (with different a's and b's) for different X's and β's. If the reinsurer chooses β, which maximizes his expected utility $\mathbf{E}u_{Re}(W_{Re} + P - R_{ab}(X))$, where R_{ab} is given in Theorem 1, and u_{Re} and W_{Re} denote the utility function and the initial wealth of the reinsurer, then the resulting reinsurance treaty is optimal for both the cedent and the reinsurer.

MAXIMIZING THE STABILITY

We now study the problem that consists in choosing a contract that minimizes a random fluctuation of the cedent's cover:

$$\inf \mathbf{E}w(X - R(X) - \mathbf{E}(X - R(X))) \quad \text{s.t.} \quad R \in \mathcal{R}, \tag{11}$$

where $w : \mathbf{R} \to \mathbf{R}$ is a harm function (see Daykin, Pentikäinen, and Pesonen, 1994) and $\mathcal{R}$ is given by (3).

Theorem 2: *Let w be strictly convex and differentiable, and let M be defined by (4). Suppose that $\mathbf{E}w((X - m^*)_+ - \mathbf{E}(X - m^*)_+) \leq \mathbf{E}w(X - \mathbf{E}X)$ for m^* satisfying $(1-\beta)\mathbf{E}(X \wedge m^*) + \beta m^* = P$. Suppose also that $\mathbf{E}X$ and $\mathbf{E}w(X - \mathbf{E}X)$ exist and are finite. The optimal contract for problem (11) is given by*

$$R_{ab}(X) = \min\left\{\left(X - \mathbf{E}X + \frac{P - \beta b}{1-\beta} - a\right)_+, b\right\}, \tag{12}$$

where $a \in \mathbf{R}$ and $b > 0$ are such that $(1-\beta)\mathbf{E}R_{ab}(X) + \beta b = P$ and

$$\min_{m \in M} \mathbf{E}w(X - R_{am}(X) - \mathbf{E}X + \mathbf{E}R_{am}(X)) = \mathbf{E}w(X - R_{ab}(X) - \mathbf{E}X + \mathbf{E}R_{ab}(X)).$$

Proof: As in the proof of Theorem 1 we obtain

$$
\begin{aligned}
&\inf_{R\in\mathcal{R}} \mathbf{E}w(X - R(X) - \mathbf{E}X + \mathbf{E}R(X)) \\
&\quad \geq \inf_{m\in M} \inf_{R\in\mathcal{R}_m^*} [\mathbf{E}w(X - R(X) - \mathbf{E}X + \mathbf{E}R(X)) + c((1-\beta)\mathbf{E}R(X) + \beta m) - cP] \\
&\quad \geq \inf_{m\in M} \left[\int \min_{0\leq y\leq x\wedge m} [w(x - y - A_m) + c(1-\beta)y]\, dF(x) + c(\beta m - P) \right], \qquad (13)
\end{aligned}
$$

where $\mathcal{R}_m^*$ is given by (5) and $A_m = \mathbf{E}X - (P - \beta m)/(1-\beta)$. It follows easily that the minimum in y is attained at

$$
R_{cm}(x) = \min\left\{(x - A_m - w^*(c(1-\beta)))_+ , m\right\}, \quad x \geq 0, \qquad (14)
$$

provided $w'(-A_m)/(1-\beta) \leq c < w'(\infty)/(1-\beta)$. Define

$$
\Psi(c) = (1-\beta)\mathbf{E}R_{cm}(X) + \beta m \quad \text{for} \quad \frac{w'(-A_m)}{1-\beta} \leq c < \frac{w'(\infty)}{1-\beta}.
$$

Clearly,

$$
\Psi\left(\frac{w'(-A_m)}{1-\beta}\right) = (1-\beta)\mathbf{E}(X\wedge m) + \beta m \geq P, \quad \Psi\left(\frac{w'(\infty)}{1-\beta}\right) = \beta m < P.
$$

