

## PRICING AND HEDGING OF DISCRETE DYNAMIC GUARANTEED FUNDS

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### ABSTRACT

We derive a risk-neutral pricing model for discrete dynamic guaranteed funds with geometric Gaussian underlying security price process. We propose a dynamic hedging strategy by adding a gamma factor to the conventional delta. Simulation results demonstrate that, when hedging discretely, the risk-neutral gamma-adjusted-delta strategy outperforms the dynamic delta hedging strategy by reducing the expected hedging error, lowering the hedging error variability, and improving the self-financing possibility. The discrete dynamic delta-only hedging not only causes potential overcharge to clients but also could be costly to the issuers. We show that a naive application of continuous-time hedging formula to a discrete-time hedging setting tends to worsen these possibilities.

### INTRODUCTION

We investigate the hedging performance of the discretely monitored dynamic guaranteed fund using our risk-neutral analytic pricing and hedging formulas that are developed by adding a gamma factor to the conventional delta under geometric Gaussian naked fund prices. We demonstrate that a gamma-adjusted-delta dynamic hedging strategy can curtail both the expected total hedging errors and the hedging uncertainty.

Equity-linked insurance products are typically linked to some reference portfolios, for example, a stock market index. Brennan and Schwartz (1976) value a single-period

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maturity guaranteed equity-linked contract as an insurance contract with an embedded put option. Boyle and Schwartz (1977) determine an optimal investment policy between investing in the reference portfolio and a riskless reserve for a fund issuer to hedge against the investment risk of these guarantees.

In previous investigation of the distribution of the discrete hedging errors of the continuously monitored Black–Scholes European options, Boyle and Emanuel (1980) show that when they use a delta hedging strategy, the mean discrete hedging error is zero. On the other hand, Leland (1985) derives discrete Black–Scholes European option pricing and the terminal-payoff-replication hedging formulas under proportional transaction costs. He shows that the transaction costs increase to infinity when the portfolio rebalancing time interval approaches zero. This result suggests that discrete dynamic hedging is not only practical but also sensible. Various methods are introduced to price and hedge options with transactions costs (e.g., Toft, 1996). Wilmott (1994) shows that the period-to-period variance-minimizing discrete hedging formula adds an adjustment factor to the delta-hedging parameter. The adjustment factor involves a market drift rate to account for the market incompleteness under discrete hedging. In this article, to be consistent with the complete market risk-neutral option valuation theory (Harrison and Kreps, 1979), we introduce the risk-neutral discrete pricing and hedging strategy by adding a gamma factor to the conventional delta.

Although static hedging strategy can be useful for replicating the terminal payoffs for some structured guaranteed products, it is not always easy to find the suitable and tradable options to replicate a long time-to-maturity complicated guarantee. Boyle and Hardy (1997) investigate the costs and benefits for hedging guarantees between using cash reserve strategy based on stochastic simulation of future investment returns and option rebalancing strategy. However, the property of the distribution of the discrete dynamic hedging errors of complicated guarantees such as a dynamic guaranteed fund has yet to be investigated. By applying the discrete probability density function of the random minimum in AitSahlia and Lai (1998), we derive both the analytic pricing and hedging formulas for the discrete dynamic guaranteed funds. These analytic formulas facilitate an accurate study of the hedging error property. We also demonstrate that the conventional delta of our discrete dynamic guaranteed fund valuation model has a call feature, while its gamma has a put feature.

In a series of articles, Gerber and Shiu (1999), Gerber and Pafumi (2000), Imai and Boyle (2001), and Gerber and Shiu (2003) introduce a dynamic guaranteed fund featured by automatic multiperiod reset guarantees. The guarantees are dynamic in the sense that the fund is upgraded through cash injection by the issuer whenever its price falls below a certain threshold during its life. Thus, the fund provides a floor protection for an equity-index linked portfolio but does not require its investors to devise sophisticated early-exercise strategies. Moreover, the replenishment to upgrade the naked fund above the guaranteed level create a leverage on its value that enables the dynamic guaranteed fund investors to benefit from the growth in the fund value in times of good market returns. Hence, it internalizes both the call option characteristics by allowing its investor to participate in an upside market with a floor protection and the put option characteristics by leveraging the downside risk with respect to the guaranteed level to enhance the investor ultimate payoff.

Hedging is usually executed at discrete-time instants, Gerber and Pafumi (2000) derive a closed-form pricing formula for the continuous dynamic guaranteed fund and the rebalancing portfolio strategy between a risky upgraded fund and a riskless asset for the fund issuers. This formulation is inconsistent with the common practice of discrete-time fund monitoring and hedging. Imai and Boyle (2001) investigate the value of the discretely monitored dynamic guaranteed fund using a Monte Carlo simulation approach.

Because of the requirement to replenish the fund value above the guaranteed level, the probability distribution of the guaranteed fund price process is a truncated distribution above the guaranteed level. We show that the expected rate of returns under conventional delta-hedging of this truncated distribution process is likely to be less than the risk-free rate of interest. This differential rate creates a negative expected total hedging error and hence overcharges the fund buyers.

A hedging error of a delta-hedged portfolio is the cost of establishing the risky and risk-free assets for the current hedging portfolio allocation minus the income from liquidating the risky and risk-free assets brought forward by the previous-period portfolio allocation. The time intervals under either the fund monitoring or the fund hedging are equal. However, the frequency of fund monitoring can be higher than that of the hedging. We assume that the naked fund price process follow a risk-neutral Brownian motion. Therefore, its expected rate of return is the risk-free rate of interest.

We explore the behavior of the hedging errors under risk-neutral pricing that we monitor monthly, but we examine discrete delta hedging by rebalancing hedging portfolio at bimonthly reset dates, over a simulated year. We show that an inconsistent model application that prices and hedges a discrete dynamic guaranteed fund by continuous formulas magnifies the overcharges. In addition, because the naked, rather than the upgraded, fund is tradable, our simulation studies show that hedging via buying at the upgraded fund value but selling at a value accompanied by a lagged growth in fund unit costs the discrete delta hedgers. Hence, the benefit of the overcharge does not necessarily accrue to the delta hedgers, but instead becomes an economic loss in the actual implementation of the discrete dynamic hedging process. However, by modifying the conventional delta to a risk-neutral gamma-adjusted-delta hedge ratio, we demonstrate that to a large extent, a fund issuer can eliminate this expected total hedging error, reduce the variability of the hedging errors, and achieve a more precise period-to-period self-financing outcome.

This article is organized as follows. The next section derives the discrete pricing and hedging strategies in an equivalent risk-neutral world of a fund monitored at discrete time instants. In the section "The Guaranteed Investment Fund," we review the theoretical structure of a dynamic guaranteed fund and discuss the pricing and hedging formulations of the discrete guaranteed fund in a geometric Brownian motion framework. The section on "Simulations" presents the simulation results. We show a possible divergence of the prices between the risk-neutral discrete and continuous guaranteed fund, and we demonstrate how the hedging performance of our discrete hedging portfolio outperforms that of Gerber and Pafumi (2000). The last section concludes.

### RISK-NEUTRAL DISCRETE PRICING AND HEDGING STRATEGIES

Similar to Wilmott (1994), we define a hedged-portfolio  $\Pi$  of holding one option  $C$  and selling a number  $\Delta$  of the underlying asset  $S$  as

$$\Pi = C - \Delta S. \quad (1)$$

A discrete-time version of the stochastic differential equation for  $S$  is

$$\delta x = \left( \mu - \frac{\sigma^2}{2} \right) \delta t + \sigma y \sqrt{\delta t}, \quad (2)$$

where  $y$  is a unit variance random variable,  $\mu$  is the market drift rate of the equity process,  $\sigma$  is the volatility of the equity process,  $\delta t$  is the small hedging time interval, and  $\delta x$  is the change of the rate of equity return over the hedging time interval  $\delta t$ . Also,

$$\delta S = S \delta x, \quad (3)$$

where  $\delta S$  represents the change of equity value over the hedging time interval  $\delta t$ .

By Taylor series expansion, the change of the hedged-portfolio return  $\delta \Pi$  is

$$\delta \Pi = \frac{\partial \Pi}{\partial S} \delta S + \frac{\partial \Pi}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 \Pi}{\partial S^2} (\delta S)^2 + o(\delta t^2), \quad (4)$$

where a function  $h(x)$  is of an order  $g(x)$ ,  $o(g(x))$ , if  $\lim_{x \rightarrow 0} |h(x)/g(x)| < \infty$ . We denote the first partial derivatives for  $S$  and  $t$  by  $\partial \Pi / \partial S$  and  $\partial \Pi / \partial t$ , respectively, and the second partial derivative of the portfolio for  $S$  by  $\partial^2 \Pi / \partial S^2$ .

#### The Risk-Neutral Hedging Parameter

We choose the gamma-adjusted delta  $\Delta$  to create on average a risk-neutral stochastic differential equation in Equation (4). It is

$$\Delta = \frac{\partial C}{\partial S} + S \frac{\partial^2 C}{\partial S^2} \sqrt{\delta t} \left( \frac{\sigma^2 [E(y^2) - 1]/2 + (\mu - \sigma^2/2) \sigma E(y) \sqrt{\delta t} + (\mu - \sigma^2/2)^2 \delta t/2}{\sigma E(y) + [(\mu - \sigma^2/2) - r] \sqrt{\delta t}} \right), \quad (5)$$

where  $E(\cdot)$  is an expectation operator based on the real-world probability distribution. We let  $\partial C / \partial S$  and  $\partial^2 C / \partial S^2$  be the conventional delta and conventional gamma, respectively. When the hedging interval  $\delta t$  approaches zero and  $\mu - \sigma^2/2 \neq r$ ,  $\Delta$  will approach the conventional delta.

Using this hedging strategy, Equation (4) becomes

$$\delta \Pi = \left( \frac{\partial C}{\partial S} - \Delta \right) \delta S + \left( \frac{\partial C}{\partial t} \right) \delta t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (\delta S)^2 + o(\delta t^2). \quad (6)$$

By equating the expected change of portfolio return to the interest earned by the portfolio, we obtain

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0, \quad (7)$$

where  $r$  is the interest rate. Thus, we can price the option  $C$  in an equivalent risk-neutral world. Equation (7) due to (5) suggests that by using the martingale mathematical concept (Harrison and Kreps, 1979), we can price an option by its risk-neutral present value of the expected payoff at expiry in an incomplete market. Unlike continuous dynamic hedging, discrete hedging involves hedging errors.

