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Evaluating Liquidation Strategies for Insurance Companies

Thomas R. Berry-Stölzle

Abstract

In this article, we examine liquidation strategies and asset allocation decisions for property and casualty insurance companies for different insurance product lines. We propose a cash-flow-based liquidation model of an insurance company and analyze selling strategies for a portfolio with liquid and illiquid assets. Within this framework, we study the influence of different bid-ask spread models on the minimum capital requirement and determine a solution set consisting of an optimal initial asset allocation and an optimal liquidation strategy. We show that the initial asset allocation, in conjunction with the appropriate liquidation strategy, is an important tool in minimizing the capital committed to cover claims for a predetermined ruin probability. This interdependence is of importance to insurance companies, stakeholders, and regulators.

Introduction

Property and casualty insurance companies, especially those insuring severe risks, often face the situation that premium inflow does not cover the outflow triggered by claims on a daily basis. This requires the liquidation of financial assets to cover all claims. But how should this be done? Which liquidation strategy should be used? Surprisingly, academia has not dealt with this basic question yet.

There are models in the finance literature focusing on the dynamic optimal selling strategy of a fixed block of securities (see, e.g., Bertsimas and Lo, 1998; Almgren and Chriss, 2000; Huberman and Stanzl, 2000). All of these articles study the optimal timing of selling securities by minimizing the expected cost of trading. But these models neglect the decision of which securities to sell. So far only Duffie and Ziegler (2003)

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analyze selling strategies that include choices between multiple securities. However, their short-term model of a leveraged financial institution, which has a company liquidating positions in falling markets to meet capital requirements, examines a special case and cannot be used for finding answers to the case of a property and casualty insurance company.

Our article is the first to study liquidation strategies for property and casualty insurance companies. We therefore combine the dynamic liquidation models from the finance literature with the standard actuarial risk model based on the fundamental work of Lundberg (1903). For an introduction to the collective risk model, see for example, Gerber (1979), Beard, Pentikäinen, and Pesonen (1984), Klugman, Panjer, and Willmot (1998), or Rolski, Schmidli, Schmidt, and Teugles (1999). We propose a cash-flow based liquidation model of an insurance company. Assuming that all premium payments have already been received at the beginning of the period under consideration and have been invested in a mix of financial assets, the insurance company has to sell securities for the settlement of insurance claims. Within this model framework, we determine the minimum required capital for a preselected safety level. Defining the minimum required capital as the objective function, we are then able to compare liquidation strategies as well as asset allocation decisions. The preferred strategy uses the least capital necessary for running the business.

At first glance, our risk-averse objective function of an insurance company, which is overly concerned with the probability of default, is not consistent with modern financial theory. According to the capital asset pricing model, the rates of return on assets represent only the undiversifiable (systematic) risk and not the diversifiable (unsystematic) risk. An insurance company faces two independent sources of risk: the underwriting return on its portfolio of insurance contracts and the return on its portfolio of investments. Risks associated with these two sources are largely diversifiable, and owners of widely held companies hence can eliminate this sort of risk by holding efficiently diversified portfolios. Unsystematic risk is by its nature diversifiable; thus, actions to reduce unsystematic risk do not create value. In the presence of transaction costs, however, modern financial theory predicts that risk management activities of a widely held insurance company will reduce the market valuation of the firm, and therefore should be omitted.

There is a broad literature on corporate risk management (see, e.g., Leland, 1978; Main, 1982; Mayers and Smith, 1982; Mayers and Smith, 1990). This strand of literature broadens the interpretation of modern financial theory by introducing corporate risk management as an important tool for reducing unsystematic risk. According to Mayers and Smith (1990) many insurance companies are closely held stock companies, mutuals, or reciprocals and are not held by efficiently diversified investors. Reducing unsystematic risk and hence total risk is rational for such firms. Other reasons for corporate risk management include taxes (Garven and Loubergé, 1996), cash management (Froot, Scharfstein, and Stein, 1993), the underinvestment problem (see, e.g., MacMinn, 1987; Garven and MacMinn, 1993), monitoring of managers (Sung, 1997), career concerns of managers (DeMarzo and Duffie, 1995), and last but not least concerns about insolvency.