Analysis similar to that in the proof of Theorem 1 shows that there exists a continuous function $M \ni m \to c(m)$ with the property that $\Psi(c(m)) = P$ for all $m \in M$. Using the identity $(x-m)_+ = (x-m) - x\wedge m, x, m \in \mathbf{R}$, we get

$$
\begin{aligned}
&\lim_{m\to P/\beta} \mathbf{E}w(X - R_{c(m)m}(X) - \mathbf{E}X + \mathbf{E}R_{c(m)m}(X)) = \mathbf{E}w(X - \mathbf{E}X) \\
&\quad \geq \mathbf{E}w\big((X - m^*)_+ - \mathbf{E}(X - m^*)_+\big) = \mathbf{E}w(X - \mathbf{E}X - X\wedge m^* + \mathbf{E}(X\wedge m^*)),
\end{aligned}
$$

where m^* is the solution of the equation $(1-\beta)\,\mathbf{E}(X\wedge m^*) + \beta m^* = P$. This gives

$$
\begin{aligned}
&\lim_{m\to P/\beta} \mathbf{E}w(X - R_{c(m)m}(X) - \mathbf{E}X + \mathbf{E}R_{c(m)m}(X)) \\
&\quad \geq \mathbf{E}w(X - R_{c(m^*)m^*}(X) - \mathbf{E}X + \mathbf{E}R_{c(m^*)m^*}(X))
\end{aligned}
$$

with $c(m^*) = w'(-A_m)/(1-\beta)$. Hence, the infimum of $\mathbf{E}w(X - R_{c(m)m}(X) - \mathbf{E}X + \mathbf{E}R_{c(m)m}(X))$ in $m \in M$ is attained at some point, say, $m = b$. Taking $c = c(b)$ completes the proof. □

Remark 3: If $w(x) = x^2$, then the inequality $\mathbf{E}w((X-a)_+ - \mathbf{E}(X-a)_+) \leq \mathbf{E}w(X - \mathbf{E}X)$ is equivalent to $\mathbf{Var}(X-a)_+ \leq \mathbf{Var}\, X$, which holds for all $a \geq 0$.

Remark 4: Define $\Psi(m) = \mathbf{E}w((X-m)_+ - \mathbf{E}(X-m)_+), m \geq 0$. If differentiation under the integral sign is permitted, then

$$
\Psi'(m) = \mathbf{E}w'\left((X-m)_+ - \mathbf{E}(X-m)_+\right)\left(-\mathbf{I}(X > m) + \mathbf{P}(X > m)\right), \qquad (15)
$$

since $\mathbf{E}(X-m)_+ = \int_m^\infty \mathbf{P}(X>t)\,dt$. We can rewrite (15) as $\mathbf{E}[h_1(X)h_2(X)]$, where $h_1(t) = w'((t-m)_+ - \mathbf{E}(X-m)_+)$ is nondecreasing and $h_2(t) = \mathbf{P}(X>m) - \mathbf{I}(t>m)$ is nonincreasing. By the inequality for associated variables (see, e.g., Devroye, Györfi, and Lugosi, 1996, p. 583), $\Psi'(m) \le \mathbf{E}\,[h_1(X)]\,\mathbf{E}\,[h_2(X)] = 0$, so the function $m \to \mathbf{E}w((X-m)_+ - \mathbf{E}(X-m)_+)$ is nonincreasing. This clearly forces $\mathbf{E}w(X-\mathbf{E}X) \ge \mathbf{E}w((X-m)_+ - \mathbf{E}(X-m)_+)$ for every $m \ge 0$.

Common measures of dispersion also include the semi-variance $\mathbf{E}(X-\mathbf{E}X)_+^2$, the mean absolute deviation $\mathbf{E}|X-\mathbf{E}X|$, and the upper mean absolute deviation $\mathbf{E}(X-\mathbf{E}X)_+$. Of course, they all have the same form as in Theorem 2, but the corresponding harm functions w (resp. $w(x)=x_+^2$, $w(x)=|x|$ and $w(x)=x_+$) are not strictly convex or differentiable. We first state the analogue of Theorem 2 for semi-variance type stability measures.

