

## **EXPLAINING LOW ANNUITY DEMAND: AN OPTIMAL PORTFOLIO APPLICATION TO JAPAN**

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### **ABSTRACT**

Using an optimizing financial planning model in the tradition of Merton and Richard we explore how individuals should determine their life insurance and annuity choices, given uncertainty about investment returns and mortality. Both consumption and bequests appear as arguments in the individual's preference function. The model explicitly recognizes the existence of social security in retirement, and of loadings on insurance premiums, due to administration costs in the life insurance and annuities markets. The model sheds light on the reasons for the thinness of voluntary life annuity markets worldwide. The relative importance of pre-existing annuitization through social security, the role of bequests, and premium loadings are quantitatively assessed within a single optimizing framework. Results are presented for a model specification calibrated to Japan.

### **INTRODUCTION**

The factors influencing decisions to purchase life annuities have received increasing attention in recent years. Renewed interest has been spurred by increasing reliance on defined-contribution (DC) plans for private retirement provision globally, and by population aging, which is leading to greater focus on the characteristics of retirement income products and markets.

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Both these developments are manifested in Japan. Now the world's oldest economy, Japan's population peaked in 2006.<sup>1</sup> Among a large number of pension reforms enacted over the last decade, and motivated in part by the rapid demographic transition being experienced there, is legislation enabling DC plans to be offered by Japanese firms. DC plans have quickly become popular, although contribution levels and account values are, thus far, low. Defined-benefit (DB) occupational pension plans are seriously underfunded and benefits can be renegotiated, sometimes for pensions in payment. Japan's social security system typically delivers less than 50 percent replacement at retirement, and large stocks of wealth are held in cash management accounts, yet the life annuity market mediates only a few thousand life annuity contracts each year. In Japan, therefore, the question of why voluntary annuity demand is so low is of special concern.

This article seeks to shed new light on the puzzle of thin voluntary life annuity markets. In sharp contrast to previous studies, it takes as its point of departure Merton's (1969, 1971) model of optimal lifetime asset allocation, and exploits Richard's (1975) theoretical extension to longevity insurance markets. The Merton–Richard framework is consistent with that used by Yaari (1965) to conclude that in the absence of a bequest motive, and given actuarially fair annuity markets, an optimizing consumer will annuitize all wealth.<sup>2</sup> Within this framework, we use stochastic control techniques to calculate optimal levels of life insurance and annuity purchases for Japanese households. We then use the model to investigate numerically the relative importance of three traditional explanations for low annuity demand: the bequest motive, the proportionate value of loadings, and the level of social security benefits. Although the results reported here have been calibrated to the case of Japan, we believe the qualitative findings to have general applicability and interest.

Our results indicate that the most important factor inhibiting annuity purchase is the bequest motive, followed by social security provision. Loadings are relatively unimportant.

The article is organized as follows. We begin in the first section with a discussion of annuity markets in Japan and the literature on thin annuity markets. The second section details, interprets and extends the Richard model—the tool we use to analyze optimal demand for annuities. The parameterization of the model is described in the third section. The fourth section reports the results of our modeling and analyzes the findings while the final section concludes.

### **ANNUITY MARKETS IN JAPAN**

Currently Japan has the highest share of people aged 65 and over in the world. It also has the highest life expectancy of any nation. Such conditions would appear ripe for a flourishing market in longevity insurance type annuities in Japan. This is not the case.

Life insurance products in Japan are marketed through numerous private sector life insurance companies, as well as Japan's publicly controlled Postal Life Insurance Service (Kampo). These companies earned 25,511 billion and 14,318 billion yen, respectively,

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<sup>1</sup> Based on the medium variant projection in National Institute of Population and Social Security Research (2002).

<sup>2</sup> Yaari (1965) did not incorporate a risky asset.

in premium income in 2002.<sup>3</sup> Both sectors offer longevity insurance in the form of whole of life and other related annuity products.

Historically, growth in sales of annuities only took off in the 1980s in Japan, even though annuities have been gazetted for Kampo since 1926. Between 1980 and 1990 premium income for private sector provided individual annuity business rose from 66 billion yen to 1,601 billion yen. It is interesting that this should occur in a decade of enormous financial turmoil in Japan. We speculate that this reflects a “flight to quality,” also observed in the U.S. annuity market during the Depression of the 1930s (Warshawsky, 1988).

In addition to getting off to a slow start in Japan, the annuity market has also played only a small role in longevity insurance. Industry sources estimate that the business of the private life insurance sector is made up of only 5 percent or less in whole life annuities (providing longevity insurance), with the remaining 95 percent consisting of term business (not providing longevity insurance). For Kampo, around 20 percent of its annuity business represents longevity insurance. Moreover, in 2002 less than 2 percent of premium income of the combined private and public life insurance sector in Japan went toward longevity insurance. In 2002 only 1 percent of the payouts from these life insurers were for longevity protecting annuities. Similarly, only a small fraction of all life insurer reserves are for longevity-insurance-type annuities—for Kampo this figure was 3.5 percent in 2002.

#### Thin Annuity Markets

Voluntary annuity markets are thin everywhere for reasons that are not clear. Possible explanations for low annuity demand, which we focus on below, are a bequest motive, the desire to hold precautionary balances to cope with uninsurable events, overannuitization through publicly provided social security, and insurance company loadings, linked either to adverse selection or administrative fees. Other explanations for low levels of annuitization include issues of expenditure on health care and long-term care needs.<sup>4</sup> What does seem clear is that life annuity prices are high relative to population life expectancy and alternative investment returns. In an early widely cited paper that computes estimated loadings Friedman and Warshawsky (1990) report that a typical male aged 65 in the United States in the 1980s would have enjoyed a premium of 4.21 percent per annum had he purchased a government bond rather than a life annuity.

High annuity prices are frequently attributed to adverse selection. Mitchell et al. (1999) report load factors on actuarially fair quotes (the difference between the premium and the expected pension benefit) of 18 percent in the United States for 1995. They attribute about half of the load factor to adverse selection. They also report a significant increase in the effect of adverse selection with age and a significantly smaller effect of adverse selection on annuity prices for women. Finkelstein and Poterba (2002) calculate that voluntary annuities for 65-year-old males in the United Kingdom provide 10–15 percent less value than such products on the basis of population mortality. Of this reduction, the authors estimate over 60 percent is due to the lighter mortality of annuitants as compared to the general population.

<sup>3</sup> These values are from both the Life Insurance Association of Japan webpage, <http://www.seiho.or.jp/english/index.html>, and *Japan Post* (2003).

<sup>4</sup> See, for example, Warshawsky, Spillman, and Murtaugh (2002).

Of the alternative explanations for low annuity demand economic analysis has focused mainly on the bequest motive. While a desire to leave bequests would no doubt discourage voluntary annuity purchase, observation suggests that even those who might be supposed not to have a strong bequest motive (e.g., the elderly with no children) do not purchase annuities. Hurd (1990) considers the interaction between private annuities and bequests in some detail and tentatively concludes that the bequest motive is not necessary to explain the lack of demand for individual annuities. Further, many people who have the possibility of securing an individual life annuity in developed countries own their own homes and their bequest motives may be satisfied through the transfer of this asset.

This assessment would appear to lead back to adverse selection as a major reason for the low demand for individual annuities. Hurd (1990), however, also reports that in at least two experimental programs elderly people in subsidized housing programs were offered the choice between a lump sum and an actuarially fair annuity. Almost all took the lump sum, even though many had no children. One plausible explanation for this is the desire to hold precautionary balances to cope with uninsurable events. A second possible explanation is that the annuity promised a flat payment path and it may be that very elderly individuals prefer lower consumption in return for higher consumption possibilities earlier in their retirement.<sup>5</sup> Finally, the participants in these programs had relatively low incomes and may have been "overannuitized" through social security.