### The Hedging Error

The hedging error  $H$  is equal to

$$H = \delta\Pi - r\Pi\delta t. \quad (8)$$

*The Risk-Neutral Hedging Error.* From the risk-neutral gamma-adjusted delta in Equation (5), we can characterize the distribution of the period-to-period hedging errors by

$$\begin{aligned} H = & \left( \frac{\partial C}{\partial S} - \Delta \right) \delta t \left[ \left( \mu - \frac{\sigma^2}{2} \right) - rS \right] - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \delta t \left[ \sigma^2 - \left( \mu - \frac{\sigma^2}{2} \right)^2 \delta t \right] \\ & + \left[ \left( \frac{\partial C}{\partial S} - \Delta \right) + \frac{\partial^2 C}{\partial S^2} S^2 \left( \mu - \frac{\sigma^2}{2} \right) \delta t \right] \sigma \sqrt{\delta t} y + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \delta t y^2 + o(\delta t^2). \end{aligned} \quad (9)$$

As the expected change of return of the hedging portfolio in Equation (9) is equal to the interest earned from the portfolio differed by  $o(\delta t^2)$ , the period-by-period hedging strategy is on average self-financing. In other words,

$$E(H) \cong 0. \quad (10)$$

We also define the variance of the hedging errors as  $\text{Var}(H)$  where  $\text{Var}(\cdot)$  is a variance operator.

*Inconsistent Pricing and Hedging Under Conventional Delta.* If we choose the conventional delta  $\Delta = \frac{\partial C}{\partial S}$  instead, then the risk-neutral pricing based on a stochastic differential Equation (7) will result in mispricing. Alternatively, given that the risk-neutral pricing is correct, the discrete hedging under conventional delta may not have zero expected hedging errors. We can deduce the distribution of the hedging error under this mispriced option  $C$  to be

$$H = \frac{1}{2} S^2 \frac{\partial^2 C}{\partial S^2} \left[ \left( \mu - \frac{\sigma^2}{2} \right)^2 (\delta t)^2 + \sigma^2 (y^2 - 1) \delta t + 2 \left( \mu - \frac{\sigma^2}{2} \right) \sigma y (\delta t)^{3/2} \right] + o(\delta t^2). \quad (11)$$

Hence,

$$E(H) \cong \frac{1}{2} S^2 \frac{\partial^2 C}{\partial S^2} \left[ \left( \mu - \frac{\sigma^2}{2} \right)^2 (\delta t)^2 + \sigma^2 (E(y^2) - 1) \delta t + 2 \left( \mu - \frac{\sigma^2}{2} \right) \sigma E(y) (\delta t)^{3/2} \right]. \quad (12)$$

For dynamic guarantees, we show that  $\partial^2 C / \partial S^2 < 0$ , and for the realistic parameters of  $\mu$ ,  $\sigma$ , and the guaranteed level,  $E(H) < 0$  in Equation (12) for large enough  $\delta t$ . The expected hedging error  $E(H)$  then becomes a differential rate. Equation (7) suggests that the risk-neutral guaranteed fund price pays and earns at the risk-free rate  $r$ . However, discrete hedging would earn an expected portfolio rate  $E[\delta \Pi / (\Pi \delta t)]$  less than this risk-free rate because of the risk from guarantee protection. Hence, the fund-issuers theoretically overcharge the risk-neutral-priced dynamic guaranteed fund if they hedge by the conventional delta. The variance of the hedging errors is

$$\text{Var}(H) = \sigma^2 S^4 \left( \frac{\partial^2 C}{\partial S^2} \right)^2 (\delta t)^2 E \left[ \frac{1}{2} \sigma (y^2 - E(y^2)) + \sqrt{\delta t} \left( \mu - \frac{\sigma^2}{2} \right) (y - E(y)) \right]^2. \quad (13)$$

## THE GUARANTEED INVESTMENT FUND

### The Dynamic Guaranteed Fund

Gerber and Pafumi (2000) investigate an investment fund with dynamic guarantees, in which they repeatedly inject fund units over the life of the fund to maintain the fund amount above its agreed-upon guaranteed level. We model the naked fund without guarantees to follow a geometric Brownian motion. Let  $F(t)$  denote the price of a naked fund at time  $t$ . Then, its price is represented by

$$F(t) = F(0) e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma W(t)}, \quad t \geq 0, \quad (14)$$

where  $F(0)$  denotes the naked fund price at time 0,  $\mu$  is the constant drift rate of the geometric Brownian motion,  $\sigma$  is the constant volatility of the geometric Brownian motion, and  $W(t)$  is a standard Wiener process. The risk-neutral process under constant risk-free interest rate  $r$  is

$$d \ln F(t) = \mu^* dt + \sigma d \tilde{W}(t), \quad t \geq 0, \quad (15)$$

where  $\mu^* = r - \sigma^2/2$ , and  $\tilde{W}(t)$  is a standard Wiener process under the risk-neutral measure.

To investigate the continuous dynamic property of the guaranteed fund, Gerber and Pafumi (2000) replace the risk-neutral geometric Brownian process given in Equation (15) by an upgraded fund price  $\tilde{F}(t)$ , which receives an injection of fund units to replenish the fund price to the constant level  $K$  whenever  $\tilde{F}(t)$  falls below  $K$ . We calculate the upgraded fund price  $\tilde{F}(t)$  as follows:

$$\tilde{F}(t) = F(t)n(t) \quad (16)$$

and

$$n(t) = \max \left\{ 1, \max_{t \geq s \geq 0} \frac{K}{F(s)} \right\},$$

where  $K$  denotes the constant level (or barrier) of the guaranteed investment fund, and  $n(t)$  denotes the total number of fund units at time  $t$  per unit of the fund invested at time 0.

On the other hand, under discrete sampling, we express the dynamics of the upgraded fund price  $\tilde{F}(t)$  as

$$\tilde{F}(t) = \tilde{F}(t - \delta t) e^{(\mu - \frac{1}{2}\sigma^2)\delta t + \sigma y \sqrt{\delta t}}, \quad (17)$$

where  $y$  is a unit variance random variable. We can define

$$p(y) = \begin{cases} 0 & \text{for } y < y^c, \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} & \text{for } y \geq y^c, \end{cases} \quad (18)$$

where  $y^c = [\ln(\frac{K}{\tilde{F}(t-\delta t)}) - (\mu - \frac{1}{2}\sigma^2)\delta t] / (\sigma\sqrt{\delta t})$ . Because  $K \leq \tilde{F}(t - \delta t)$ ,  $y^c < 0$  when  $\mu > \sigma^2/2$ . The probability density function of  $y$  is  $f(y) = p(y) / \int_{-\infty}^{\infty} p(y) dy$ . As a result,  $E(y) > 0$ . It turns out that for realistic parameters of  $K$ ,  $\mu$ , and  $\sigma$ , simulation studies suggest that when  $\delta t$  is sufficiently large,  $E(H) < 0$  in Equation (12).

We note that for our simulation studies, we use  $E(y^2) = 1$  and  $E(y) = 0$  when  $t = 0$ , that is, we assume a normal prior distribution for the first observation. When the guaranteed level grows at a rate  $\gamma$ ,  $K$  in Equations (16) and (18) is replaced by  $Ke^{\gamma t}$ .

Let  $A(F(0), T)$  denote the discounted expected value of the guaranteed investment fund given in Equation (19) at time 0, where  $T$  is the maturity date of the guaranteed investment fund. A risk-neutral expectation of the discounted price of the continuous guaranteed fund at maturity time  $T$  leads to equation (2.2) in Gerber and Pafumi (2000), which is

$$A(F(0), T) = e^{-rT} E^*[\tilde{F}(T)], \quad (19)$$

where  $E^*[\cdot]$  represents the expectation operator under the risk-neutral measure given in Equation (15). The value of the fund protection at time  $t$  ( $T \geq t \geq 0$ ) is

$$V(t) = A(\tilde{F}(t), T) - \tilde{F}(t) \quad (20)$$

with  $V(T) = 0$ .

Gerber and Pafumi (2000) derive a closed-form formula as a result of Equation (19) as follows:

$$\begin{aligned}
A(F(0), T) = & Ke^{-rT} \left( 1 - \frac{\sigma^2}{2r} \right) \Phi \left( \frac{-\ln(F(0)/K) - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) \\
& + \frac{K\sigma^2}{2r} \left( \frac{K}{F(0)} \right)^{\frac{2r}{\sigma^2}} \Phi \left( \frac{-\ln(F(0)/K) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) \\
& + F(0) \Phi \left( \frac{\ln(F(0)/K) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right), \tag{21}
\end{aligned}$$

where  $\Phi(\cdot)$  represents the cumulative standard normal distribution function. The formula for a continuous dynamic guaranteed fund with a guaranteed level growing exponentially at a constant rate  $\gamma$  is the same as Equation (19) but the interest rate  $r$  is replaced by  $r - \gamma$ .

### Dynamic Guarantees Under Sampling

Although continuously sampled closed-form formulas offer exact and efficient valuation of the options, Broadie, Glasserman, and Kou (1999) and Heynen and Kat (1995) illustrate the possible mispricing if the closed-form formulas are mistakenly used to price an option that is actually monitored at fixed discrete dates. To formulate this discrete feature, we develop analytic pricing and hedging formulas of the discrete dynamic guaranteed fund. We base these formulas on the hindsight option pricing analysis in AitSahlia and Lai (1998).

*The Discrete Guaranteed Fund Pricing Formula.* We monitor the guaranteed fund at discrete time points  $n\delta t$  for  $n = 0, 1, \dots, m$ , where the consecutive time points are separated by equal time interval  $\delta t = T/m$ , and  $m$  denotes the total number of monitorings. The random price of the naked fund at time  $n\delta t$  is  $F(0)e^{U_n}$ ;  $U_n$  is a random value under stationary Gaussian process at time  $n\delta t$  for  $n = 0, 1, \dots, m$ ; and  $U_n \sim \phi(\mu_n, \sigma_n^2)$  where  $\mu_n = \mu^*n\delta t$ ,  $\sigma_n = \sigma\sqrt{n\delta t}$ , and  $\phi(\cdot)$  is a univariate normal probability density function. Because  $F(0)e^{U_0} = F(0)$ , it is clear that  $U_0 = 0$ . We denote the minimum value of  $U_n$  over the time interval  $[0, T]$  by  $M_m$ , that is,

$$M_m = \min\{U_n : m \geq n \geq 0\}.$$

Imai and Boyle (2001) highlight an alternative representation of the terminal price of the upgraded fund in their Equation (9) as follows:

$$\tilde{F}(T) = F(T) \max(1, \tilde{K}e^{-M_m}),$$

where  $\tilde{K} = \frac{K}{F(0)}$  for  $\gamma = 0$ . Therefore, the value of the discrete guaranteed fund at time 0,  $\tilde{A}(F(0), T)$ , is

$$\tilde{A}(F(0), T) = e^{-rT} E^*[F(T) \max(1, \tilde{K}e^{-M_m})].$$

The value of the discrete dynamic guaranteed fund protection at time  $t$  is

$$\tilde{V}(t) = \tilde{A}(\tilde{F}(t), T - t) - \tilde{F}(t)$$

with  $\tilde{V}(T) = 0$ .