Theorem 3: *Assume that $w:\mathbf{R}\to\mathbf{R}$ is differentiable, strictly convex on the positive half axis, and $w(x)=0$ for $x\le 0$. If $\mathbf{E}X$ and $\mathbf{E}w(X-\mathbf{E}X)$ exist and are finite, then the limited stop loss treaty (12) is a solution to problem (11).*

Proof: Let M be defined by (4), and let $A_m = \mathbf{E}X - (P-\beta m)/(1-\beta)$, $m\in M$. Arguing as in (13) it can be shown that

$$\inf_{R\in\mathcal{R}} \mathbf{E}w(X-R(X)-\mathbf{E}X+\mathbf{E}R(X)) \\ \ge \inf_{m\in M}\left[\int \min_{0\le y\le x\wedge m}[w(x-y-A_m)+c(m)(1-\beta)y]\,dF(x) + c(m)(\beta m - P)\right] \tag{16}$$

for any function $c(m)\ge 0$. Let

$$M_0 = \{m\in M; (1-\beta)\mathbf{E}\min((X-A_m)_+, m) + \beta m < P\},$$

and $M_1 = M\backslash M_0$. If $m\in M_1$, then $c(m)$ can be defined as in the proof of Theorem 2. Set $c(m)=0$ for all $m\in M_0$. If $m\in M_0$, then the minimum in (16) is attained at $y_a(x) = \min((x-a)_+, m)$, where a is an arbitrary number from the interval $[0, A_m]$. It is easy to check that there exists a continuous function $M_0 \ni m \to a(m)$ with the property that $(1-\beta)\mathbf{E}y_{a(m)}(X) + \beta m = P$. Putting

$$R_m(x) = \begin{cases} R_{c(m)m}(x), & m\in M_1 \\ y_{a(m)}(x), & m\in M_0, \end{cases}$$

we conclude that there exists a continuous function $M\ni m\to c(m)$ such that $(1-\beta)\times \mathbf{E}R_m(X) + \beta m = P$ and

$$\inf_{R\in\mathcal{R}} \mathbf{E}w(X-R(X)-\mathbf{E}(X-R(X))) \ge \inf_{m\in M}\mathbf{E}w(X-R_m(X)-\mathbf{E}X+\mathbf{E}R_m(X)).$$

It remains to prove that

$$\mathbf{E}w((X-m)_+ - \mathbf{E}(X-m)_+) \le \mathbf{E}w(X-\mathbf{E}X) \tag{17}$$

for any $m \geq 0$. We see at once that

$$((x-m)_+ - \mathbf{E}(X-m)_+)_+ \leq (x - \mathbf{E}X)_+ \quad \text{for} \quad x, m \geq 0. \tag{18}$$

Combining this with the fact that w is nondecreasing and $w(x_+) = w(x)$ for every $x \in \mathbf{R}$ gives (17). The rest of the proof is similar to that of Theorem 2. □

If the cedent's objective is to minimize the upper mean absolute deviation of his cover, the following problem must be solved:

$$\inf \mathbf{E}(X - R(X) - \mathbf{E}(X - R(X)))_+ \quad \text{s.t.} \quad R \in \mathcal{R}. \tag{19}$$

Theorem 4: *Let M be defined by (4). Suppose that $\mathbf{E}X$ exists and is finite. The limited stop loss treaty $R_{ab}(X) = \min\{(X-a)_+, b\}$, is a solution to problem (19), where $a \geq 0, b > 0$ are such that $(1-\beta)\mathbf{E}R_{ab}(X) + \beta b = P$ and*

$$\mathbf{E}(X - R_{ab}(X) - \mathbf{E}X + \mathbf{E}R_{ab}(X))_+ = \min_{m \in M} \mathbf{E}(X - R_{am}(X) - \mathbf{E}X + \mathbf{E}R_{am}(X))_+.$$