## MODEL

Building on Merton's (1969, 1971) treatment of optimal consumption and investment, Richard (1975) extended the model to include markets for life insurance and annuities.<sup>6</sup> Yaari's (1965) work established that, in the presence of actuarially fair versions of these markets, with information about the probability distributions of future lifetimes available as public information, a risk-averse individual with no bequest motive would hold his assets (liabilities) as a life annuity (life insurance).<sup>7</sup>

Richard (1975) models a multiperiod utility maximizing investor with objective<sup>8</sup>

$$\max E \left[ \int_{\tau}^T U(C(t), t) dt + B(Z(T), T) \right], \quad (1)$$

<sup>5</sup> The inefficiency of constant annuities is further considered in Yagi and Nishigaki (1993).

<sup>6</sup> For an empirical treatment of the Richard model in a Japanese context, see Chuma (1994).

<sup>7</sup> Yaari's (1965) work has recently been extended by Davidoff, Diamond, and Brown (2005).

<sup>8</sup> One problem that has been discussed concerning this objective function (Borch, 1990, pp. 257–260) has been that it does not allow for the spouse or beneficiary predeceasing the insured. For simplicity, we assume that in the event the spouse dies before the insured, the insured immediately finds someone whom he or she wishes to insure at the same amount. That is, we are assuming that the insured immediately re-marries on the death of his or her spouse—and that the new spouse is the same age as the previous spouse.

The resolution of this issue is not straightforward; Borch (1990) does not attempt it. We leave the issue for future research. However, it must be borne in mind that our solution results in overinsurance. We also note that in the section "Numerical Specification" we are able to interpret the optimal consumption decision rule as choosing consumption such that total wealth provides a reversionary annuity. This suggests the matter can be resolved by resorting to the theory of ordered deaths (Bowers et al., 1997, chap. 17).

where  $T$  is the investor's uncertain time of death, and  $U$ ,  $C$ ,  $Z$ , and  $B$  are the investor's utility, consumption, legacy at death and utility from bequest. The investor is able to choose between two securities, one risky and one risk free, with the price of the risky asset,  $Q$ , following geometric Brownian motion

$$\frac{dQ(t)}{Q(t)} = \alpha dt + \sigma dq(t), \quad (2)$$

where  $dq(t)$  is a Wiener increment.

The investor's change in wealth is given by the stochastic differential equation

$$\begin{aligned} dW(t) = & -C(t)dt - P(t)dt + Y(t)dt + rW(t)dt \\ & + (\alpha - r)\pi(t)W(t)dt + \sigma\pi(t)Wdq(t), \end{aligned} \quad (3)$$

where  $P(t)$ ,  $Y(t)$ ,  $W(t)$  are, respectively, the investor's life insurance premium paid, income (assumed to be nonstochastic), and wealth at time  $t$ . From Equation (2), the mean return on risky investment is  $\alpha$ , with standard deviation  $\sigma$ , while the risk-free investment returns  $r$ ; the investor places a proportion  $\pi$  of wealth in the risky asset.

Richard's model necessarily incorporates the probability of death of an investor. Let the investor's age at death,  $X$  (a continuous random variable), have a cumulative distribution function given by  $F(x)$  and probability density function of  $f(x)$ . The survival function,  $S(x)$ , given by  $1 - F(x)$ , yields the probability that the investor lives to age  $x$ .

The investor buys instantaneous term life insurance to the amount of  $Z(t) - W(t)$ , if this difference is positive. For life insurance, a premium of  $P(t)$  is paid. If we denote the instantaneous probability of death<sup>9</sup> by  $\mu(t)$ , then the amount of premium paid for actuarially fair insurance will be<sup>10</sup>

$$P(t) = \mu(t)[Z(t) - W(t)]. \quad (4)$$

Here Richard has extended Yaari's ideas in his formulation of insurance/annuity demand. By imposing Equation (4), should wealth levels lie below the desired bequest, then the individual must purchase insurance to cover this deficit.

Alternatively, Equation (4) also requires that any wealth in excess of the desired bequest level be annuitized, and negative values of  $P(t)$  represent the annual rate of life annuity payments to the investor. The type of immediate life annuity implied by

<sup>9</sup> Strictly speaking, this is the conditional instantaneous probability of death (the probability the investor dies at exact age  $x$ , having survived to that age) and is given by  $f(x)/S(x)$ . It is known as the *force of mortality* by demographers and actuaries, or as the *hazard rate* or *intensity rate* by reliability theorists (Elandt-Johnson and Johnson, 1980).

<sup>10</sup> Richard actually considered the more general case of there being some sort of "loading" to mortality. This means mortality rates are increased to above their true levels, ensuring profitability for the life insurer. For present purposes, the simpler case of actuarial fairness is sufficient, with the alternate "loaded" model being considered in the section "Numerical Specification."

this model is similar to an immediate variable life annuity (United States) or a with profits immediate life annuity (United Kingdom or Germany). Richard has in fact also captured both the well-known illiquidity of annuity contracts and the issue of timing of annuitization (Milevsky and Young, 2003; Stabile, 2006). Annuitization via Equation (4) implies the investor must annuitize when  $W(t) > Z(t)$  and that the death of the annuitant involves the loss of the annuitized capital,  $W(t) - Z(t)$ , to the annuity provider: illiquidity. Further, the switch from positive  $P(t)$  to negative values gives the timing of the annuity purchase. We expand on these ideas in the "Results" section.

For an annuitant, the (continuously compounding rate of) return on wealth in bonds is (following from Equations (3) and (4))  $r + (1 - (Z(t)/W(t)))\mu(t)$  and the return on wealth in equities is  $\alpha + \sigma dq(t) + (1 - (Z(t)/W(t)))\mu(t)$ . Thus, for the case of no bequest motive ( $Z(t) \equiv 0$ ) we get  $r + \mu(t)$  as the return on bonds and  $\alpha + \sigma dq(t) + \mu(t)$  as the return on equities—(much) the same result as Yaari (1965). For a bequest motive that is equivalent to the wealth on hand ( $Z(t) \equiv W(t)$ ), the benefits of mortality pooling in annuity purchase are extinguished—as they should be if the annuity provider must return the reserves backing an annuity to an annuitant's beneficiary.

### Re-Expressing the Objective

The investor's problem is to solve Equation (1), subject to budget constraint (3) and initial wealth condition  $W(0) = W_0$ , by optimal choice of controls  $C$ ,  $\pi$ , and  $Z$ . Utility,  $U$ , is assumed to be strictly concave in  $C$  and  $B$  is assumed strictly concave in  $Z$ . Equation (1) can be re-expressed as

$$J(W, \tau) = \max_{C, Z, \pi} E_{\tau} \int_{\tau}^{\omega} \frac{S(T)}{S(\tau)} \mu(T) \left[ \int_{\tau}^T U(C(t), t) dt + B(Z(T), T) \right] dT, \quad (5)$$

where  $\omega$  represents the limiting age of the underlying mortality table; i.e., one's age at death lies in the range from zero to  $\omega$ . Equation (5) may be further simplified to the following:<sup>11</sup>

$$J(W, \tau) = \max_{C, Z, \pi} E_{\tau} \int_{\tau}^{\omega} \frac{S(t)}{S(\tau)} [\mu(t)B(Z(t), t) + U(C(t), t)] dt. \quad (6)$$

The Hamilton–Jacobi–Bellman equation can thus be shown to be

$$0 = \max_{C, Z, \pi} \left\{ \mu(t)B(Z(t), t) + U(C(t), t) - \mu(t)J + J_t \right. \\ \left. + [\pi\alpha W + (1 - \pi)rW + Y - C - P]J_W + \frac{1}{2}\sigma^2\pi^2 W^2 J_{WW} \right\}. \quad (7)$$

<sup>11</sup> Proceed by using Fubini's theorem to swap the order of integration over the triangle  $t \leq T < \omega$ ,  $\tau \leq t < \omega$  in  $\mathbb{R}^2$ . For a proof that Fubini's theorem can be used in stochastic integration (see, e.g., Protter, 1990, theorem IV.45).