Concerns about insolvency arise because of the associated costs. In addition to the rather low explicit costs of bankruptcy (see, e.g., Warner, 1977), insurance companies face significant costs for getting into financial distress. The mounting financial problems lead the financial monitoring services to downgrade the insurer's solvency rating. Such companies may be forced to offer their policies at lower prices than better rated insurance companies (Doherty and Tinic, 1981). Since most buyers of insurance are individuals or small businesses, they cannot efficiently diversify their insurance coverage over multiple insurance companies. Thus, even risk-neutral policyholders will demand a price discount for policies exposed to solvency risk (Berger, Cummins, and Tennyson, 1992). Insurance regulation can also magnify the costs of financial distress. These regulators primary objective is to protect the claims of the policyholders. Therefore they are likely to seize an insurance company with declining solvency margin at a certain trigger point. They will then force the firm to take actions to protect claimant's rights, not to protect the interest or even continued existence of the insurance company. Modern regulatory systems like the risk-based capital system in force in the United States, or Pillar One of the Solvency II system currently discussed in the European Union, build on the basic idea that the trigger point for intervention is tied to a minimum safety level or maximal tolerated probability of insolvency.

Insurance companies hold capital as a buffer against adverse loss development or investment fluctuations. Premiums and equity capital are substitute mechanisms for managing the company's overall risk exposure. In our model, we minimize the sum of the premiums and the equity capital subject to a given probability of insolvency. Assuming a constant premium income, this objective function can be viewed as minimizing the cost of equity capital subject to a safety constraint reflecting the risk management activity of the insurance company. Hence, our objective function is in line with the general result of Greenwald and Stiglitz (1990) that in the presence of bankruptcy costs firms will act in a risk-averse manner.

Liquidity risk is typically addressed in the market microstructure literature, and its equilibrium models are of considerable conceptual importance (see, e.g., Glosten and Milgrom, 1985; Grossman and Miller, 1988; Kyle, 1985). However, existing models are quite simple, and solving for an equilibrium in a more realistic dynamic model is a rather complex task. Consequently, to analyze liquidation strategies and asset allocation decisions of property and casualty insurance companies in illiquid markets, this article takes a partial equilibrium perspective.

In our model, all premium payments have already been received at the beginning of the period under consideration and have been invested in a mix of financial assets. Thus, the insurance company has to sell securities to settle insurance claims while facing an inelastic demand for these securities. We characterize the illiquidity of securities with a stochastic model of bid-ask spreads, and compare different initial asset allocations and liquidation strategies within this framework. Our model setup implicitly assumes that premium income and, hence, insurance demand is independent of the insurer's asset allocation and liquidation strategy. This assumption is often used in insurance models (see, e.g., MacMinn and Witt, 1987; Rees, Gravelle, and Wambach, 1999), since neither policyholders nor potential costumers can observe the asset management activities of an insurance company. We further assume that claims and asset prices are stochastically independent, that there are no taxes, and that the insurance company cannot raise additional capital. This restriction on external financing reflects the financial pecking order of funding options: debt financing is more costly than internally generated funds, and outside equity is even more costly than using debt instruments.[1] Companies prefer to finance their projects with retained earnings and, hence, should manage their cash flow to meet their planned investments (Froot, Scharfstein, and Stein, 1994). We incorporate this view in our model with the binding constraint on refinancing.

Our analysis is divided into three sections. First, we show that the optimal liquidation strategy depends on the basic asset allocation of the insurance company at the beginning of the liquidation process. Therefore, an optimization has to take the liquidation strategy and the basic asset allocation into account simultaneously. So this article analyzes not only the liquidation strategies of insurance companies, but also their asset allocation decisions in a simple and intuitive simulation model. Second, we study the influence of different bid-ask spread magnitudes and negative return-spread correlations on the minimum capital requirement and the optimal asset allocation and liquidation strategy. Third, we compare the optimal asset allocation and liquidation strategy for different types of insurance business.

Our analysis provides three main results: First, we find an optimal solution set consisting of a liquidation strategy and an initial asset allocation. Second, the optimal solution set differs for different insurance product lines. Third, the stochastic characteristics of bid-ask spread processes do not influence the optimal solution set but expected spread width does.

Although this is the first article to analyze liquidation strategies for property and casualty insurance companies, our research can be related to three strands of literature. First, we extend the literature on dynamic liquidation models by examining selling strategies, which include choices between multiple securities in a long-term insurance context. Second, we contribute to the literature on asset liability management by analyzing optimal asset allocation and liquidation decisions for an insurance company with liabilities. Third, we extend the literature on the financial management of insurance companies by introducing a cash-flow-based liquidation model for the first time.