Proof: Set $A_m = \mathbf{E}X - (P - \beta m)/(1-\beta)$ and observe that $A_m \geq 0$ for $m \in M$. Analysis similar to that in the proof of Theorem 2 shows that

$$\begin{aligned} &\inf_{R \in \mathcal{R}} \mathbf{E}(X - R(X) - \mathbf{E}X + \mathbf{E}R(X))_+ \\ &\geq \inf_{m \in M} \left[\int \min_{0 \leq y \leq x \wedge m} [(x - y - A_m)_+ + c(m)(1-\beta)y] \, dF(x) + c(m)(\beta m - P) \right] \end{aligned}$$

for any $c(m) \geq 0$. Define $c(m) = 0$ for $m \in M_0$ and $c(m) = 1/(1-\beta)$ for $m \in M_1$, in which

$$M_0 = \{m \in M; (1-\beta)\mathbf{E} \min\{(X - A_m)_+, m\} + \beta m < P\} \tag{20}$$

and $M_1 = M \backslash M_0$. If $m \in M_0$, then the minimum in y is attained at $y_a(x) = \min\{(x - a)_+, m\}$, where a is an arbitrary number from the interval $[0, A_m]$. If $m \in M_1$, the minimum is attained at $y_a(x) = \min\{(x-a)_+, m\}$, where a is an arbitrary real greater than or equal to A_m. By the definitions of M_0 and M_1, there exists $a(m)$ such that $a(m) \in [0, A_m]$ for $m \in M_0$, $a(m) \in [A_m, \infty)$ for $m \in M_1$, and $(1-\beta)\,\mathbf{E}y_{a(m)}(X) + \beta m = P$. Hence $A_m = \mathbf{E}X - \mathbf{E}y_{a(m)}(X)$. Putting $R_m(x) = y_{a(m)}(x)$ we get

$$\inf_{R \in \mathcal{R}} \mathbf{E}(X - R(X) - \mathbf{E}X + \mathbf{E}R(X))_+ \geq \inf_{m \in M} \mathbf{E}(X - R_m(X) - \mathbf{E}X + \mathbf{E}R_m(X))_+.$$

By (18), $\mathbf{E}(X - \mathbf{E}X)_+ \geq \mathbf{E}\,((X-m)_+ - \mathbf{E}(X-m)_+)_+$ for any $m \geq 0$. Arguing similarly as in the proof of Theorem 2 we can assert that the minimum in y is attained at some point $b \in M$. The proof is complete. □

Remark 5: The contract provided by Theorem 4 is also optimal in the case when the stability is measured by the mean absolute value of the cedent's cover. It follows immediately from the identity $\mathbf{E}|X - \mathbf{E}X| = 2\mathbf{E}(X - \mathbf{E}X)_+$.

MINIMIZING THE RUIN PROBABILITY

Suppose the cedent wants to minimize the ruin probability for a fixed reinsurer's premium $P > 0$ calculated via the maximal claims principle (1):

$$\inf \mathbf{P}(W - P - X + R(X) < 0) \quad \text{s.t.} \quad R \in \mathcal{R}, \tag{21}$$

where $W \geq P$ and $\mathcal{R}$ is defined by (3). Suppose X has a distribution function $F(t) = \mathbf{P}(X \leq t)$ of the form

$$F(t) = p\,\mathbf{I}(t \geq 0) + (1 - p)\int_0^t f(x)\,dx\,\mathbf{I}(t > 0),$$

in which $0 \leq p < 1$ and f is a positive density function on $(0, \infty)$. Let m_0 be the unique solution of the equation

$$(1 - \beta)\mathbf{E}\left[(X - W + P)_+\mathbf{I}(X < W - P + m)\right] + \beta m = P, \quad m > 0. \tag{22}$$

Theorem 5: *The truncated stop loss contract*

$$R^*(X) = (X - W + P)_+\mathbf{I}(X < W - P + m_0)$$

is a solution to problem (21).

Proof: We can proceed analogously to the proof of Theorem 1. For an arbitrary function $c(m) \geq 0$,

$$\begin{aligned}
&\inf_{R\in\mathcal{R}} \mathbf{P}(W - P - X + R(X) < 0) \\
&\quad \geq \inf_{m\in M} \inf_{R\in\mathcal{R}^*_m} \left[\mathbf{EI}(W - P - X + R(X) < 0) + c(m)((1-\beta)\mathbf{E}R(X) + \beta m) - c(m)P\right] \\
&\quad \geq \inf_{m\in M}\left[\int \min_{0\leq y\leq x\wedge m}\left[\mathbf{I}(W - P - x + y < 0) + c(m)(1-\beta)y\right]dF(x) + c(m)(\beta m - P)\right].
\end{aligned} \tag{23}$$