### Deterministic Labor Income

We may eliminate  $Y(t)$  in Equations (3) and (7) as it is nonstochastic. Richard demonstrates that (7) is equivalent to an equation involving capitalized  $Y(t)$ . Let  $b(t)$  be defined as the capitalized value of future income:

$$b(t) = \int_t^\omega Y(\theta) \frac{S(\theta)}{S(t)} e^{-r(\theta-t)} d\theta, \quad (8)$$

and further define *adjusted wealth* as

$$\tilde{W}(t) \equiv W(t) + b(t). \quad (9)$$

Implicitly we are treating capitalized income as a traded asset. That is, markets are complete to the extent that future income can be perfectly replicated by using traded securities. This manipulation of our stochastic control problem allows us to remove  $Y(t)$  from (7) and substitute  $\tilde{W}(t)$  for  $W(t)$ .

### Solution

Richard provides an algebraic solution to the above model for CRRA utility. The derivation of the following results follows from the solution of the partial differential Equation (7) above (with the  $Y$  term removed). By using the CRRA form for the utility and bequest function one is able to guess a functional form for  $J(\tilde{W}, t)$  in (7). Substitution of this functional form into (7) yields an ordinary differential equation of Bernoulli form, which is readily solved to provide  $C^*$ ,  $Z^*$  and  $\pi^*$ .

Richard demonstrates that when

$$U(C(t), t) = h(t) \frac{C(t)^\gamma}{\gamma}, \quad \gamma < 1, h > 0, C > 0 \quad (10)$$

and

$$B(Z(t), t) = m(t) \frac{Z(t)^\gamma}{\gamma}, \quad \gamma < 1, h > 0, Z > 0 \quad (11)$$

the optimal controls are given by<sup>12</sup>

$$C^*(\tilde{W}, t) = \left( \frac{h(t)}{\hat{a}(t)} \right)^{1/(1-\gamma)} \tilde{W}(t), \quad (12)$$

$$Z^*(\tilde{W}, t) = \left( \frac{m(t)}{\hat{a}(t)} \right)^{1/(1-\gamma)} \tilde{W}(t), \quad (13)$$

<sup>12</sup> Note that equation (13) simplifies equation (32) of Richard, as the current work assumes actuarially fair insurance. See the discussion in footnote 10.

and

$$\pi^*(\tilde{W}, t)W = \frac{\alpha - r}{(1 - \gamma)\sigma^2} \tilde{W}(t), \quad (14)$$

where

$$\hat{a}(t) = \left\{ \int_t^\omega k(\theta) \frac{S(\theta)}{S(t)} \exp \left[ \frac{\gamma}{1 - \gamma} \left( \frac{(\alpha - r)^2}{2(1 - \gamma)\sigma^2} + r \right) (\theta - t) \right] d\theta \right\}^{1-\gamma} \quad (15)$$

and<sup>13</sup>

$$k(t) = \left\{ \left[ \frac{1}{\mu(t)} \right]^{\gamma/(1-\gamma)} [\mu(t)m(t)]^{1/(1-\gamma)} + h(t)^{1/(1-\gamma)} \right\}. \quad (16)$$

The solutions are linear in adjusted wealth, a familiar result for hyperbolic absolute risk aversion (HARA) utility functions (Merton, 1971). Interestingly, for  $h(t) = m(t)$  optimal consumption (Equation (12)) and bequest amounts (Equation (13)) will be identical. Indeed if  $m(t) \equiv 0$ , the case of no bequest motive (and hence no demand for life insurance) then Equation (15) becomes an equation for a expected present value of a life annuity and the solution for the optimal level of consumption is that one annuitizes adjusted wealth—a prescription consistent with Yaari and, indeed, both the traditional life cycle or permanent income models of consumption.

As it stands, for  $m(t)$  not uniformly zero, Equation (15) not only annuitizes current wealth to provide for consumption but also sets aside a portion of wealth to provide the desired bequest (through life insurance) in the event of death. We expand on this issue in the section “Numerical Specification” below.

The solution for  $\pi^*$  (Equation (14)) indicates investment in the risky asset should be a constant fraction of adjusted wealth. This is an example of the well-known result that optimal investment behavior over the life cycle, for utility functions that display constant relative risk aversion, is “myopic,” with individuals always investing a constant proportion of wealth in the risky asset and ignoring the future distribution of asset returns and current age.

The results for optimal life insurance and annuitization are more complicated as they depend on the chosen bequest motive—an issue not addressed by Richard. We postpone our discussion of these issues to the sections “Numerical Specification” and “Results.” Indeed, through the use of numerical simulation we are able to present a clear picture of the workings of all aspects of the Richard model.

### Numerical Specification

A critical step in producing numerical results for the model is parameter choice. Although Richard has partially parameterized his model, by presenting a solution to

<sup>13</sup> Equation (39) of the Richard paper, which is the formula for  $k(t)$ , contains a typographical error. Equation (16) above is our corrected version—for the case of actuarially fair life insurance.

the model for isoelastic utility and utility of bequest, key parts of the model remain unspecified. We start by considering the discount functions  $h(t)$  and  $m(t)$ , associated with the utility and bequest functions and introduced in Equations (10) and (11). We advance plausible functional forms for each. We then discuss the inclusion of premium loadings into the model. The appropriate Japanese parameter values are treated. The section concludes with an intuitive interpretation of the Richard model.

We continue to use constant relative risk aversion utility as it is a standard benchmark in economics and readily facilitates comparison with the earlier literature.

*Bequest Function.* A plausible choice of  $h(t)$  is  $e^{-\rho t}$ , where  $\rho$  is the rate of the investor's time preference. The choice of  $m(t)$  is not clear, however. If  $m(t)$  is set equal to  $h(t)$ , consideration of Equations (12) and (13) readily yields that such a model would produce optimal consumption and bequest amounts that are identical. This result is not particularly appealing. Prevailing social norms seem to indicate a reasonable value to leave a surviving spouse would be an amount sufficient to provide two-thirds of the deceased's current income for the life of the survivor.<sup>14</sup> This notion has in fact been enshrined in pension benefits regulations in Canada, for example, where the surviving spouse of a deceased pensioner is provided with between 50 percent and 66<sup>2</sup>/<sub>3</sub> percent pension continuation.<sup>15</sup>

Thus we may desire to provide an optimal legacy of  $Z^*(t) = {}^{2/3}C^*(t) \times \int_t^\omega (\hat{S}(\theta)/\hat{S}(t)) \exp(-r(\theta - t)) d\theta = \phi(t)C^*(t)$ , say, where  $\hat{S}(t)$  is the survival function associated with investor's spouse and where we have  $\phi(t)$  as  ${}^{2/3} \int_t^\omega (\hat{S}(\theta)/\hat{S}(t)) \exp(-r(\theta - t)) d\theta$ . Before choosing this as our value for  $\phi(t)$  let us reflect on its meaning and implications. From actuarial mathematics, this choice for  $\phi(t)$  is equivalent to the net single premium of a life annuity commencing immediately and paying \$0.67 continuously from the current age of the spouse,  $t$ .<sup>16</sup> To achieve such an optimal legacy implies  $m(t) \equiv e^{-\rho t} \phi(t)^{1-\gamma}$ , from Equations (12) and (13) above. While this would seem a highly appropriate choice for  $m(t)$ , the inclusion of this functional form into the Richard model is not at all straightforward, due to the interaction of survival probabilities of both the consumer and beneficiary.<sup>17</sup>

Instead we assume that the investor wants to leave a term certain annuity to his surviving spouse, which pays  ${}^{2/3}C^*(t)$  from the date of death to the limiting age of the mortality table. Such policies exist and are known as a family income benefit or family income insurance (Bowers et al., 1997, pp. 537–539). To provide such a benefit we set

$$Z^*(t) = {}^{2/3}C^*(t) \int_t^\omega e^{-r(\theta-t)} d\theta,$$

<sup>14</sup> We ignore dynastic style intergenerational bequests. Horioka (2002) provides a comprehensive treatment of intergenerational bequest behavior in the Japanese context.

<sup>15</sup> See, for example, section 45(2) of Ontario's Pensions Benefits Act, 1987.

<sup>16</sup> See, for example, Bowers et al. (1997, chap. 5).