By applying the discrete probability density function of the random minimum in AitSahlia and Lai (1998), we recast the expected value  $E^*[\tilde{F}(T)]$  into a series of multivariate probability values. We derive the pricing formula for the value of the discrete dynamic guaranteed fund  $\tilde{A}(F(0), T)$  as follows.

**Proposition 1:** *The pricing formula of the discrete dynamic guaranteed fund  $\tilde{A}(F(0), T)$  over  $[0, T]$  monitored at  $m$  time points of equal time subintervals in  $[0, T]$  is expressed as<sup>1</sup>*

$$\tilde{A}(F(0), T) = F(0) \left\{ \max(0, \tilde{K} - 1) \beta_m + \sum_{v=1}^m \beta_{m-v} [e^{-rv\delta t} \tilde{K} B_v^2 - B_v^1] + 1 \right\}, \quad (22)$$

where

$$\begin{aligned} \beta_0 &= 1, \\ \beta_v &= \int_{-\infty}^{d_1(1)} \cdots \int_{-\infty}^{d_1(v)} C_v e^{-\frac{1}{2} \mathbf{x}_v' Z_v^{-1} \mathbf{x}_v} d\mathbf{x}_v, \\ B_v^1 &= \int_{-\infty}^{-d_1(1)} \cdots \int_{-\infty}^{-d_1(v-1)} \int_{-\infty}^{-d_1^*(v)} C_v e^{-\frac{1}{2} \mathbf{x}_v' Z_v^{-1} \mathbf{x}_v} d\mathbf{x}_v, \\ B_v^2 &= \int_{-\infty}^{-d_2(1)} \cdots \int_{-\infty}^{-d_2(v-1)} \int_{-\infty}^{-d_2^*(v)} C_v e^{-\frac{1}{2} \mathbf{x}_v' Z_v^{-1} \mathbf{x}_v} d\mathbf{x}_v, \end{aligned}$$

$\mathbf{x}'_v = (x_1, \dots, x_v)$ ,  $m = \frac{T}{\delta t}$ ,  $C_v = (2\pi)^{-\frac{v}{2}} |Z_v|^{-\frac{1}{2}}$ ,  $Z_v = [\rho_{ij}]_{i,j=1,v}$ , and  $\rho_{i,j} = \frac{i}{\sqrt{ij}}$  for  $i, j = 1, \dots, v$  and  $j \geq i$ ,  $Z_v^{-1}$  is the inverse of  $Z_v$ ,  $d_2(i) = \frac{\mu_i}{\sigma_i}$ ,  $d_1(i) = d_2(i) + \sigma_i$ , for  $i = 1, \dots, v$ ,  $d_2^*(v) = \frac{-v^* + \mu_v}{\sigma_v}$ ,  $d_1^*(v) = d_2^*(v) + \sigma_v$ ,  $v^* = \min[0, \ln(\tilde{K})]$ , and  $\tilde{K} = \frac{K}{F(0)}$ .

**Proof:** See the first section of the Appendix.

The term  $F(0) \max(0, \tilde{K} - 1) \beta_m$  represents the discounted expected protection value when  $F(0)$  is the minimum value over the life  $[0, T]$  of the fund. The value  $\beta_m$  is the probability that  $F(0)$  is the minimum value. The discounted expected protection value is derived on the assumption that the terminal value  $F(T)$  is not less than the minimum value  $F(0)$ . However, this expected protection value occurs only when it needs to pay  $\tilde{K} - 1$  additional units for the fund protection over the life of the fund. Hence, the fund protection leads to the multiplication factor  $\max(0, \tilde{K} - 1)$ .

The term  $F(0) \sum_{v=1}^m \beta_{m-v} [e^{-rv\delta t} \tilde{K} B_v^2 - B_v^1]$  represents the discounted expected protection value when the minimum value occurs at the first and continues to the  $m$ th monitoring fixed time points. The value  $\sum_{v=1}^{m-1} \beta_{m-v} B_v^2$  denotes the sum of the joint probabilities for the occurrences of the minimum value  $M_m$  in the time interval  $[T/m, (m-1)T/m]$ . The terminal payoff  $\max(F(T)\tilde{K}e^{-M_m} - F(T), 0)$  suggests that additional fund units will be injected for fund protection only when  $M_m < \ln(\tilde{K})$ . In addition, the minimum value  $M_m$  has to be less than the initial value  $U_0 = 0$ . Consequently,  $M_m < \min[0, \ln(\tilde{K})]$ .

<sup>1</sup> Similar to the discussion in section 4 of Gerber and Pafumi (2000), the formula for the discrete dynamic guaranteed fund with guaranteed level growing exponentially at constant rate  $\gamma$  can be deduced from replacing  $r$  in Equation (22) by  $r - \gamma$ .

The term  $F(0)[e^{-rT}\tilde{K}B_m^2 - B_m^1]$  represents the discounted expected protection value when the minimum value occurs at the  $m$ th monitoring time point. The probability that the minimum value occurs at the terminal date is  $B_m^2$ . The terminal payoff under such an occurrence is  $\max(F(0)\tilde{K} - F(T), 0)$ . As  $F(0)\tilde{K}$  is not random, the payoff  $\max(F(0)\tilde{K} - F(T), 0)$  is that of a European put option with strike price  $F(0)\tilde{K}$ . This analysis suggests that the dynamic guaranteed fund would guarantee at least the strike value  $F(0)\tilde{K}$ , even if the minimum value  $F(0)e^{M_m} = F(T)$  occurs at the terminal date  $T$ .

In sum, a dynamic guaranteed fund protects its investor against losses at a strike level  $F(0)\tilde{K}$ , but it leverages the fund gain through the contractually required injection of additional fund units based on the minimum value attained over the life of the dynamic guaranteed fund from time 0 to time  $(m - 1)T/m$ . Thus, an investor would not be required to exercise early his fund to capture the gain, unlike in the American-style one-time protection guaranteed fund structure.

*A Modified Numerical Procedure.* Our analytic formula in Equation (22) is an exact valuation formulation for discrete dynamic guaranteed funds. However, its degree of accuracy still depends on the choice of an appropriate procedure for delivering accurate probability values. Tse, Li, and Ng (2001) introduce a numerical procedure to compute multivariate Gaussian probability to high precision. To improve its speed, we modify the numerical procedure in Tse, Li, and Ng for our discrete dynamic guaranteed fund calculation. We impose a fixed integral cutoff limit directly on the standardized multivariate probability distribution given in Equation (22).

An efficient computational procedure is necessary for two reasons. First, our discrete dynamic guaranteed fund formula in Equation (22) requires the consideration of the occurrence of minimum naked fund price at every monitoring date, which could lead to a long series of expected value calculations. Second, a fund by its nature is usually of longer maturity than an option. This characteristic can proportionally increase the number of monitorings. Therefore, an efficient computational technique is necessary to cut down on computational time while retaining an acceptable degree of accuracy.

For multivariate probability computation based on Brownian motion, Tse, Li, and Ng (2001) demonstrate that its order of computational complexity is  $(p + (m - 1)p^2)$  where  $p$  is the number of computational points per integral of the Gaussian quadrature method. However,  $p$  grows nonlinearly in value with the growth in the length of the integration range between the lower and upper integral cutoff limits, which lengthens as the number of monitorings  $m$  increases. Because of fixed cutoff-limit range, the number of computational points per layer,  $p$ , of our modified procedure grows roughly linearly. So, this enables the calculation of a fairly accurate probability or expected payoff value of frequent monitoring in a much shorter time.

*Error Analysis of the Modified Numerical Procedure.* As compared with Tse, Li, and Ng's (2001) procedure, our procedure trades error control for efficiency. We identify the numerical integrity of our procedure by the following proposition.

**Proposition 2:** *For a continuous function*

$$G(x_1, \dots, x_m) = G_1(x_1)G_2(x_1, x_2) \cdots G_m(x_{m-1}, x_m),$$

the absolute error between its  $m$ -variate integral  $G(x_1, \dots, x_m)$  and its  $m$ -variate numerical sum is bounded below by  $m\varepsilon$ , that is,

$$m\varepsilon \geq \left| \int_{I_1} \cdots \int_{I_m} G(x_1, \dots, x_m) d\mathbf{x}_m - \sum_{\mathbf{x}_m} G(\mathbf{x}_m) \right|,$$

where

$$\varepsilon = \left| \int_{I_i} G_i(x_{i-1}, x_i) dx_i - \sum_{x_i} G_i(x_{i-1}, x_i) \right|,$$

and is a constant,  $|\cdot|$  is an absolute value operator,  $I_i$  is the integral domain for the random variable  $x_i$ ,  $G_i$  is a recursive functional of the random variables  $x_{i-1}$  and  $x_i$ , and  $G$  is a continuous functional of the random variables  $x_m = (x_1, \dots, x_m)$ .

**Proof:** See the second section of the Appendix.

Proposition 2 is useful for error estimate only if  $\varepsilon = |\int_{I_i} G_i(x_{i-1}, x_i) dx_i - \sum_{x_i} G_i(x_{i-1}, x_i)|$ . This implies that the integration limit and the numerical integration error for the underlying function over the integration range per layer should be the same. Under this fixed limit, Proposition 2 suggests that the overall numerical errors including cutoff and numerical integration errors will at most grow at a linear rate as  $m$  increases.

The Tse, Li, and Ng (2001) procedure with fixed cutoff limit provides a reasonable environment for Proposition 2 to apply because it not only has a fixed cutoff limit but also allows us to control the numerical integration errors by the computational point estimation formula given in equation (12) in Tse, Li, and Ng. To identify the efficiency and the degree of accuracy of our results, we calculate individual probability values benchmarked by the counterpart values using the limit-adjusting procedure given in Tse, Li, and Ng.

In Table 1, the numerical figures under columns  $TLN$  are computed by the variable cutoff-limit numerical procedure in Tse, Li, and Ng, whereas the numerical figures under the columns  $TLN_f$  are computed by the fixed cutoff-limit modified procedure of Tse, Li, and Ng. The computational errors of the probability values under columns  $TLN$  are controlled to be accurate to 10 decimal places. By choosing the lower cutoff limit at  $-6.75$ , the Tse, Li, and Ng probability values under fixed limit are accurate to nine decimal places with 52 monitorings. A comparison of the corresponding figures under columns  $TLN$  and  $TLN_f$  shows that the computational errors grow roughly at a linear rate in accordance with the growth in the number of monitorings  $m$ . Thus, the probability values with 260 monitorings should be accurate to at least seven decimal places.