Set $c(m) = 0$ for $m < m_0$ and $c(m) > 0$ for $m \geq m_0$, where m_0 is the unique solution to (22). Since $\beta m < P$ and

$$(1 - \beta)\mathbf{E}\left[(X - W + P)_+\mathbf{I}(X < W - P + m)\right] + \beta m \leq (1 - \beta)\mathbf{E}(X \wedge m) + \beta m$$

for any $m \in M$, we have $m_0 \in M$. It is easy to check that the minimum in (23) is attained at $y = R_{c(m)m}(x)$, where

$$R_{cm}(x) = (x - W + P)_+\mathbf{I}\left(x < W - P + \min\left\{\frac{1}{c(1-\beta)}, m\right\}\right), \quad x \geq 0, \tag{24}$$

in which, by convention, $1/0 = \infty$. Define $\Psi(c) = (1-\beta)\mathbf{E}R_{cm}(X) + \beta m$. Clearly,

$$\Psi\left(\frac{1}{(1-\beta)m}\right) = (1-\beta)\mathbf{E}\left[(X - W + P)_{+}\mathbf{I}(X < W - P + m)\right] + \beta m \geq P$$

and $\Psi(\infty) = \beta m < P$ for $m \geq m_0$. Analysis similar to that in the proof of Theorem 1 shows that there exists an increasing function $c(m)$ on $M_1 = M \cap [m_0, \infty)$ such that $c(m_0) = [(1-\beta)m_0]^{-1}$, $\Psi(c(m)) = P$ and $c(m) \geq [(1-\beta)m]^{-1}$ for $m \in M_1$. Put $M_0 = M \cap [0, m_0]$. From (23) we obtain

$$\begin{aligned}
\inf_{R \in \mathcal{R}} \mathbf{P}(W - P - X + R(X) < 0) &\geq \inf_{m \in M} \mathbf{P}(W - P - X + R_{c(m)m}(X) < 0) \\
&= \inf_{m \in M} \mathbf{P}\left(X > W - P + \min\left\{\frac{1}{c(m)(1-\beta)}, m\right\}\right) \\
&= \min\left\{\inf_{m \in M_1} \mathbf{P}\left(X > W - P + \frac{1}{c(m)(1-\beta)}\right), \inf_{m \in M_0} \mathbf{P}(X > W - P + m)\right\}.
\end{aligned} \tag{25}$$

Since $c(m)$ is increasing, $[c(m)(1-\beta)]^{-1} \leq [c(m_0)(1-\beta)]^{-1} = m_0$ for every $m \geq m_0$. Combining this with (25) gives

$$\begin{aligned}
\inf_{R \in \mathcal{R}} \mathbf{P}(W - P - X + R(X) < 0) &\geq \mathbf{P}(X > W - P + m_0) \\
&= \mathbf{P}(W - P - X + R^*(X) < 0),
\end{aligned}$$

which completes the proof. □

Remark 6: There is a formal analogy between the reinsurer's share of the claim in a truncated stop loss contract $R(X) = X\mathbf{I}(X \leq a)$ and the claim size net of the franchise deductible of a (see Daykin, Pentikäinen, and Pesonen, 1994; Dhane and Denuit, 1999).

CONCLUDING REMARKS

Raviv (1979) divides insurance analysis into those in which the insurance policy was exogenously specified (Mossin, 1968; Smith, 1968; Gould, 1969) and those in which it was not (Borch, 1960, 1962; Arrow, 1971; Aase, 2004). In the article we proceed with the study of the former approach proposed by Borch (1961) and Arrow (1963), and developed by Borch (1975), Hasselager (1990), Schmitter (2001), Gajek and Zagrodny (2004 a,b), and Kaluszka (2004a,b), among others. We prove that a limited stop loss contract maximizes the expected utility (Theorem 1) and the stability (Theorems 2–4) of the cedent for a fixed reinsurance premium calculated according to the maximal possible claims principle, which is a convex combination of expected loss and maximal loss principles. This type of reinsurance has been commonly used in practice for more than 100 years. The premium principle belongs to the class of coherent risk measures (see Artzner et al., 1999; Yang and Siu, 2001; Wang, 2002) and can be treated as a limit of zero utility principle and Wang's principle. In the case when the cedent's expected utility is maximized, Theorem 1 extends the classical result given by Arrow (1963).