<sup>17</sup> These complications have been noted by other authors (Borch, 1990, pp. 257–260); see footnote 8 above.

implying that

$$m(t) = e^{-\rho t} \left( \frac{2}{3} \int_t^\omega e^{-r(\theta-t)} d\theta \right)^{1-\gamma}.$$

Henceforth, we set  $m(t) = e^{-\rho t} \phi(t)^{1-\gamma}$ , where we have adopted  $\frac{2}{3} \int_t^\omega e^{-r(\theta-t)} d\theta$  as the value of  $\phi(t)$ .

As per the constraint given in Equation (4), we maintain that the investor always wants such a policy over his lifetime. Thus we are implicitly assuming that should his spouse die he immediately remarries a woman of the same age. This is a more generous bequest function than the ideal, and this overprovision—which could be thought of as a intergenerational bequest—should be borne in mind when considering the results below.

*Consequences of Functional Forms.* Adopting the functional forms discussed above will have consequences for the rest of the model—in particular, the optimal values of the consumption, given above in Equation (12). In particular, let us look at the  $(h(t)/\hat{a}(t))^{1/(1-\gamma)}$  term. Consider the reciprocal of this term (here we draw on Equations (15) and (16)):

$$\begin{aligned} \left\{ \frac{\hat{a}(t)}{h(t)} \right\}^{\frac{1}{1-\gamma}} &= h(t)^{\frac{1}{\gamma-1}} \int_t^\omega \left[ \left( \frac{1}{\mu(\theta)} \right)^{\frac{\gamma}{1-\gamma}} (\mu(\theta)m(\theta))^{\frac{1}{1-\gamma}} + h(\theta)^{\frac{1}{1-\gamma}} \right] \\ &\quad \times \frac{S(\theta)}{S(t)} e^{\Omega(\theta-t)} d\theta \\ &= h(t)^{\frac{1}{\gamma-1}} \int_t^\omega \left[ \mu(\theta)m(\theta)^{\frac{1}{1-\gamma}} + h(\theta)^{\frac{1}{1-\gamma}} \right] \frac{S(\theta)}{S(t)} e^{\Omega(\theta-t)} d\theta \\ &= \int_t^\omega \left[ \mu(\theta) \left( \frac{m(\theta)}{h(\theta)} \right)^{\frac{1}{1-\gamma}} + 1 \right] \frac{S(\theta)}{S(t)} e^{\Omega(\theta-t)} d\theta \\ &= \int_t^\omega \left[ \frac{2}{3} \mu(\theta) \left( \int_\theta^\omega e^{-r(s-t)} ds \right) + 1 \right] \frac{S(\theta)}{S(t)} e^{\Omega'(\theta-t)} d\theta, \end{aligned} \quad (17)$$

where  $\Omega = (\gamma/(1-\gamma))((\alpha-r)^2/2(1-\gamma)\sigma^2 + r)$  (from Equation (15)),  $\Omega' = \Omega - \rho/(1-\gamma)$ , and in the last line we have substituted the functional forms discussed above.

Equation (17) has an elegant interpretation. Of the two terms in the equation the first represents the net single premium of a two-thirds reversionary term certain annuity. The second term is the net single premium of a life annuity commencing at the investor's current age  $t$ .<sup>18</sup> Both annuities are paid continuously, and together they approximate a joint and two-thirds survivor life annuity.

This means the level of optimal consumption in the Richard model,  $C^*(t)$ , has a delightfully consistent interpretation: the investor chooses his consumption level such

<sup>18</sup> See, for example, Bowers et al. (1997, chaps. 5, 9, and 18)

**TABLE 1**

Summary of Japanese Economic and Financial Market Experience, 1970–2000. All Series From DX Data (2000) CEIC Database

|            | WAgess <sup>a</sup> | Prices <sup>b</sup> | Nikkei <sup>c</sup> | Bill Rate <sup>d</sup> |
|------------|---------------------|---------------------|---------------------|------------------------|
| Mean       | 5.9%                | 3.9%                | 6.4%                | 4.7%                   |
| Volatility | 2.2%                | 2.5%                | 19.9%               | 0.9%                   |
| Range      | (−1.9%, 29.1%)      | (−1.1%, 24.7%)      | (−41.1%, 99.4%)     | (0.5%, 12.2%)          |

<sup>a</sup>Japanese nominal wage index, all industries, including bonuses, monthly, January 1970–June 2000.

<sup>b</sup>Japanese consumer price index, monthly, January 1970–June 2000.

<sup>c</sup>Japanese index of Nikkei stock average 225, monthly, February 1970–July 2000.

<sup>d</sup>Japanese three-month Bank of Japan bill rate, monthly, January 1970–June 2000.

**TABLE 2**

Parameters Used in Numerical Simulation of the Model

|                  |                            |
|------------------|----------------------------|
| $\alpha = 0.025$ | Mortality: JLT18 (male)    |
| $r = 0.005$      | $\omega = 110$             |
| $\rho = 0.005$   | $\gamma = -0.5$ or $-4$    |
| $\sigma = 0.2$   | $Y = W(0) = 4,375,686$ yen |

that his current level of total wealth is sufficient to provide both a flow of  $C^*(t)$  for his lifetime as well as a term-certain annuity of  $^{2/3} C^*(\tau)$  to his spouse following his death at random time  $\tau$ . This exactly matches our bequest motive outlined above, approximating the value of a joint and  $^{2/3}$  survivor life annuity.

*Parameter Values.* The economic and financial data we use to parameterize the model are summarized in Table 1. In Table 2 we set out the parameters we adopt for the model. The values we adopt reflect real values of asset accumulation; hence,  $\alpha$ , the rate of return on the risky asset, is chosen as the real rate of return on the Nikkei:  $(1.064/1.039) - 1 \approx 0.025$ . The safe rate,  $r$ , is similarly chosen. We set the rate of time preference equal to the safe rate.<sup>19</sup>

We adopt Japanese male population mortality given by the Ministry of Health and Welfare (1995) Japanese life table, number 18, excluding the effects of the Kōbe earthquake.<sup>20</sup> Values of  $\gamma$  of  $-0.5$  and  $-4$  reflect an individual who is somewhat risk averse

<sup>19</sup> For a careful treatment of the issues of parameterizing a life cycle model, see Engen et al. (1999). Within the life cycle consumption literature there has been considerable debate about the relationship between the rate of time preference and the discount rate and its implications for consumption growth. See, for example, Deaton (1992).

<sup>20</sup> This is a period table, not a cohort table. Mitchell et al. (1999) argue persuasively that cohort tables should be used for annuity calculations. We leave the cohortization of this table and re-analysis to future research.

and quite risk averse, respectively.<sup>21</sup> Full information on mortality is assumed to be known by all agents, implying no adverse selection. We set the individual's yearly earnings to 12 times the average monthly cash earnings of regular employees for calendar year 1999:  $12 \times 364,638 = 4,375,656$  yen.<sup>22</sup>

We assume this amount remains constant and safe over the employee's working life. The employee works from age 30 to 65, at which point he must retire.

*Loadings.* Up to now our consideration of the Richard model has been from the perspective of actuarially fair insurance and annuity provision. It is well known that insurers add loadings to their products to cover administrative costs and make provision against unforeseen fluctuations. Here we adjust the model to take these factors into account.

While the Richard model does consider the case of loadings to mortality rates we do not adopt this approach. Rather, we consider the modification of premia in the presence of loadings. This is the approach to treating expenses commonly adopted by actuaries (Bowers et al., 1997).

From the consumer's perspective an insurance premium consists of the actuarially fair premium plus loadings. We thus model expenses in this way:

$$\hat{P}(t) = \mu(t)\{Z(t) - W(t)\} + \Gamma(t),$$

where  $\hat{P}$  is the loaded premium and  $\Gamma(t)$  is the loading amount applied to the premium at time  $t$ .<sup>23</sup> This adjustment to the Richard model entails minimum change to its solution; the optimum solution given by Equations (12) and (14) remains unchanged.

Note that, formally, loadings apply only to the mortality component of charges and returns. The capitalization of the individual's income is still done at the (unloaded) actuarially fair rates.