### The Synthetic Hedging Strategy

To protect  $A(F(0), T)$  at level  $K$  for a duration  $[0, T]$ , Gerber and Pafumi (2000) introduce the idea of replicating portfolio as a hedging strategy to replicate the terminal value of the dynamic guaranteed fund which requires the issuer to continuously rebalance

**TABLE 1**  
A Comparison of Probability Values<sup>a</sup>

| $m^b$ | -1.5 <sup>c</sup>                      |                           | 0                          |                           | 1.5                        |                           |
|-------|----------------------------------------|---------------------------|----------------------------|---------------------------|----------------------------|---------------------------|
|       | $TLN^d$                                | $TLN_f^e$                 | $TLN$                      | $TLN_f$                   | $TLN$                      | $TLN_f$                   |
| 1     | 0.066807201219<br>(0.032) <sup>f</sup> | 0.066807201261<br>(0.000) | 0.499999999950<br>(0.032)  | 0.499999999993<br>(0.00)  | 0.933192798681<br>(0.031)  | 0.933192798724<br>(0.000) |
| 2     | 0.028590881592<br>(0.032)              | 0.028590881577<br>(0.000) | 0.375000000000<br>(0.031)  | 0.374999999985<br>(0.000) | 0.894976479054<br>(0.016)  | 0.894976479040<br>(0.000) |
| 7     | 0.005738568673<br>(0.083)              | 0.005738568628<br>(0.063) | 0.209472656250<br>(0.078)  | 0.209472656202<br>(0.063) | 0.809452586866<br>(0.093)  | 0.809452586817<br>(0.063) |
| 12    | 0.002836380302<br>(0.344)              | 0.002836380241<br>(0.126) | 0.161180257797<br>(0.343)  | 0.161180257724<br>(0.160) | 0.770331628955<br>(0.343)  | 0.770331628881<br>(0.155) |
| 36    | 0.000666743355<br>(11.162)             | 0.000666743274<br>(1.478) | 0.093705675297<br>(11.145) | 0.093705675153<br>(1.163) | 0.692132063952<br>(11.121) | 0.692132063798<br>(1.289) |
| 52    | 0.000409848596<br>(46.475)             | 0.000409848515<br>(2.514) | 0.078051172374<br>(46.475) | 0.078051172203<br>(2.577) | 0.667031883073<br>(59.505) | 0.667031882885<br>(2.646) |

<sup>a</sup>The probability values are calculated assuming that the underlying stochastic processes follow Brownian motion.

<sup>b</sup>The values  $m$  stand for the number of monitorings.

<sup>c</sup>We use the same upper limit values for all the monitoring dates.

<sup>d</sup>The values under columns  $TLN$  are computed based on the computational procedure in Tse, Ng, and Li (2001). We set the cutoff error tolerance at  $10^{-10}$  and the numerical integration error tolerance at  $10^{-13}$  for our calculation using this procedure.

<sup>e</sup>The probability values calculated under column  $TLN_f$  are based on the computational procedure in Tse, Ng, and Li (2001) with fixed cutoff limit. We set the cutoff at  $-6.75$  and the numerical integration error tolerance at  $10^{-13}$  for our calculation using this procedure.

<sup>f</sup>The values in parentheses underneath their probability values are the computational time in seconds for computing these probability values using MATLAB version 6.5 software and a 1.6 GH Pentium IV personal computer.

his portfolio between the risky asset  $\tilde{F}(t)$  and a riskless asset. The strategy calls for an allocation at time  $t$  the amount

$$\tilde{F}(t)\Delta(\tilde{F}(t), T - t)$$

to the risky asset, and the amount

$$A(\tilde{F}(t), T - t) - \tilde{F}(t)\Delta(\tilde{F}(t), T - t)$$

to the riskless asset, where  $\Delta(\cdot)$  denotes the delta and is equal to the first partial derivative of  $A(\tilde{F}(t), T - t)$  with respect to  $\tilde{F}(t)$ .<sup>2</sup>

Following the continuous sampling formulation of Gerber and Pafumi (2000), we deduce from our synthetic portfolio rebalancing strategy under discrete sampling that at time  $t$  allocates an amount

$$\tilde{F}(t)\tilde{\Delta}(\tilde{F}(t), T - t)$$

to the risky asset  $\tilde{F}(t)$ , and an amount

$$\tilde{A}(\tilde{F}(t), T - t) - \tilde{F}(t)\tilde{\Delta}(\tilde{F}(t), T - t)$$

to the riskless asset. The discrete delta is  $\tilde{\Delta} = \tilde{A}_f(\tilde{F}(t), T - t) + \theta \tilde{A}_{f^2}\tilde{F}(t)$  where  $\theta = 0$  or  $\theta = \sqrt{\delta t}(\frac{\sigma^2[E(y^2) - 1]/2 + (\mu - \sigma^2/2)\sigma E(y)\sqrt{\delta t} + (\mu - \sigma^2/2)^2\delta t/2}{\sigma E(y) + [(\mu - \sigma^2/2) - r]\sqrt{\delta t}})$ , respectively for an unadjusted and a gamma-adjusted-delta (calculation shown below) hedging portfolio. We derive the analytic delta  $\tilde{A}_f(\tilde{F}(t), T - t)$  as follows:

$$\tilde{A}_f(\tilde{F}(t), T - t) = 1 - \sum_{v=1}^{m'} \beta_{m'-v} B_v^1 \quad \text{for } 1 \geq \tilde{K}, \quad (23)$$

where

$$\begin{aligned} \beta_0 &= 1, \\ \beta_v &= \int_{-\infty}^{d_1(1)} \cdots \int_{-\infty}^{d_1(v)} C_v e^{-\frac{1}{2} \mathbf{x}'_v Z_v^{-1} \mathbf{x}_v} d\mathbf{x}_v, \\ B_v^1 &= \int_{-\infty}^{-d_1(1)} \cdots \int_{-\infty}^{-d_1(v-1)} \int_{-\infty}^{-d_1^*(v)} C_v e^{-\frac{1}{2} \mathbf{x}'_v Z_v^{-1} \mathbf{x}_v} d\mathbf{x}_v, \end{aligned}$$

$\mathbf{x}'_v = (x_1, \dots, x_v)$ ,  $m' = \frac{T-t}{\delta t}$ ,  $C_v = (2\pi)^{-\frac{v}{2}} |Z_v|^{-\frac{1}{2}}$ ,  $Z_v = [\rho_{ij}]_{i,j=1,v}$ , and  $\rho_{i,j} = \frac{i}{\sqrt{ij}}$  for  $i, j = 1, \dots, v$ , and  $j \geq i$ ,  $Z_v^{-1}$  is the inverse of  $Z_v$ ,  $d_2(i) = \frac{\mu_i}{\sigma_i}$ ,  $d_1(i) = d_2(i) + \sigma_i$ , for  $i = 1, \dots, v$ ,  $d_2^*(v) = \frac{-v^* + \mu_v}{\sigma_v}$ ,  $d_1^*(v) = d_2^*(v) + \sigma_v$ ,  $v^* = \ln(\tilde{K})$ , and  $\tilde{K} = \frac{K}{\tilde{F}(t)}$ .

<sup>2</sup> For synthetic investment fund protection of the continuously sampled dynamic guaranteed fund, see equations (5.6) to (5.7) in Gerber and Pafumi (2000).

Based on the analytic delta formula in Equation (23), we can also calculate the analytic gamma formula, which is the second partial derivatives  $\tilde{A}_{f2}(\cdot)$  of  $\tilde{A}(\tilde{F}(t), T - t)$  with respect to  $\tilde{F}(t)$ . The gamma formula is

$$\tilde{A}_{f2}(\tilde{F}(t), T - t) = - \sum_{v=1}^{m'} \beta_{m'-v} b_v^1 \quad \text{for } 1 \geq \tilde{K}, \quad (24)$$

where  $\mathbf{x}_v^{*'} = (x_1, \dots, x_{v-1}, -d_1^*(v))$ , and

$$\begin{aligned} \beta_0 &= 1, \\ \beta_v &= \int_{-\infty}^{d_1(1)} \cdots \int_{-\infty}^{d_1(v)} C_v e^{-\frac{1}{2} \mathbf{x}_v^{*'} Z_v^{-1} \mathbf{x}_v} d\mathbf{x}_v, \\ b_v^1 &= \frac{1}{\sigma_v \tilde{F}(t)} \int_{-\infty}^{-d_1(1)} \cdots \int_{-\infty}^{-d_1(v-1)} C_v e^{-\frac{1}{2} \mathbf{x}_v^{*'} Z_v^{-1} \mathbf{x}_v} d\mathbf{x}_{v-1}. \end{aligned}$$

In Figure 1(a), we plot the values of the discrete dynamic guaranteed fund with the guaranteed level  $K$  set at 100. The value increases with the increase in the upgraded fund price  $\tilde{F}(t)$ . Figure 1(b) shows that the analytic delta value is between zero and one. Hence, it is a delta of a call feature. In Figure 1(c), the analytic gamma value is negative, suggesting that it is a gamma of a put feature.

As discussed above, we require a delta-hedging portfolio consisting of risky and riskless assets to replicate the terminal value of the fund. However, the portfolio allocation strategy between the risky and riskless assets need not be a self-financing strategy. Under discrete dynamic hedging, this period's hedging error comes from the difference between the income brought forward from liquidating the preceding period's hedging portfolio and the cost of constructing the current period's hedging portfolio. The total hedging error is the sum of individual period's hedging errors. To simplify notations, we denote  $n(t)$ ,  $F(t)$ ,  $\tilde{F}(t)$ ,  $\tilde{A}(\tilde{F}(t), T - t)$ , and  $\tilde{\Delta}(\tilde{F}(t), T - t)$ , respectively, by  $n_t$ ,  $F_t$ ,  $\tilde{F}_t$ ,  $\tilde{c}_t$ , and  $\Delta_t$ . The  $i$ th period's hedging error of the delta-hedged portfolio at the  $(i-1)$ th period is

$$H_{i\delta t}(\theta) = \tilde{c}_{i\delta t} - \Delta_{(i-1)\delta t} \tilde{F}_{i\delta t} - e^{r\delta t} [\tilde{c}_{(i-1)\delta t} - \Delta_{(i-1)\delta t} \tilde{F}_{(i-1)\delta t}]. \quad (25)$$