A popular risk measure used by banking and casual insurance industries is the Value at Risk (cf. Artzner et al., 1999; Wang, 2002), which is based on ruin probability. In 2004, Gajek and Zagrodny found out that the contract that maximizes survival probability of the insurer under the expected value type pricing rule is a truncated stop loss treaty. We prove (Theorem 5) that a truncated stop loss contract also maximizes survival probability of the cedent under maximal possible loss principle. As was pointed out by one of the referees, this reinsurance treaty can give an opportunity for fraud. Of course, the risk of fraud cannot be completely eliminated, but it can be decreased, e.g., by employing reliable inspectors and using effective and efficient audit instruments. Moreover, an opportunity for fraud is not frequent because extremely large claims are rare.

APPENDIX

In this section, we indicate which of the desirable properties catalogued by Young (2004) are satisfied by the maximal possible claims premium principle. Let $\mathcal{X}$ denote a set of nonnegative random variables on the probability space $(\Omega, \mathcal{S}, \mathbf{P})$, and let $\pi : \mathcal{X} \to \mathbf{R} \cup \{\infty\}$ be defined as in (1), i.e., $\pi(X) = \mathbf{E}X + \beta(\sup X - \mathbf{E}X)$. The premium principle π has the following properties.

1. *Independence*: For any $X \in \mathcal{X}, \pi(X)$ depends only on the cumulative distribution function of X.
2. *Risk loading*: $\pi(X) \geq EX$ for all $X \in \mathcal{X}$.
3. *No unjustified risk loading*: If a risk $X \in \mathcal{X}$ is equal to a constant $c \geq 0$ (almost everywhere), then $\pi\ (X) = c$.
4. *No rip-off*: $\pi(X) \leq \sup X$ for all $X \in \mathcal{X}$.
5. *Translation invariance*: $\pi(X + a) = \pi(X) + a$ for all $X \in \mathcal{X}$ and all $a \geq 0$.
6. *Scale invariance*: $\pi(bX) = b\pi(X)$ for all $X \in \mathcal{X}$ and all $b \geq 0$.
7. *Subadditivity*: $\pi(X + Y) \leq \pi(X) + \pi(Y)$ for all $X, Y \in \mathcal{X}$.
8. *Additivity for independent risks*: $\pi(X + Y) = \pi(X) + \pi(Y)$ for all $X, Y \in \mathcal{X}$ such that X and Y are independent.
9. *Additivity for comonotonic risks*: $\pi(X + Y) = \pi(X) + \pi(Y)$ for all $X, Y \in \mathcal{X}$ such that X and Y are comonotonic.
10. *Monotonicity*: If $X, Y \in \mathcal{X}$ and $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, then $\pi(X) \leq \pi(Y)$.
11. *Preserves first stochastic dominance ordering*: If $X, Y \in \mathcal{X}$ and $\mathbf{P}(X > t) \leq \mathbf{P}(Y > t)$ for all $t \geq 0$, then $\pi(X) \leq \pi(Y)$.
12. *Preserves stop loss ordering*: If $X, Y \in \mathcal{X}$ and $\mathbf{E}(X - d)_+ \leq \mathbf{E}(Y - d)_+$ for all $d \geq 0$, then $\pi(X) \leq \pi(Y)$.
13. *Continuity*: $\lim_{a \to 0+} \pi((X - a)_+) = \pi(X)$ and $\lim_{a \to \infty} \pi(\min(X, a)) = \pi(X)$ for all $X \in \mathcal{X}$.

This means that the maximal possible claims principle only does not satisfy two properties from Young's catalogue, namely,

14. *Additivity*: $\pi(X+Y) = \pi(X) + \pi(Y)$ for all $X, Y \in \mathcal{X}$.
15. *Superadditivity*: $\pi(X+Y) \geq \pi(X) + \pi(Y)$ for all $X, Y \in \mathcal{X}$.