*Social Security.* We introduce social security into the model by making use of the following contrivance. Since social security entitlements depend on employment, and since employment in this model is fixed by assumption, a social security scheme is equivalent to a modification of the human capital depreciation schedule. Wage income is adjusted to take account of contingent social security entitlements, such that the present value of total wealth at the beginning of the modeled life cycle is unchanged. This allows appropriately calibrated comparison of optimal financial choices with and without social security.

Japan's social security system was established in the 1940s with different groups of workers being covered by different programs. In 1985 it was restructured to resemble

<sup>21</sup> Kato (1998) used a coefficient of relative risk aversion of 5 ( $\gamma = -4$ ) in his work on the Japanese economy; other studies have used lower values. Miles and Černý (2005) review the literature in terms of choosing a coefficient of relative risk aversion for Japan.

<sup>22</sup> These values are from the Japanese Institute of Labor webpage, located at <http://www.jil.go.jp/estatis/e0301.htm>.

<sup>23</sup> Formally,  $\Gamma(t) = |\lambda P(t)|$ , where  $\lambda$  is the given loading percentage.

something close to the current system, although further reforms were enacted in 1994 and 1999. Until recently benefit levels have progressively increased since its introduction.

The replacement rate for a full-time worker on average earnings for 40 years is about 55 percent if he retires at 65. Our calculations suggest that the replacement rate after 30 years is 33.3 percent, again assuming retirement at 65. At this point, life expectancy is 17.1 years.<sup>24</sup> We have stylized this in our model by using as our base case a 50 percent replacement rate after 35 years continuous employment. Since no taxes, other than an implicit social security tax, are levied, this replacement rate should be thought of as a net-of-income-tax rate.

Our social security paradigm comprises a life annuity for the retired worker equal to a given proportion of his wage. Our base case sets this at 50 percent but we model a 20 percent wage income replacement as well for some simulations. There is no provision for survivor insurance within this stylized social security program—bequests are unaffected by the introduction of the policy. Implicitly a payroll tax that raises revenue on an actuarially fair basis is levied on wage income. This follows from our calibration of the model in which we set initial total wealth to a common value under the alternative model specifications.

### Solution Method

The Richard model gives the optimal behavior of an individual in a stochastic environment. Thus no unique wealth or consumption path is produced; an infinity of results are possible. For us to make meaningful comments on the model we need a way of characterizing its results. Here we propose to base our analysis on the expected paths of the dynamic state and control variables. Such values give us a feel for the “typical” results of the model and provides some check with intuition. Also, further investigation is possible by examining confidence bounds around these paths.

Given the Richard model has a closed-form solution for its controls it would be appropriate to adopt an analytic approach to determining the expected values of the state and control variables. Richard has not done this, but in equation (47) of his paper (Equation (18) below) he determines a stochastic differential equation representation of the growth in *total* wealth. Using this equation he goes on to determine stochastic integral representations for total wealth (his equation (48); Equation (20) below) and optimal consumption (his equation (50)) at any point in time. From these equations one might expect to be able to proceed analytically to find the expected paths of the state and control variables—due to their tractable functional form.

Our analysis of these results of Richard, however, suggest that such an approach is not possible. We detail our reasoning in Appendix A. The analytic calculation of the expected values of the state and control variables actually results in an expression that is more or less intractable, and expected paths must therefore be numerically evaluated. This is carried out using numerical simulation. Indeed, it is well known

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<sup>24</sup> Ministry of Health and Welfare, Japan.

that Monte Carlo methods can be used for numerical integration.<sup>25</sup> Details of the simulation approach we adopted are given in Appendix B.

## RESULTS

In this section we present and interpret the results generated by the model. It is convenient to start with some simple observations. First, in an entirely *deterministic* world standard preference maximization—in which individuals are assumed to maximize a time-separable lifetime power utility function—generates a level consumption stream. When uncertain lifetimes are introduced this result continues to hold, provided that actuarially fair annuities are available. In the absence of bequests individuals will annuitize completely and there will, of course, be no demand for life insurance. Introducing a bequest motive does not alter the level consumption result, provided that actuarially fair life insurance contracts are available. That is, the introduction of uncertain lifetimes makes no difference to economic behavior, and in particular individual preference for level consumption, provided this uncertainty can be insured against at actuarially fair prices.

When a risky asset is introduced into this framework individuals maximize expected utility through some exposure to the risky asset. That is, in this setting most individuals will prefer *not* to insure completely against investment risk. The result is, in general, that the expected consumption stream will not be level. It does, however, grow at a constant rate. Under the assumption that the rate of time preference is equal to the safe rate of return, the expected consumption path will rise through time.<sup>26</sup> An individual who is extremely risk averse, and invests only in safe assets, will enjoy a certain and level consumption stream—the intuition of the previous paragraph continues to hold.

Overall, in the optimal solution to the financial planning problem (Equation (1)), the proportion of total wealth invested in the risky asset will be constant, a result first derived by Merton (1969). This holds not only through the accumulation phase of the life cycle but through the retirement phase as well. This feature of optimal financial planning has profound implications for the nature of retirement securities that are marketed. Recall that in the present model, human capital is assumed a safe asset. It therefore follows that early in the life cycle, when the value of human capital is high, the proportion of financial wealth exposed to the risky asset is very high. As human capital declines with age more financial wealth is allocated to the safe asset, so as to maintain overall constancy of exposure to the risky asset.

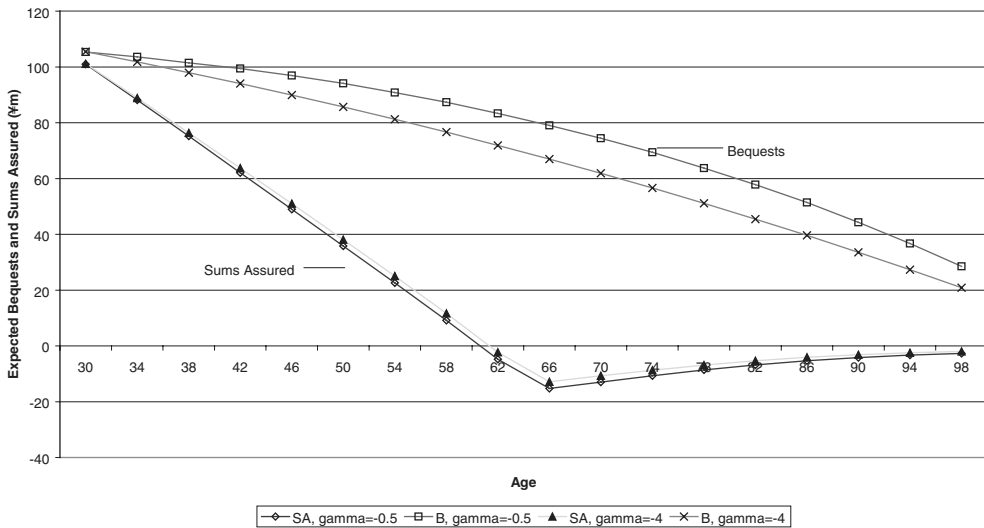
This age-phasing result has been analyzed most thoroughly by Bodie, Merton, and Samuelson (1992). They provide two explanations for this pattern of lifetime financial investment. The first is based upon the Merton (1969) derivation alluded to above. The second is that, typically, households have much more flexibility earlier in the life cycle and are therefore better placed to adapt to a negative shock than they would be later in the life cycle. Only the first of these is relevant here, since there is no

<sup>25</sup> See, for example, Press et al. (1992, pp. 304–328) or Talay (1995).

<sup>26</sup> This is in contrast to traditional life cycle models of consumption with *no* investment uncertainty. In such models, the equivalence of the discount rate and the rate of time preference implies level consumption. See, for example, Deaton (1992) or Büttler (2001).

**FIGURE 1**

Expected Bequests and Sums Assured (B and SA, respectively, in Legend)



choice variable reflecting flexibility in our model. The standard control variable used in formal analysis of this problem is labor supply, which we are assuming fixed.