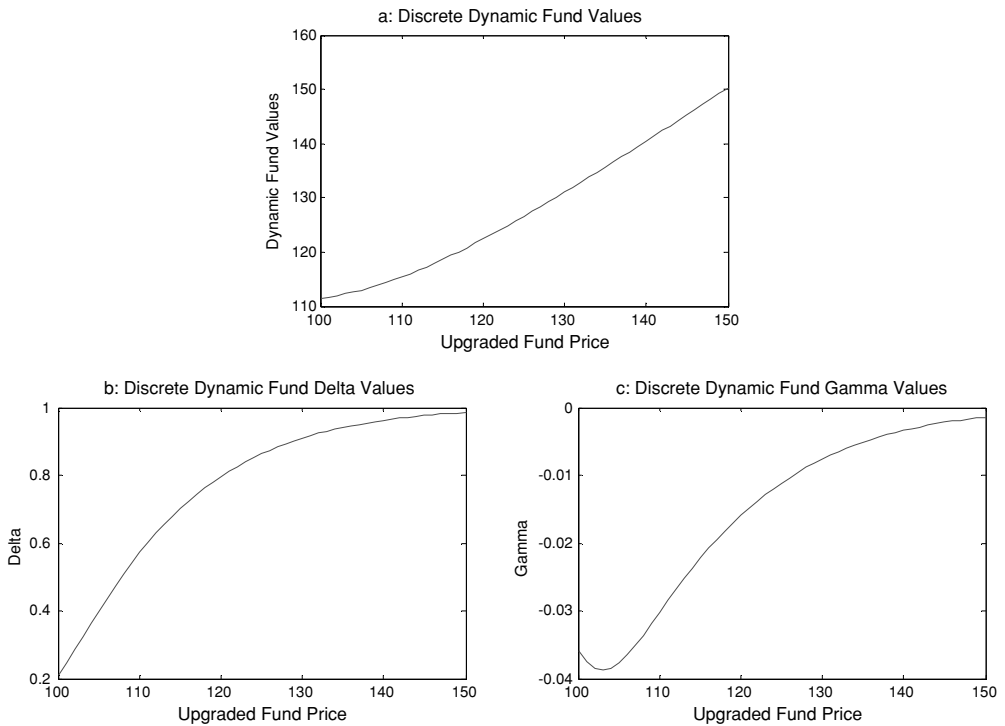
To investigate its hedging performance when the portfolio is subject to the tradable asset constraint, we modify Equation (25) to Equation (26):

$$H_{i\delta t}(\theta) = \tilde{c}_{i\delta t} - \Delta_{(i-1)\delta t} n_{(i-1)\delta t} F_{i\delta t} - e^{r\delta t} [\tilde{c}_{(i-1)\delta t} - \Delta_{(i-1)\delta t} \tilde{F}_{(i-1)\delta t}]. \quad (26)$$

This hedging error adjustment requires an agreement to buy the upgraded fund at  $\tilde{F}_{i\delta t}$  but to sell the upgraded fund at  $n_{(i-1)\delta t} F_{i\delta t}$ , even though  $n_{(i-1)\delta t} F_{i\delta t} \leq \tilde{F}_{i\delta t}$ . As the naked, instead of the upgraded, fund is the tradable asset, practically, the portfolio hedging strategy in Equation (26) is more applicable than that specified in Equation (25).

**FIGURE 1**

Behaviors of Discrete Dynamic Guaranteed Fund and Its Hedging Parameters

**SIMULATIONS****Analytic Valuation of Dynamic Guaranteed Fund Protection**

To justify the validity of our valuation formula in Equation (22), we compare the numerical results of the protection values derived from our valuation formula under similar terms to those in Imai and Boyle (2001).

In Table 2, using the analytic formula in Gerber and Pafumi (2000), we calculate the continuously sampled dynamic guaranteed fund protection values and list them under column CDG-GP. The values under column MC-IB are Monte Carlo simulation results copied from table 10 of Imai and Boyle (2001). Imai and Boyle derive the approximate formula from an extension of Broadie, Glasserman, and Kou (1999). As Imai and Boyle compute their discrete dynamic guaranteed fund pricing formula by Monte Carlo simulation, we report a confidence interval. Using our analytic formula given in Equation (22), we calculate the values of the discrete dynamic guaranteed fund protection under column DDG.

First, we find that when the number of monitorings declines from weekly monitoring to monthly monitoring, the fund protection values obtained by our analytic valuation are consistently smaller. This finding conforms to our intuition that the amount of expected total guarantee injections decreases when we reduce the number of monitorings.

**TABLE 2**  
Monte Carlo Compared to Analytic Valuation<sup>a</sup>

| <i>T</i> | <i>K</i> | CDG-GP <sup>b</sup> | Weekly             |                  | Monthly |         |
|----------|----------|---------------------|--------------------|------------------|---------|---------|
|          |          |                     | MC-IB <sup>c</sup> | DDG <sup>d</sup> | MC-IB   | DDG     |
| 1        | 100      | 14.7931             | 13.053             | 13.0389          | 11.375  | 11.3608 |
|          |          |                     | 12.977             |                  | 11.096  |         |
|          | 90       | 6.0120              | 5.196              | 5.1801           | 4.461   | 4.4446  |
|          |          |                     | 5.121              |                  | 4.197   |         |
|          | 80       | 1.7709              | 1.494              | 1.4811           | 1.254   | 1.2414  |
| 3        | 100      | 23.8741             | 1.451              |                  | 1.119   |         |
|          |          |                     | 21.993             | 21.9430          | 20.060  | 20.0089 |
|          | 90       | 13.4646             | 21.915             |                  | 19.890  |         |
|          |          |                     | 12.338             | 12.2866          | 11.194  | 11.1429 |
|          | 80       | 6.6443              | 12.253             |                  | 10.999  |         |
| 5        | 100      | 29.1716             | 6.054              | 6.0054           | 5.357   | 5.3966  |
|          |          |                     | 5.981              |                  | 5.295   |         |
|          | 90       | 18.0257             | 27.130             | 27.1462          | 25.097  | 25.3963 |
|          |          |                     | 27.097             |                  | 25.021  |         |
|          | 80       | 10.1373             | 16.709             | 16.7063          | 15.395  | 15.3963 |
|          |          |                     | 16.682             |                  | 15.294  |         |
|          |          |                     | 9.340              | 9.3441           | 8.559   | 8.5645  |
|          |          |                     | 9.326              |                  | 8.487   |         |

<sup>a</sup>Option parameters:  $F(0) = 100$ ,  $r = 0.04$ ,  $\gamma = 0$ , and  $\sigma = 0.2$ .

<sup>b</sup>CDG-GP denotes the continuously sampled dynamic guaranteed fund protection values calculated using the formula in Gerber and Pafumi (2000).

<sup>c</sup>MC-IB denotes the Monte Carlo results copied from Imai and Boyle (2001), table 10.

<sup>d</sup>DDG denotes the discretely sampled dynamic guaranteed fund protection values calculated using the formula in Equation (22).

Second, compared with the numerical intervals of Imai and Boyle (2001), our numerical values for the discrete dynamic guaranteed fund values stay within their intervals for short maturity and close to the money. Those of our values that go out of their numerical ranges are higher, but at most by a second decimal place difference.

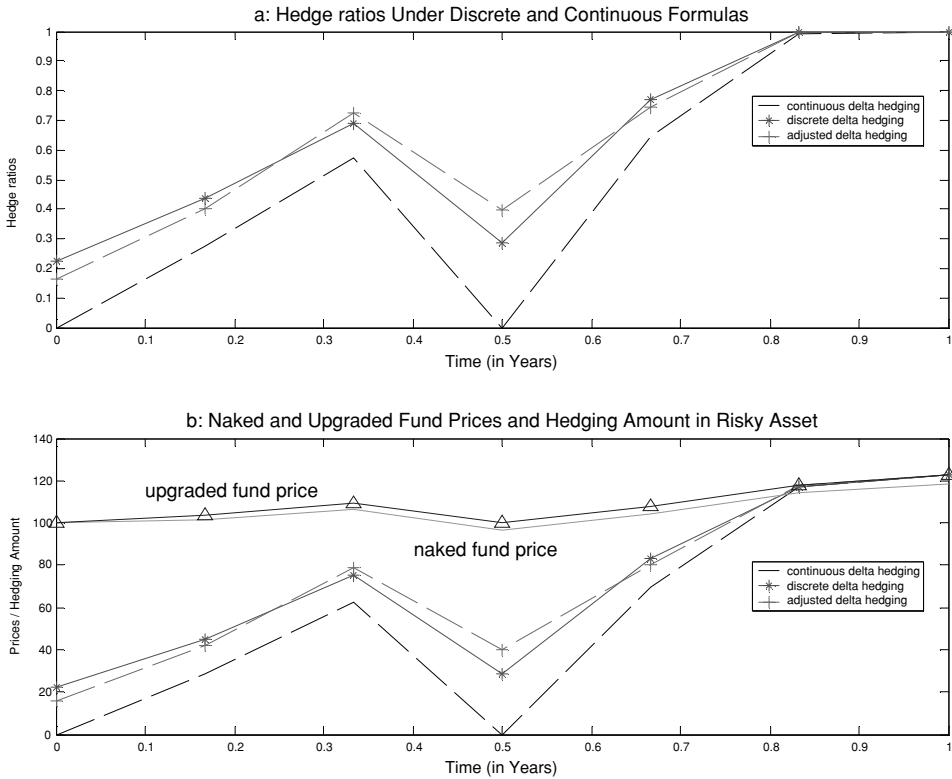
Third, although the continuous formula correction approach is conceptually different from our discrete probability valuation approach, the consistency of the two sets of results mutually support both conceptual frameworks. However, our analytic valuation formula enables an easy derivation of the analytic delta- and gamma-hedging formulas of the discrete dynamic guaranteed fund.

### The Hedging Errors of Dynamic Guaranteed Funds

We assume zero hedging error in the initial period for delta dynamic hedging. Usually, a synthetic hedging portfolio is created when the corresponding option is sold. Consequently, the initial option value covers the cost of synthetic portfolio creation.

Based on section 5 in Gerber and Pafumi (2000), we compute the amount invested in the risky reference portfolio  $R_C$  and the amount invested in the risk-free asset  $RF_C$  at

**FIGURE 2**  
Discrete Versus Continuous Hedging Behaviors



the fixed time point  $t$ . Henceforth, subscripts  $C$ ,  $D$ , and  $G$  refer to continuous, discrete, and gamma-adjusted hedging, respectively. We calculate the continuous dynamic guaranteed fund hedging error  $H_C$ , which is the income of previous period's risky and risk-free assets brought forward subtracted from the costs of risky and risk-free assets in this period.