It is worth noting that properties 14 and 15 do not hold for such premium principles as the variance principle, standard deviation principle, exponential principle, Esscher principle, proportional hazards principle, Wang principle, Swiss principle, and Dutch principle.

Another interesting property of the maximal possible claims premium is that it can be treated as a limit of a sequence of some well-known premium principles. Examples are

1. Exponential type principle:

$$\pi_a(X) = \mathbf{E}X + \beta\left(\frac{1}{a}\log \mathbf{E}e^{aX} - \mathbf{E}X\right) \to \mathbf{E}X + \beta(\sup X - \mathbf{E}X) \quad \text{as} \quad a \to \infty.$$

   Hereafter, $\beta \in (0, 1)$.
2. Quantile type principle:

$$\pi_\varepsilon(X) = \mathbf{E}X + \beta\left(F_X^{-1}(1-\varepsilon) - \mathbf{E}X\right) \to \mathbf{E}X + \beta(\sup X - \mathbf{E}X) \quad \text{as} \quad \varepsilon \to 0+,$$

   where F_X^{-1} denotes the quantile function, i.e., $F_X^{-1}(t) = \inf\{x \in \mathbf{R}; F_X(x) \geq t\}$.
3. Absolute deviation from the median type principle:

$$\pi_\varepsilon(X) = \mathbf{E}X + \beta\left|F_X^{-1}(1-\varepsilon) - \mathbf{E}X\right| \to \mathbf{E}X + \beta(\sup X - \mathbf{E}X) \quad \text{as} \quad \varepsilon \to 0+.$$

4. Absolute deviation type principle:

$$\pi_p(X) = \mathbf{E}X + \beta(\mathbf{E}|X - \mathbf{E}X|^p)^{1/p} \to \mathbf{E}X + \beta(\sup X - \mathbf{E}X) \quad \text{as} \quad p \to \infty.$$

5. Upper mean deviation type principle:

$$\pi_p(X) = \mathbf{E}X + \beta\left(\mathbf{E}(X - \mathbf{E}X)_+^p\right)^{1/p} \to \mathbf{E}X + \beta(\sup X - \mathbf{E}X) \quad \text{as} \quad p \to \infty.$$

6. Wang's principle:

$$\pi_n(X) = \mathbf{E}X + \beta(\mathbf{E}X_{n:n} - \mathbf{E}X) \to \mathbf{E}X + \beta(\sup X - \mathbf{E}X) \quad \text{as} \quad n \to \infty,$$

   where $X_{n:n}$ is the maximal order statistic from the independent and identically distributed observations $X_1, \ldots, X_n$, $n \geq 2$, and X is a random variable with the same distribution as X_1. Indeed, since

$$\mathbf{E}X_{n:n} = \int_0^\infty \mathbf{P}(X_{n:n} > t)\, dt = \int_0^\infty \left[1 - (1 - \mathbf{P}(X > t))^n\right] dt,$$

we have

$$\pi_n(X) = \int_0^\infty [(1-\beta)\mathbf{P}(X > t) + \beta(1 - (1 - \mathbf{P}(X > t))^n)]\, dt$$
$$= \int_0^\infty g(\mathbf{P}(X > t))\, dt$$

with $g(x) = (1-\beta)x + \beta(1-(1-x)^n)$, $x \in [0, 1]$.

7. Expected tail loss type principle:

$$\pi_\alpha(X) = \mathbf{E}X + \beta\left(\frac{1}{\mathbf{P}\big(X \geq F_X^{-1}(\alpha)\big)}\mathbf{E}\left[X\mathbf{I}\big(X \geq F_X^{-1}(\alpha)\big)\right] - \mathbf{E}X\right)$$
$$= \mathbf{E}X + \beta\left(\frac{1}{1-\alpha}\int_\alpha^1 F_X^{-1}(t)\, dt - \mathbf{E}X\right)$$

tends to $\mathbf{E}X + \beta(\sup X - \mathbf{E}X)$ as $\alpha \to 1$, where $\alpha \in (0, 1)$ (see Embrechts, Klüppelberg, and Mikosch, 1997).

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