Bodie, Merton, and Samuelson (1992) also point out that in practice the purchase of owner-occupied housing provides a ready mechanism for debt-financed exposure to a risky asset. An exposure to risky assets equal to, roughly, 1,000 percent of net financial wealth corresponds to a 90 percent mortgage, which is a fairly standard financial position among young home buyers.<sup>27</sup>

Model results depicting expected time paths of variables of interest have been generated through numerical simulation in which the model is solved many times. Here we typically take the average of 10,000 simulations.<sup>28</sup>

#### Life Insurance, Annuities and Bequests

Figure 1 contains the time profiles of both intended bequests and sums assured for households with the two reference risk aversion parameters. The intended bequest at any point is the sum of the sum assured (the life insurance payout at that point of time in the event of death) and accumulated wealth. The central specification of bequests in the model is the amount necessary to purchase a level and certain income stream equal to two-thirds of consumption at that point in time and paying benefits for a period of years equal to 110 less the current age of the life insurance purchaser. Intended bequests thus gradually fall with age.

<sup>27</sup> This is currently not the case in Japan. With declining house prices over the last decade or so, banks have required larger deposits.

<sup>28</sup> With a maximum effective future length of life of 80 years and discrete time periods spanning two weeks, more than 20,000,000 separate solutions are obtained for each model specification.

**FIGURE 2**

Expected Insurance Premia Paid (+ Values) and Annuity Payments Received (– Values)

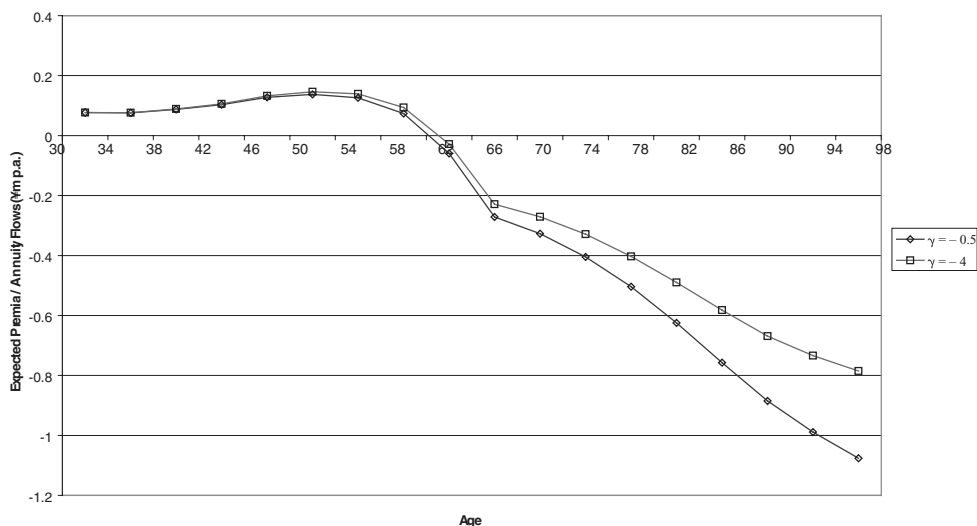


Figure 1 illustrates the symmetry of life insurance and annuities described in the subsection above and neatly exploited by the Richard model. The sum assured becomes negative as the consumer enters his 60s. A negative sum assured represents the capital sum released to the annuity provider and upon which annuity payments are based. After the age of 65, when the individual is forced to retire, the excess wealth gradually diminishes. The consumer is able to gradually run down his wealth, holding enough in reserve to meet his (diminishing) bequest target. In each period he annuitizes wealth not earmarked for either bequests or current consumption.

Figure 2 reports the patterns of life insurance premia and annuity payments. Life insurance premia begin at a low level because the risk of mortality is small at young ages. The mortality effect more than offsets the large required death-contingent benefit. With time, however, increasing mortality forces the premium up, even though the required sum assured is declining. As wealth accumulation approaches the bequest target, the requirement of life insurance diminishes and with it the premium paid.

Analogous to Figure 1, the insurance premium paid transforms into an annuity payout for the individual in his early 60s. At 65 there is a kink in the payout at retirement, but it then increases over time, along with mortality, even though the stock of financial wealth funding the annuity payments gradually diminishes. At the age of 90, on average annuity payouts finance one quarter of all consumption expenditures, with the rest financed from wealth decumulation.

One surprise to arise from our analysis is the suggestion that annuitization would be far more widespread if actuarially fair variable life annuities were available. Table 3 indicates that the contribution of annuity income to consumption would be considerable, with about 11 percent of consumption funded from annuity income at age 70, rising to about 25 percent at age 90. This contrasts sharply with observed voluntary life annuity purchases, which in Japan and in most other countries are negligible.

**TABLE 3**Capital Used for Annuity Purchase Together With Resulting Annuity Flow ( $\gamma$  Varying)

| Age | $\gamma = -0.5$                 |                    | $\gamma = -4$                   |                    |
|-----|---------------------------------|--------------------|---------------------------------|--------------------|
|     | Capital for<br>Annuity Purchase | Annuity<br>Payment | Capital for<br>Annuity Purchase | Annuity<br>Payment |
| 60  | N/A                             | N/A                | N/A                             | N/A                |
| 70  | 12.95                           | 0.33               | 10.72                           | 0.27               |
| 80  | 7.69                            | 0.56               | 6.09                            | 0.44               |
| 90  | 4.16                            | 0.89               | 3.15                            | 0.67               |
| 100 | 2.42                            | 1.12               | 1.75                            | 0.81               |

Note: Figures in yen (millions). Figures for annuity payments are annual rates.

The thinness of voluntary annuity markets has puzzled analysts for many years. Among possible explanations is the presence of social security, which provides a form of mandatory annuitization for retired workers. The importance of social security in modifying personal financial choices has been underestimated by the personal finance industry worldwide. It is to this issue that we turn next.

#### Social Security and Loadings

Our treatment of social security in the model, described in the section “Numerical Specification” above, is equivalent to mandating a safe saving flow. The household compensates for this by saving less privately and by investing a higher proportion of personal financial wealth in risky assets. In the specifications considered here, it is possible for the consumer to offset the impact of social security completely.

The proportion of expected consumption financed by social security varies with the degree of risk aversion of the household. With relatively low risk aversion ( $\gamma = -1/2$ ), social security payments finance about 60 percent of retirement consumption. With high risk aversion ( $\gamma = -4$ ) social security is even more dominant. This is because expected private wealth accumulation is inversely related to the degree of risk aversion.

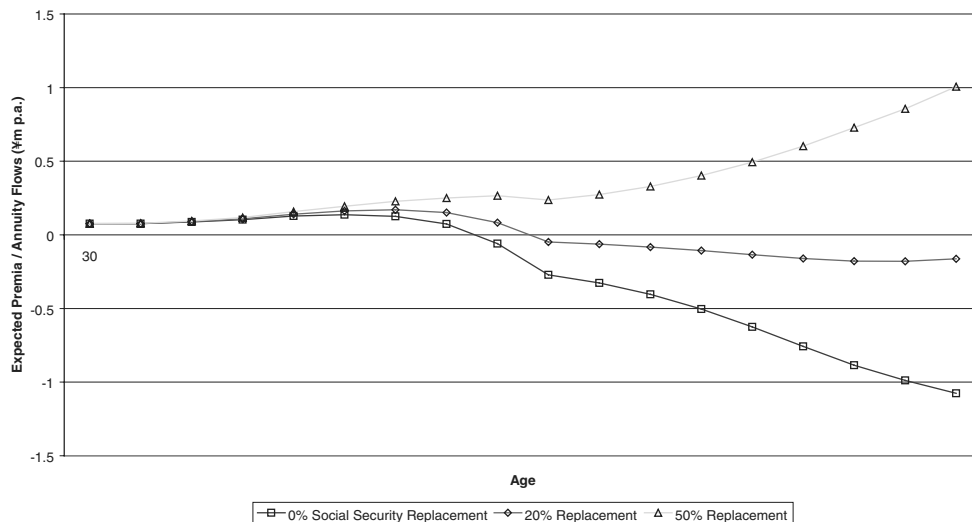
In addition, life insurance and annuity purchases are affected by the presence of social security. Figure 3 shows that as social security becomes more important, the household will buy more life insurance (because social security as specified does not carry a survivor’s benefit) and less annuities, and will postpone annuity purchase. For the case where social security is set at 50 percent of preretirement income life annuities are not purchased at all.