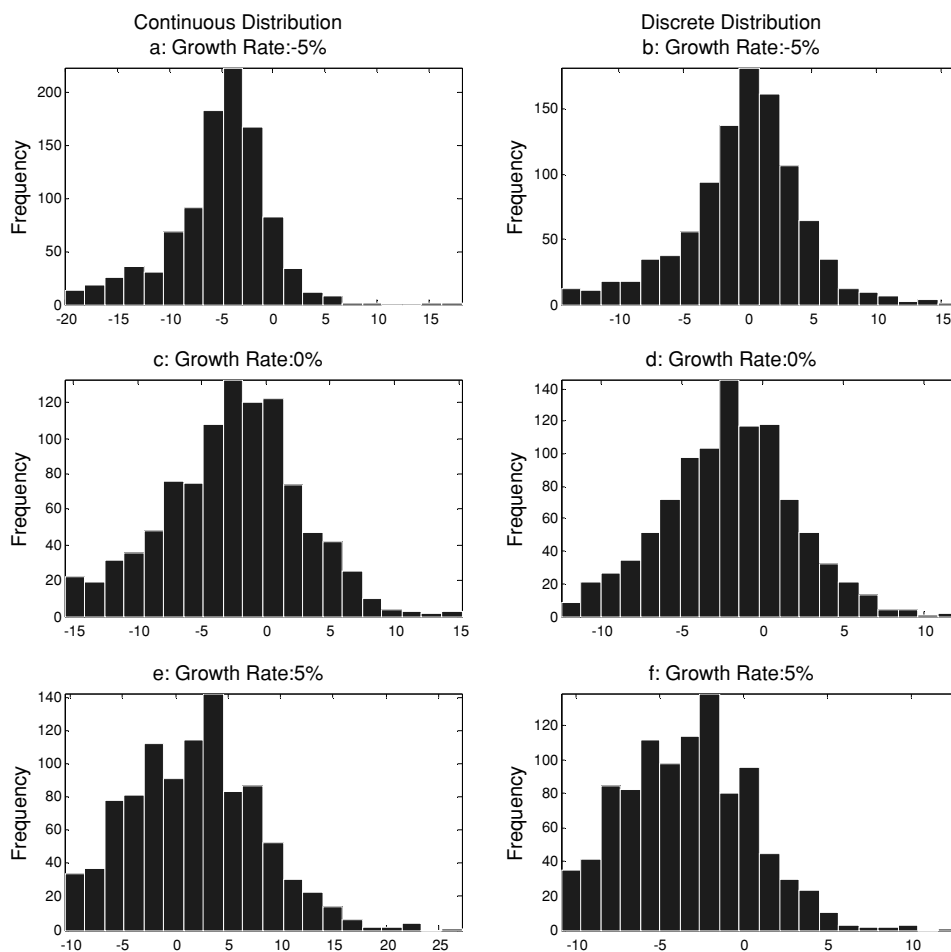
Our formulation in Equation (23) suggests an alternative synthetic portfolio under discrete sampling consisting of the risky reference portfolio  $R_D$  and a risk-free reserve  $RF_D$  to hedge against the uncertainty of the upgraded fund  $\tilde{F}(t)$ . However, over time, the patterns of fluctuations of the delta hedge ratios under the discrete and continuous hedging formulas rise and fall together (see Figure 2a).

Under the continuous hedging formula in Gerber and Pafumi (2000), the delta hedge ratio will be zero whenever the upgraded fund price is at the guaranteed level. The situation usually occurs when the naked fund price is drifting down. If this continues, then the guaranteed fund pays the nonstochastic guaranteed fund price  $K$  at maturity. This implication is the rationale for holding risk-free asset only before fund maturity when the upgraded fund price is at the guaranteed level  $K$ .

The zero hedge ratio effect creates a wide gyration in the hedging amount of the risky asset (see Figure 2b), which can suddenly move between zero to values close to

**FIGURE 3**

Distributions of Continuous and Discrete Delta Hedging Errors



the upgraded fund prices. Nonetheless, this high degree of gyration is not the case under our discrete hedging formula, because its hedge ratio is always above the ratio calculated under the continuous hedging formula in Gerber and Pafumi (2000).

Figure 3 plots the frequency distributions of the total hedging errors  $\Sigma_t H_C$  and  $\Sigma_t H_D$  for  $\gamma = -0.05, 0$ , and  $0.05$ . The frequency distributions are generated by 1,000 Monte Carlo simulation runs of  $F(t)$  over the simulated year hedged bimonthly and monitored monthly under  $\mu = 0.08$ ,  $\sigma = 0.16$ , and the other parameters as in the section "Analytic Valuation of Dynamic Guaranteed Fund Protection." Each run creates one data point for  $\Sigma_t H_D$  and  $\Sigma_t H_C$ . Figures 3a, 3c, and 3e, are for  $\Sigma_t H_C$  with  $\gamma = -0.05, 0$ , and  $0.05$ . Figures 3b, 3d, and 3f are for  $\Sigma_t H_D$  with  $\gamma = -0.05, 0$ , and  $0.05$ , respectively. In Table 3, we list the means of the distributions and their standard deviations in Figures 3a, 3c, and 3e) under columns  $\mu_C^{(25)}$  and  $\sigma_C^{(25)}$  for Equation (25), and  $\mu_C^{(26)}$  and  $\sigma_C^{(26)}$  for Equation (26), respectively. The table also lists the means of the distributions

**TABLE 3**Simulation Results of Continuous and Discrete Delta Hedging<sup>a</sup>

| Continuous Delta Hedging |                 |                    |                         |                |                   |                        |
|--------------------------|-----------------|--------------------|-------------------------|----------------|-------------------|------------------------|
| $\gamma$                 | $\mu_C^{(25)b}$ | $\sigma_C^{(25)c}$ | $\mu_{C, RMSE}^{(25)d}$ | $\mu_C^{(26)}$ | $\sigma_C^{(26)}$ | $\mu_{C, RMSE}^{(26)}$ |
| -5%                      | -5.2113         | 4.7373             | 1.5987                  | 0.1126         | 5.8607            | 1.7204                 |
| 0%                       | -2.6117         | 5.2744             | 1.5768                  | 1.1486         | 6.3288            | 1.6937                 |
| 5%                       | 1.8090          | 6.0902             | 2.1566                  | 3.6940         | 7.8090            | 2.5590                 |
| Discrete Delta Hedging   |                 |                    |                         |                |                   |                        |
| $\gamma$                 | $\mu_D^{(25)}$  | $\sigma_D^{(25)}$  | $\mu_{D, RMSE}^{(25)}$  | $\mu_D^{(26)}$ | $\sigma_D^{(26)}$ | $\mu_{D, RMSE}^{(26)}$ |
| -5%                      | -0.3521         | 4.4957             | 1.3321                  | 4.8819         | 5.4549            | 1.6656                 |
| 0%                       | -2.1104         | 3.9929             | 1.1958                  | 4.3999         | 4.1636            | 1.3753                 |
| 5%                       | -3.2747         | 3.7584             | 1.2273                  | 4.3941         | 3.8170            | 1.5135                 |

<sup>a</sup>Option parameters:  $\mu = 0.08$ ,  $r = 0.04$ ,  $\sigma = 0.16$ ,  $K = 100$ ,  $\delta t = 1/6$ , and  $T = 1$ . Given the same  $\gamma$ , the continuous, and discrete delta hedging are subject to the same set of simulated stochastic paths for the naked fund price.

<sup>b</sup> $\mu_C^{(q)}$  and  $\mu_D^{(q)}$  are the means of the total hedging errors, generated using equation (q), based on continuous, and discrete delta hedging, respectively.

<sup>c</sup> $\sigma_C^{(q)}$  and  $\sigma_D^{(q)}$  are the standard deviations of the total hedging errors, generated using equation (q), based on continuous, discrete delta hedging, respectively.

<sup>d</sup> $\mu_{C, RMSE}^{(q)}$  and  $\mu_{D, RMSE}^{(q)}$  are the average root mean squared errors of the period-to-period hedging errors, generated using equation (q), based on continuous and discrete delta hedging, respectively.

and their standard deviations in Figures 3b, 3d, and 3f under columns  $\mu_D^{(25)}$  and  $\sigma_D^{(25)}$  for Equation (25), and  $\mu_D^{(26)}$  and  $\sigma_D^{(26)}$  for Equation (26), respectively.

In Table 3, we show that an inconsistent application of the continuous delta intended to discretely hedge a risk-neutral-priced dynamic guaranteed fund results in an overcharge more serious than the discrete delta hedging (indicated by  $\mu_C^{(25)} < \mu_D^{(25)} < 0$ ) when  $\gamma \leq 0$ , and an undercharge (indicated by  $\mu_C^{(25)} > 0$ ) when  $\gamma > 0$ . In all these cases, the variances of the frequency distributions of  $\Sigma_t H_C$  are larger than their counterpart values of  $\Sigma_t H_D$ . So, to calculate the rebalancing portfolio, applying our analytic hedging formula rather than the Gerber and Pafumi (2000) formula reduces the variability of the discrete hedging errors.

*The Conventional Delta and Gamma-Adjusted-Delta Hedging.* In Table 4, we compare the hedging performance between the delta hedging that uses Equation (25) and the gamma-adjusted-delta hedging that uses Equation (26) of the dynamic guaranteed fund at bimonthly time instants,  $\delta t = 1/6$ , over a simulated year. The dynamic path of the naked fund is generated by Monte Carlo simulation under  $\mu = 0.08$ ,  $\sigma = 0.16$ . The parameters of time-to-maturity and the guaranteed level are the same as in the section "Analytic Valuation of Dynamic Guaranteed Fund Protection."

We calculate the number of units  $n(t)$  and the upgraded fund price  $\tilde{F}(t)$  at time  $t$  by using Equation (16), where  $T \geq t \geq 0$ . Following Gerber and Pafumi (2000), we split

**TABLE 4**

Hedging Performance of the Synthetic Investment Fund Under Delta Hedging and Gamma-Adjusted-Delta Hedging<sup>a</sup>

| $t$            | $F(t)^b$ |        | $n(t)^c$ | $\tilde{F}(t)^d$ | Delta Hedging <sup>e</sup> |        |         | Gamma-Adjusted-Delta Hedging |        |       |
|----------------|----------|--------|----------|------------------|----------------------------|--------|---------|------------------------------|--------|-------|
|                |          |        |          |                  | $RF_D^f$                   | $R_D$  | $H_D^g$ | $RF_D$                       | $R_D$  | $H_D$ |
| 0              | 0.00     | 100.00 | 1.000    | 100.00           | 86.20                      | 22.46  | 0.00    | 92.49                        | 16.17  | 0.00  |
| 1/12           | 97.44    | 101.25 | 1.026    | 103.90           | 63.98                      | 45.17  | -0.96   | 67.39                        | 41.76  | -0.33 |
| 3/12           | 107.36   | 106.12 | 1.026    | 108.90           | 36.15                      | 75.09  | -0.51   | 32.11                        | 79.13  | -0.37 |
| 5/12           | 103.92   | 96.77  | 1.033    | 100.00           | 77.53                      | 28.30  | 0.49    | 66.03                        | 39.80  | 1.35  |
| 7/12           | 96.98    | 104.29 | 1.033    | 107.76           | 25.98                      | 82.94  | 0.37    | 28.92                        | 80.01  | -0.44 |
| 9/12           | 107.02   | 113.98 | 1.033    | 117.78           | 0.51                       | 117.28 | 0.98    | 0.07                         | 117.72 | 1.24  |
| 11/12          | 118.72   | 118.49 | 1.033    | 122.44           | 0.00                       | 122.44 | 0.01    | 0.00                         | 122.44 | -0.01 |
| Hedging error: |          |        |          |                  | $\sum_t H_D = 0.38$        |        |         | $\sum_t H_D = 1.43$          |        |       |

<sup>a</sup>Option parameters:  $\mu = 0.08$ ,  $r = 0.04$ ,  $\sigma = 0.16$ ,  $\gamma = 0$ ,  $K = 100$ ,  $\delta t = 1/6$ , and  $T = 1$ .

<sup>b</sup> $F(t)$  is the naked fund value at time  $t$  that follows a geometric Brownian motion, that is,  $F(t) = F(0)e^{U_t}$ ,  $U_t \sim \phi((\mu - \sigma^2/2)t, \sigma^2 t)$ . This specific stochastic series is generated by a Monte Carlo stochastic simulation of a monthly interval. The second column is at the end of the month, and the third column is at the beginning of every 2-month period.

<sup>c</sup> $n(t) = \max\{1, \max_{0 \leq s \leq t} \frac{Ke^{\gamma s}}{F(s)}\}$ . We report only the beginning of the bimonthly figures.

<sup>d</sup> $\tilde{F}(t) = n(t)F(t)$ . We report only the beginning of the bimonthly figures.

<sup>e</sup>Columns 6 to 8 are for the conventional delta hedging;  $\theta = 0$ . Columns 9 to 11 are for the gamma-adjusted-delta hedging:

$$\theta = \sqrt{\delta t} \left( \frac{\sigma^2 [E(y^2) - 1]/2 + (\mu - \sigma^2/2)\sigma E(y)\sqrt{\delta t} + (\mu - \sigma^2/2)^2 \delta t/2}{\sigma E(y) + [(\mu - \sigma^2/2) - r]\sqrt{\delta t}} \right).$$

<sup>f</sup> $R_D(t)$  and  $RF_D(t)$  denote the amount invested in the risky and riskless assets, respectively.

$$^g H_D(t) = [RF_D(t) - e^{r\delta t} RF_D(t - \delta t)] + \left[ R_D(t) - \frac{R_D(t - \delta t)F(t)}{F(t - \delta t)} \right].$$

our synthetic fund investment into an investment in the reference portfolio  $R_D$  and an investment in the risk-free asset  $RF_D$ .

Under realistic parameters, Equations (10) and (12) imply that the expected total hedging error of delta hedging is lower than that of the gamma-adjusted-delta hedging. Our simulation run in Table 4 has similar numerical order between the two total hedging errors, though their values are both positive. The total hedging error of this synthetic hedging portfolio  $\sum_t H_D$  for  $\gamma = 0$  under delta hedging is 0.38 and under gamma-adjusted delta hedging is 1.43. With the gamma adjustment in Figure 2, it suggests that the effective delta can fluctuate above or below the conventional delta.

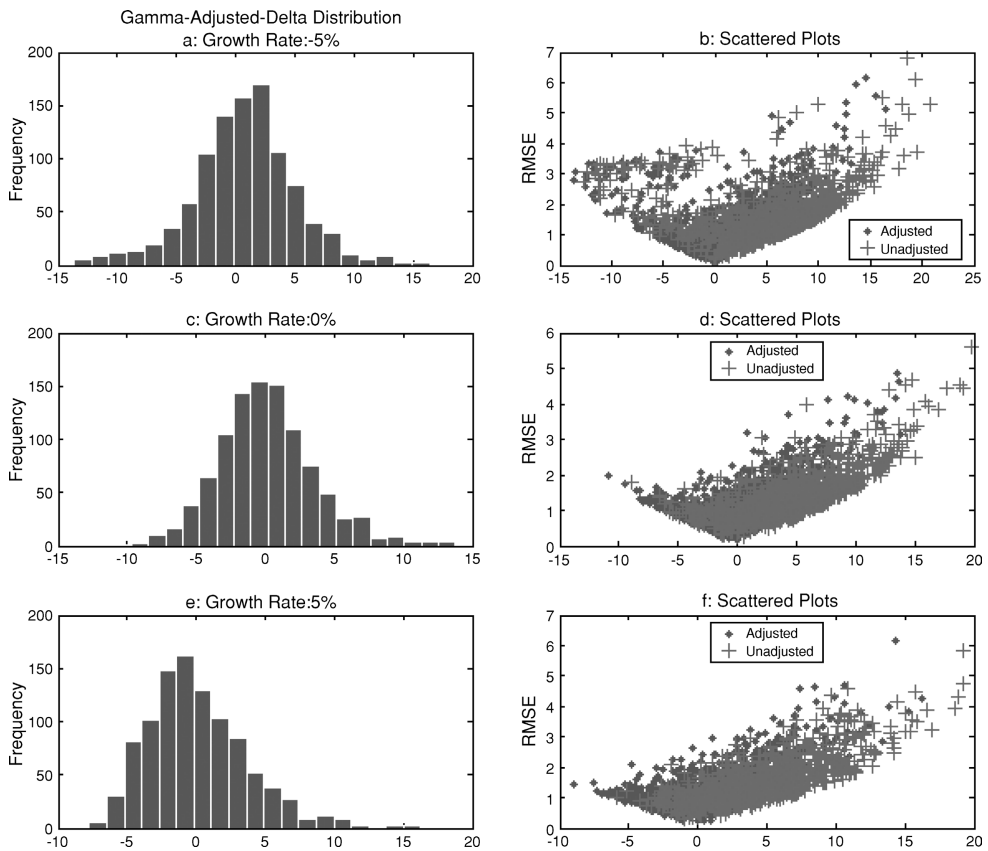
In Table 4, although the pattern of fluctuations between the conventional delta and the gamma-adjusted-delta hedge ratios are similar, the adjustment sometimes causes opposite movements of the hedging errors  $H_D$  between the delta and gamma-adjusted-delta hedging at various time points of the simulated year. Moreover, when the naked

fund price decreases sharply, using Equation (25), the increase in the number of fund units prevents a rise in the value of the hedging error under the discrete delta hedging.

In contrast, using Equation (26), the gamma-adjusted-delta hedging facilitates the rise in the hedging error when the naked fund price declines. Therefore, the two-sided period-to-period hedging error variation under the gamma-adjusted-delta hedging that uses Equation (26) minimizes the negative bias of the expected total hedging error.

*The Gamma-Adjusted-Delta Hedging.* To further investigate the difference in performance between conventional delta and our gamma-adjusted-delta hedging that uses Equation (26), we apply the same 1,000 stochastic paths of the naked fund price in Table 3 to Figures 4a, 4c, and 4e. The figures show the distributions of the total hedging errors of the gamma-adjusted-delta hedging strategy, which correspond to those in Figures 4b, 4d, and 4f. Based on the same set of simulated paths of the naked fund price, Figures 4b, 4d, and 4f compares on their own graphs the root mean squared errors

**FIGURE 4**  
Hedging Errors Under Tradable Asset Constraint



**TABLE 5**  
Simulation Results of Delta Hedging and Gamma-Adjusted-Delta Hedging<sup>a</sup>

| Discrete Delta Hedging       |                 |                    |                         |                |                   |                        |
|------------------------------|-----------------|--------------------|-------------------------|----------------|-------------------|------------------------|
| $\gamma$                     | $\mu_D^{(25)b}$ | $\sigma_D^{(25)c}$ | $\mu_{D, RMSE}^{(25)d}$ | $\mu_D^{(26)}$ | $\sigma_D^{(26)}$ | $\mu_{D, RMSE}^{(26)}$ |
| -5%                          | -0.3521         | 4.4957             | 1.3321                  | 4.8819         | 5.4549            | 1.6656                 |
| 0%                           | -2.1104         | 3.9929             | 1.1958                  | 4.3999         | 4.1636            | 1.3753                 |
| 5%                           | -3.2747         | 3.7584             | 1.2273                  | 4.3941         | 3.8170            | 1.5135                 |
| Gamma-Adjusted-Delta Hedging |                 |                    |                         |                |                   |                        |
| $\gamma$                     | $\mu_G^{(25)}$  | $\sigma_G^{(25)}$  | $\mu_{G, RMSE}^{(25)}$  | $\mu_G^{(26)}$ | $\sigma_G^{(26)}$ | $\mu_{G, RMSE}^{(26)}$ |
| -5%                          | -0.3025         | 4.4466             | 1.3347                  | 0.7281         | 4.3687            | 1.3785                 |
| 0%                           | -1.9644         | 4.0293             | 1.2081                  | 0.1624         | 3.5485            | 1.1803                 |
| 5%                           | -2.9575         | 3.8949             | 1.2577                  | 0.1386         | 3.6284            | 1.3148                 |

<sup>a</sup>Option parameters:  $\mu = 0.08$ ,  $r = 0.04$ ,  $\sigma = 0.16$ ,  $K = 100$ ,  $\delta t = 1/6$ , and  $T = 1$ . Given the same  $\gamma$ , the discrete and gamma-adjusted-delta hedging are subject to the same set of simulated stochastic paths for the naked fund price.

<sup>b</sup> $\mu_D^{(q)}$  and  $\mu_G^{(q)}$  are the means of the total hedging errors, generated using equation (q), based on discrete and gamma-adjusted-delta hedging, respectively.

<sup>c</sup> $\sigma_D^{(q)}$  and  $\sigma_G^{(q)}$  are the standard deviations of the total hedging errors, generated using equation (q), based on discrete and gamma-adjusted-delta hedging, respectively.

<sup>d</sup> $\mu_{D, RMSE}^{(q)}$  and  $\mu_{G, RMSE}^{(q)}$  are the average root mean squared errors of the period-to-period hedging errors, generated using equation (q), based on discrete and gamma-adjusted-delta hedging, respectively.

(RMSE) of the bimonthly hedging errors over the simulated year between the delta and gamma-adjusted-delta hedging. The RMSE measures the degree of departure of a hedging strategy to achieve an ideal period-to-period self-financing strategy.

In Table 5, we compare the hedging performances of both the delta and gamma-adjusted-delta hedging. We derive the means, standard deviations, and the average RMSE of the gamma-adjusted-delta total hedging errors from Equation (26) and list them under columns  $\mu_G^{(26)}$ ,  $\sigma_G^{(26)}$ , and  $\mu_{G, RMSE}^{(26)}$ , respectively. Using the same equation, we generate the means, standard deviations, and the average RMSE of the delta total hedging errors and we list them under columns  $\mu_D^{(26)}$ ,  $\sigma_D^{(26)}$ , and  $\mu_{D, RMSE}^{(26)}$ , respectively. In Table 5, we find that  $\mu_D^{(25)} < 0$  and  $\mu_D^{(25)} < \mu_G^{(25)}$ , where we generate the means of the total hedging errors  $\mu_G^{(25)}$  from Equation (25) for the gamma-adjusted-delta hedging. This finding is consistent with the theoretical derivation in Equations (10) and (12). The simulation results in Table 5 suggest that the theoretical delta hedging overcharge will not accrue to the discrete delta hedgers, because  $\mu_D^{(26)} > 0$ . However, especially when  $\gamma \geq 0$ , we observe a marked decrease toward zero in the expected total hedging errors of the gamma-adjusted-delta hedging when we use Equation (26) rather than Equation (25). Further, simulation studies suggest that this reduction in expected total hedging errors and their variability for hedging the risk-neutral-priced dynamic guaranteed fund is valid for most of the realistic parameters for  $\mu$ ,  $\sigma$ , and  $r$ .