The proportion of wealth invested in risky increases to compensate for the mandated safe investment embodied in social security. For  $\gamma = -1/2$ , the proportion of financial wealth invested in risky assets at age 40 rises from 135 percent in the no social security case to 165 percent for the 50 percent case. At age 50 the increase is more dramatic—from 67 percent to 86 percent.

For a household on average earnings the pattern of private wealth accumulations and insurance holdings revealed here is more likely to approximate the optimal pattern than the estimates reported for the no social security case, since most developed economies have well-developed social security systems offering generous retirement

**FIGURE 3**

Expected Insurance Premium Paid (+ Values) and Annuity Payments (– Values), Social Security Replacement Rates Varying



benefits. An implication of the model is that for such a typical household expected financial accumulations at retirement are nearly 25 percent less in the presence of social security. This suggests a much lower private saving rate through working life in the presence of social security.

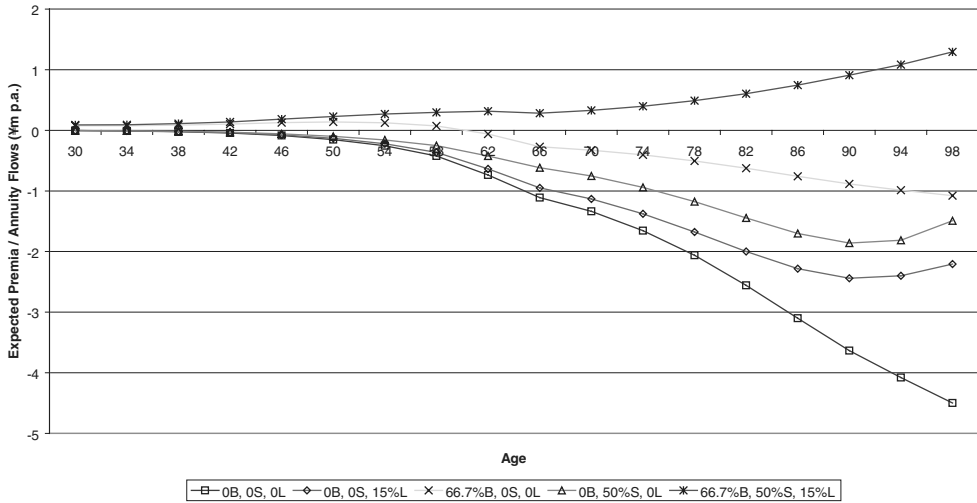
We have also investigated the impact of loadings on expected consumption. As we suggested earlier, the expected consumption stream (with actuarially fair life insurance and annuity markets) will be level or will grow at a constant rate with exposure to investment risk. When life and longevity insurance contracts are not actuarially fair, however, households will tend to self-insure in some degree. This effect is seen in the expected consumption path (contingent on survival) with and without loadings. Without loadings, expected consumption increases seemingly linearly from 3.06 to 3.75 million yen between ages 65 and 100. With a 15 percent and 30 percent loading placed on the absolute value of the actuarially fair mortality premium, the expected consumption path is increasingly concave and rises from 3.04 to 3.51 million yen, and 3.03 to 3.31 million yen, respectively, over the same age range. The 30 percent loading path is sufficiently concave to peak in the consumer's mid-90s and declines thereafter.

## Discussion

The optimizing model we have developed is able to shed new light on the question of the which factors are the most important in determining annuity demand. We are able to manipulate bequest targets, social security, and administrative loadings in the specification of the model and generate the optimal time paths for life insurance premiums and annuity payments under various combinations of these parameters. This allows comparison of optimal behavior toward annuity purchase under alternative specifications to be compared so that the relative importance of alternative explanations for low annuity demand can be weighed.

**FIGURE 4**

Expected Insurance Premia Paid (+Values) and Annuity Payments Received (–Values)—Varying Social Security Replacement, Bequest Motives, and Expense Loadings



Note: Legend gives S for social security, B for bequest annuity, and L for the absolute value of the mortality premium loading.

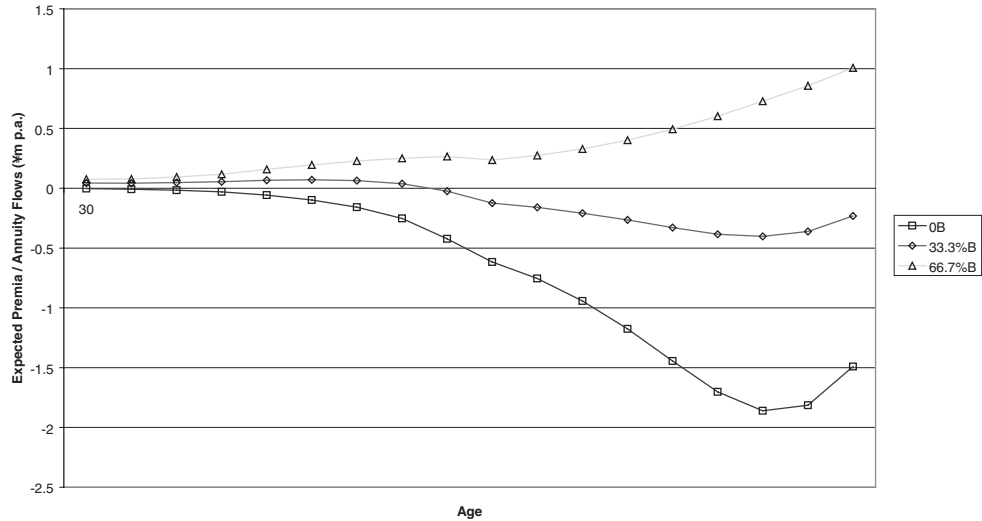
Figure 4 depicts time paths for five such combinations. When all three parameters of interest are set to zero—that is, there is no bequest motive and no social security and the annuity quotes are actuarially fair—there is a gradual increase in annuity purchase throughout life. For the specifications we have chosen the introduction of bequests reduces annuity demand the most. Social security comes next, followed by administrative loadings. When all these factors are present together, annuity demand is zero.

In Figure 5 we explore the impact of bequests further, this time for our benchmark case in which 50 percent social security is present. (Loadings are kept at zero.) Because of the presence of social security individuals reduce their voluntary annuity purchases late in life and begin to draw down their wealth at a rate faster than full annuitization permits. This reduction in wealth leads to diminished demand for annuities (recall that annuity demand is given by  $\min(0, \mu(Z^* - W))$ ). It reaches quite low levels in the presence of bequests guaranteeing a one-third consumption stream to survivors. Annuity purchase is completely eliminated with a bequest motive of a two-thirds annuity purchase.

These results are probably sensitive to the stylized social security plan in the model, which has no survivor benefits. A social security scheme offering generous survivor benefit may well blunt the importance of the bequest specification as a determinant of annuity demand. It would in principle be possible to specify a social security scheme offering such generous survivor benefits that the household response would be increased annuity purchase, rather than increased life insurance purchase.

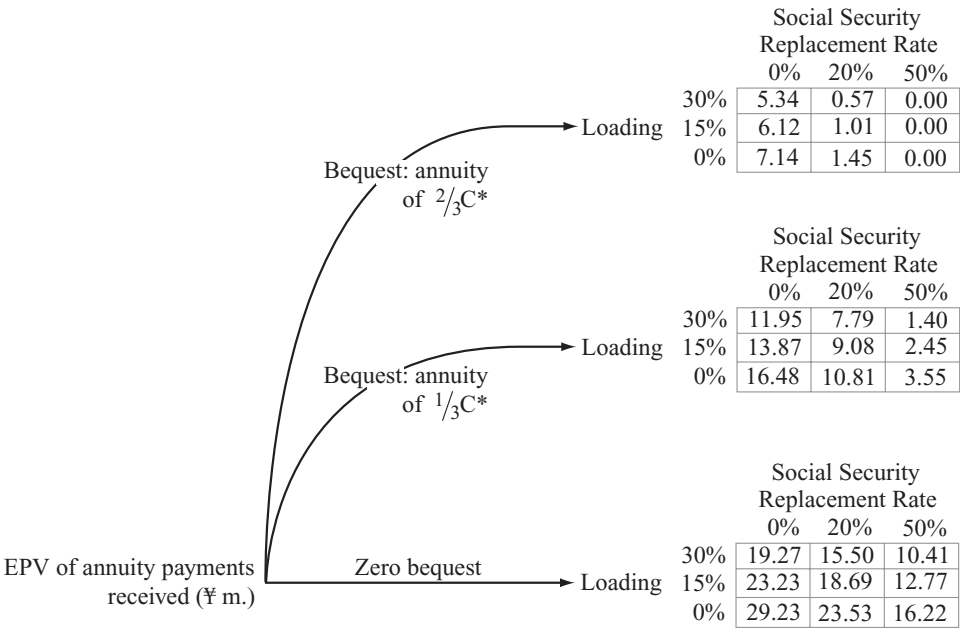
Figure 6 reports the value of annuity payouts received by an investor from age 65, capitalized to age 65. These values reflect the effects already demonstrated in the time profiles. With no bequest, no social security, and no loading, the value of annuity

**FIGURE 5**  
Expected Insurance Premium Paid (+ Values) and Annuity Payments Received (– Values),  
Varying by Bequests (50% Social Security Replacement)



Note: In legend, B represents bequest annuity.

**FIGURE 6**  
Expected Present Value of Future Annuity Payments Received, From Age 65, Expense  
Loading and Bequest Motive Varying



Note: Figures are in yen (millions). Loadings are expressed as a percentage of the absolute value of the actuarially fair mortality premium.

purchase stands at 29.2 million yen. This is an underestimate of the optimal lifetime expected value of annuity purchases under these particular assumptions, however. Under this specification, a household will annuitize throughout working life.

Interestingly, Figure 6 shows the importance of expenses in diminishing annuity flows decreases with stronger bequest motives. With no bequest motive, a 15 percent expense loading (and no expense loading) results in marginally less annuity flows than a 20 percent social security replacement rate (with no social security). With positive bequest motives, this result is reversed, with a 20 percent social security replacement rate (and no expense loading) resulting in clearly less annuity flows than both the 15 percent and the 30 percent expense loading (with no social security).

## CONCLUSION

In this article we have examined how households might optimally allocate their resources between different kinds of assets for investment purposes, between life insurance and longevity insurance and between saving and consumption in a framework in which both investment returns and mortality are uncertain. Households faced a variety of policy and market specifications, including varying social security payouts and a range of administrative loadings on life insurance and annuity purchases. Household preferences were varied across a range of risk aversion and bequest parameters.

We use the model to explore why voluntary annuity markets are so thin, not just in Japan, but globally. There are several possible explanations for this phenomenon. Among them are the existence of a bequest motive, the loadings charged by insurance companies on private annuities, thus rendering this kind of insurance expensive, and the possibility that social security already provides annuity flows that are sufficient to meet demand. Results suggest that the bequest motive is the strongest single deterrent to annuity purchase, followed by social security. In our idealized setting, administrative loadings do not on their own lead to dramatic reductions in annuity flows.

Our modeling has been undertaken using a constant relative risk aversion utility function, which may affect the generality of our results. Also, comparison of our predicted outcomes with reality is difficult due to the lack of good data on life annuitants, both in Japan and around the world. Finally, the major simplifying assumptions in the model as developed thus far revolve around the specification of labor. It is assumed that human capital is a safe asset and that lifetime labor supply is fixed. These are clearly unrealistic assumptions—wages are uncertain, involuntary spells of unemployment occur, and the choice between labor and leisure is very flexible over the life cycle, particularly with respect to secondary labor force participation and choice of retirement age. These are interesting issues for further research.

## APPENDIX A: REVISITING RICHARD (1975)

Richard (1975) chooses to represent total wealth  $W(t) + b(t)$  by the variable  $X(t)$ . In his equation (47) he derives a stochastic differential equation for the dynamics of total wealth:

$$\frac{dX}{X} = \left[ \frac{(\alpha - r)^2}{\delta \sigma^2} + r + \mu(t) - \frac{k(t)}{a(t)^{1/\delta}} \right] dt + \frac{\alpha - r}{\delta \sigma} dq \quad (A1)$$

(where we have made the adjustments necessary to present the actuarially fair case). To determine the corresponding stochastic equation for  $X(t)$  from (A1) we first use Itô's lemma to determine  $d \ln X$ :

$$d \ln X = \left[ \frac{(\alpha - r)^2}{\delta \sigma^2} + r + \mu(t) - \frac{k(t)}{a(t)^{1/\delta}} \right] dt - \frac{1}{2} \left( \frac{\alpha - r}{\delta \sigma} \right)^2 dt + \frac{\alpha - r}{\delta \sigma} dq. \quad (A2)$$

The key to a tractable result here is the ability to cleanly integrate  $k(t)/\hat{a}(t)^{1/\delta}$ . Ideally  $k(t)$  would be the differential of  $\hat{a}(t)^{1/\delta}$ , yielding a nice log integral. In fact this appears to be what Richard found. However correctly applying Leibniz's rule<sup>29</sup> (which Richard correctly did earlier in moving from his equation (25) to equation (30)) yields

$$\frac{d}{dt} \hat{a}(t)^{1/\delta} = -k(t) + \int_t^\omega k(\theta) \frac{\partial}{\partial t} \left\{ \frac{S(\theta)}{S(t)} \exp \left[ \frac{\gamma}{1-\gamma} \left( \frac{(\alpha - r)^2}{2(1-\gamma)\sigma^2} + r \right) (\theta - t) \right] \right\} d\theta$$

and not just  $-k(t)$ .

One concludes that the integration of (A2) leads to integrals with no tractable closed form, and not

$$X(t) = X(0) \left[ \frac{\hat{a}(t)}{\hat{a}(0)} \right]^{1/\delta} \exp \left\{ \left( \nu + \frac{r}{\delta} \right) t + \frac{\alpha - r}{\delta \sigma} \int_0^t dq \right\} \quad (A3)$$

(where we have again presented the result for the case of actuarially fair insurance) as Richard claims in his equation (48). They must then be numerically evaluated, which becomes a messy and complicated business.

To ease these problems we adopt a simulation approach to determine the expected state and control variables of the Richard model. As mentioned above, such an approach implicitly performs the necessary numerical integration, but in a less involved manner.

## APPENDIX B: SIMULATION

We determine the expected values of the state and control variables over an investor's lifetime by simulation. Drawing on a standard result in mathematical finance, it can be shown that a valid approximation to Equation (2) is given by

$$\frac{\Delta Q(t)}{Q(t)} = \alpha \Delta t + \sigma \sqrt{\Delta t} \epsilon_t,$$

where  $\epsilon_t \sim N(0, 1)$  and independent.

We measure time units in fortnights ( $\Delta t = 1/26$  of a year), and analyze a life span from ages 30 at 110.<sup>30</sup> This generates 2,080 periods.

<sup>29</sup> See, for example, Rudin (1976, pp. 236–238).

<sup>30</sup> One hundred and ten is the limiting age,  $\omega$ , of Japanese life table number 18 for males.

Using the approximation to  $dq(t)$  of  $\sqrt{\Delta t}\epsilon_t$  we can also simulate the path of the state variable,  $W(t)$  in Equation (3) over an individual's lifetime. The investor's optimal behavior is given by Equations (12) to (14), which may be evaluated analytically given simulated values of  $W(t)$  and an initial value,  $W(0)$ . For the results we present below, we calculate expected values of the state and control variables at each of the 2,080 discrete time points of the individual's lifetime. These expected values are based on 10,000 simulations of the individual's lifetime. Although we do not report variances and confidence intervals, we have calculated them in our work and this number of simulations is sufficient to produce acceptable results. We have also repeated the simulations using a wide range of parameter values and found the patterns reflected in Figures 1–5 to be robust.

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