The means in Figures 4a, 4c, and 4e are 0.7281, 0.1624, and 0.1386, respectively, and their standard deviations are 4.3687, 3.5485, and 3.6284, respectively. The corresponding means under delta hedging are 4.8819, 4.3999, and 4.3941, respectively, and their standard deviations are 5.4549, 4.1636, and 3.8170, respectively. The gamma-adjusted-delta hedging strategy that uses Equation (26) is also closer to the self-financing benchmark than that of the discrete delta hedging strategy. The average *RMSE* of the gamma-adjusted-delta hedging  $\mu_{G, RMSE}^{(26)}$  are 1.3785, 1.1803, and 1.3148, respectively. The corresponding average *RMSE* of the delta hedging  $\mu_{D, RMSE}^{(26)}$  are higher at 1.6656, 1.3753, and 1.5135, respectively.

## CONCLUSIONS

In this article, we investigate the analytic valuation and hedging strategy of a discrete dynamic guaranteed fund with geometric Gaussian naked fund prices. A guaranteed fund is an innovative product designed by insurance companies to offer their investors higher returns and protect them against the downside risk of these returns. A dynamic guaranteed fund does not require the investor to actively monitor the fund value to make prudent early exercise and rollover decisions. Our pricing and hedging formulations are based on the risk-neutral expected payoff approach. Although a continuously sampled option formula offers an efficient reference for the value of an option, a discretely sampled option formula is the theoretically accurate formulation for practical applications.

As the terminal value of a dynamic guaranteed fund is leveraged by injections of incremental units that upgrade the naked fund value so that it remains above the guaranteed level over its life, the total value injection depends on the minimum value of the naked fund during its life. Therefore, we make use of the probability density function for discrete lookback options in AitSahlia and Lai (1998) to develop our pricing and hedging formulas for the discrete dynamic guaranteed fund. Our formulas are a result of the risk-neutral expected terminal value of the upgraded fund conditional on the possible occurrence of the minimum value at various fixed discrete time instants.

Imai and Boyle (2001) provide a pricing interval for the discrete dynamic guaranteed fund. However, we are able to accurately calculate the analytic value of a discrete dynamic guaranteed fund. This type of fund is cheaper than its continuous counterpart. Furthermore, based on our Monte Carlo simulation experiments, we show that inconsistent application of the continuously, rather than a discretely, sampled hedging formula for hedging a discrete dynamic guaranteed fund at bimonthly monitoring points in time overcharges the fund buyers and enlarges the variance of the total hedging error. Nonetheless, because of the unidirectional upward movement of the guaranteed fund units, this overcharge persists even for the discrete delta dynamic hedging of the risk-neutral-priced discrete dynamic guaranteed fund.

To account for the fact that the naked, instead of the upgraded, fund is tradable, our simulation studies show that the benefit of an overcharge does not necessarily accrue to the discrete delta hedgers, but loses in the actual implementation of the dynamic hedging process. By modifying the conventional delta to a risk-neutral gamma-adjusted-delta hedge ratio, we demonstrate that to a large extent, a fund issuer can

eliminate this expected total hedging errors, cut down their variability, and achieve a more precise period-to-period self-financing outcome.

## APPENDIX

### Proof of Proposition 1

We define

$$\begin{aligned} d_2^*(v) &= \frac{-v^* + \mu_v}{\sigma_v}, \\ d_1^*(v) &= d_2^*(v) + \sigma_v, \\ d_2(v) &= \frac{\mu_v}{\sigma_v}, \\ d_1(v) &= d_2(v) + \sigma_v, \end{aligned}$$

for  $v = 1, \dots, m$  where  $v^* = \min[0, \ln(\tilde{K})]$ . Also, we define

$$\beta_v = \int_{-\infty}^{d_1(1)} \cdots \int_{-\infty}^{d_1(v)} C_v e^{-\frac{1}{2} \mathbf{y}_v' Z_v^{-1} \mathbf{y}_v} d\mathbf{y}_v$$

and where  $C_v = (2\pi)^{-v/2} |Z_v|^{-1/2}$ ,  $Z_v$  is a  $v \times v$  full rank matrix arising from Brownian motion,  $\mathbf{y}_v = (y_1, \dots, y_v)$ .

Under risk-neutral expectation, the value of the discrete guaranteed fund at time 0 is

$$\begin{aligned} &\tilde{A}(F(0), T) \\ &= e^{-rT} E^*[F(T) \max(\tilde{K} e^{-M_m}, 1)] \\ &= e^{-rT} E^*[F(T) \max(\tilde{K} e^{-M_m} - 1, 0) + F(T)] \\ &= e^{-rT} F(0) \{ E^*[e^{U_m} E^*(\max(\tilde{K} e^{-x} - 1, 0) | x = M_m) | U_m = M_m] \\ &\quad + E^*[e^{U_m} E^*(\max(\tilde{K} e^{-x} - 1, 0) | x = M_m) | U_m > M_m] + E^*[e^{U_m}] \} \\ &= e^{-rT} F(0) \left\{ \int_{U_m}^{\infty} \cdots \int_{U_m}^{\infty} \int_{-\infty}^0 e^{U_m} (\max(\tilde{K} e^{-U_m} - 1, 0)) f(U_m | U_m = M_m) \phi(U_m = M_m) d\mathbf{U} \right. \\ &\quad + \sum_{i=1}^{m-1} \int_{U_i}^{\infty} \cdots \int_{U_i}^{\infty} \int_{-\infty}^0 \int_{U_i}^{\infty} \cdots \int_{U_i}^{\infty} e^{U_m} (\max(\tilde{K} e^{-U_i} - 1, 0)) f(U_m | U_i = M_m) \phi(U_i = M_m) d\mathbf{U} \\ &\quad \left. + \int_0^{\infty} \cdots \int_0^{\infty} e^{U_m} (\max(\tilde{K} e^{-U_0} - 1, 0)) f(U_m | U_0 = M_m) \phi(U_0 = M_m) d\mathbf{U} + E^*[e^{U_m}] \right\}, \end{aligned}$$

where  $f(U_m | U_i = M_m)$  is the probability density function of  $U_m$  conditional on the occurrence of minimum  $M_m$  at  $U_i$ , and  $\phi(U_i = M_m)$  is the normal probability density function of  $U_i$  under Brownian motion. As  $U_0 = 0$  and  $e^{-rT} E^*[e^{U_m}] = 1$ , by applying the joint probability results in Ait-Sahalia and Lai (1998), the value of the discrete guaranteed fund can be written as

$$\begin{aligned}
& \tilde{A}(F(0), T) \\
&= e^{-rT} F(0) \left\{ \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_{-\infty}^{\min(0, \ln \tilde{K})} (\tilde{K} - e^{U_m}) \phi(0 > U_{m-i}, U_m \in dU_m; i < m) d\mathbf{U}_{i,m} \right. \\
&\quad + \sum_{i=1}^{m-1} \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_{-\infty}^{\min(0, \ln \tilde{K})} e^{U_i} (\tilde{K} e^{-U_i} - 1) \phi(0 > U_{i-j}, U_i \in dU_i; j < i) d\mathbf{U}_{1,i} \\
&\quad \times \int_0^\infty \cdots \int_0^\infty e^{U_{m-i}} \phi(U_{k-i} > 0; k > i) d\mathbf{U}_{1,m-i} \\
&\quad \left. + \max(\tilde{K} - 1, 0) \int_0^\infty \cdots \int_0^\infty e^{U_m} \phi(U_k > 0; k > 0) d\mathbf{U}_{1,m} + e^{rT} \right\} \\
&= e^{-rT} F(0) \left\{ \int_{-\infty}^{-\frac{\mu_1}{\sigma_1}} \cdots \int_{-\infty}^{-\frac{\mu_{m-1}}{\sigma_{m-1}}} \int_{-\infty}^{\frac{v^* - \mu_m}{\sigma_m}} (\tilde{K} - e^{\sigma_m y_m + \mu_m}) C_m e^{-\frac{1}{2} y'_m Z_m^{-1} y_m} d\mathbf{y}_m \right. \\
&\quad + \sum_{i=1}^{m-1} \int_{-\infty}^{-\frac{\mu_1}{\sigma_1}} \cdots \int_{-\infty}^{-\frac{\mu_{i-1}}{\sigma_{i-1}}} \int_{-\infty}^{\frac{v^* - \mu_m}{\sigma_m}} (\tilde{K} - e^{\sigma_i y_i + \mu_i}) C_i e^{-\frac{1}{2} y'_i Z_i^{-1} y_i} d\mathbf{y}_i \\
&\quad \times \int_{-\frac{\mu_1}{\sigma_1}}^\infty \cdots \int_{-\frac{\mu_{m-i}}{\sigma_{m-i}}}^\infty e^{\sigma_{m-i} y_{m-i} + \mu_{m-i}} C_{m-i} e^{-\frac{1}{2} y'_{m-i} Z_{m-i}^{-1} y_{m-i}} d\mathbf{y}_{m-i} \\
&\quad \left. + \max(\tilde{K} - 1, 0) \int_{-\frac{\mu_1}{\sigma_1}}^\infty \cdots \int_{-\frac{\mu_m}{\sigma_m}}^\infty e^{\sigma_m y_m + \mu_m} C_m e^{-\frac{1}{2} y'_m Z_m^{-1} y_m} d\mathbf{y}_m + e^{rT} \right\} \\
&= F(0) \left\{ 1 + \max(0, \tilde{K} - 1) \beta_m \right. \\
&\quad \left. + \sum_{v=1}^m e^{-rv\Delta} \beta_{m-v} \int_{-\infty}^{-d_2(1)} \cdots \int_{-\infty}^{-d_2(v-1)} \int_{-\infty}^{-d_2^*(v)} (\tilde{K} - e^{\sigma_v y_v + \mu_v}) C_v e^{-\frac{1}{2} y'_v Z_v^{-1} y_v} d\mathbf{y}_v \right\}.
\end{aligned}$$

We can rewrite the above expression as

$$\begin{aligned}
\tilde{A}(F(0), T) &= F(0) \left\{ 1 + \max(0, \tilde{K} - 1) \beta_m \right. \\
&\quad + \sum_{v=1}^m e^{-rv\Delta} \tilde{K} \beta_{m-v} \int_{-\infty}^{-d_2(1)} \cdots \int_{-\infty}^{-d_2(v-1)} \int_{-\infty}^{-d_2^*(v)} C_v e^{-\frac{1}{2} y'_v Z_v^{-1} y_v} d\mathbf{y}_v \\
&\quad \left. - \sum_{v=1}^m \beta_{m-v} \int_{-\infty}^{-d_1(1)} \cdots \int_{-\infty}^{-d_1(v-1)} \int_{-\infty}^{-d_1^*(v)} C_v e^{-\frac{1}{2} y'_v Z_v^{-1} y_v} d\mathbf{y}_v \right\}.
\end{aligned}$$

□

Proof of Proposition 2

The proof will be furnished upon request.

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