The organization of the article is as follows. Our model is presented in "Model." "Optimization Procedure" explains the optimization algorithm used. In "Liquidation Strategies in Illiquid Markets" we study different bid-ask spread magnitudes and negative return-spread correlations and analyze their influence on the minimum capital requirement and the optimal asset allocation and liquidation strategy. "Liquidation Strategies for Different Business Lines" focuses on the optimal asset allocation and liquidation strategy for different types of insurance business. The final section concludes and puts forth additional research topics.

[1] When relaxing the strict assumptions of the Modigliani–Miller world, different ways of financing are associated with different costs (Myers, 1977; Myers and Majluf, 1984; Myers, 1984).

MODEL

In this section, we introduce the theoretical foundation for our model. This model is a selective simplification of an insurance company as a pure liquidation enterprise. Thus the focus is on cash flows only.

For this model we make a number of simplifying assumptions. The time of consideration is a 1-year period. We assume that the company underwrites the same amount and type of insurance business every year. In other words, the insurance business stays constant over time. Under this assumption the (expected) claim payments within the 1 year under consideration for business written in this year as well as in previous years approximately equals the (expected) payments for claims from this year's business, which may be settled this year or in future years. So, if we ignore the time value of money, we can substitute the claim payments within the period under consideration for the claim payments of the business written in this period.[2]

Considering the claim payments, we assume that the sequence of interoccurrence times $\{T_i, i \geq 1\}$ consists of independent random variables with an exponential distribution $Exp(\lambda)$, $\lambda > 0$. The claim payments $\{U_i, i \geq 1\}$ are independent and identically distributed and independent of the sequence $\{T_i\}$ of interoccurrence times. These assumptions lead to the classical collective risk model of actuarial risk theory, where the cumulative claim amount $\{Y_t, t \geq 0\}$ follows a compound Poisson process.[3] Using the notation $\vartheta_i = \sum_{k=1}^{i} T_k$ for the points of time, at which claim payments occur, the aggregated amount to be payed out for claims in the interval $(0, t]$ can be expressed as

$$Y_t = Y(t) = \sum_{i=1}^{\infty} U_i \mathbb{I}[\vartheta_i \leq t] = \sum_{i=1}^{N(t)} U_i, \tag{1}$$

where $\mathbb{I}(.)$ is the indicator function and $\{N(t), t \geq 0\}$ the counting process given by

$$N(t) = \sum_{i=1}^{\infty} \mathbb{I}(\vartheta_i \leq t). \tag{2}$$

We further assume that all premium payments have been received at the beginning of the year and are already invested in financial assets. Therefore, securities have to be sold for the settlement of insurance claims. With these assumptions we set the

[2] The inaccuracy we introduce here is quite small, especially for short-tailed lines of business.

[3] Note that the collective model of risk theory usually builds upon the claims per period and describes the aggregated claim amounts. Especially in long-tailed business lines, there is a significant difference between the time the claims occur and the actual cash settlement. Thinking, for example, of large product liability court claims, insurers may be able to manage the timing of the settlement to a certain extent. To abstract from this additional element of strategy for an insurance company, we model the claim payments directly using the collective model of risk theory. Our interpretation of the collective model is, thus, slightly different as the one in standard actuarial textbooks.

framework of a pure liquidation model. At the beginning of the year, the insurance company is invested in a mix of three different assets: cash, a relatively liquid asset, and an illiquid asset. The liquid and illiquid assets can be thought of as common stocks (or other risky assets), which differ in the magnitude of their bid-ask spreads. Cash earns a continuously compounded (constant) rate of return r. Therefore, the value of the cash position $S_{0,t}$ at time t can be written as

$$S_{0,t} = S_{0,0} \exp(rt), \tag{3}$$

where $S_{0,0}$ is the amount originally invested at time $t = 0$.

We model the mid-prices of the liquid and the illiquid asset as geometric Brownian motions. The mid-price of the liquid asset at time t is given by

$$S_{1,t} = S_{1,0} \exp(\mu_1 t + \sigma_1 B_{1,t}), \tag{4}$$

where $S_{1,0}$ is the amount originally invested in the asset at time $t = 0$, μ_1 and σ_1 denote the expected return and volatility and $\{B_{1,t}\}$ is a standard Brownian motion. The mid-price process of the illiquid asset follows

$$S_{2,t} = S_{2,0} \exp[\mu_2 t + \sigma_2(\rho B_{1,t} + \sqrt{1-\rho^2} B_{2,t})], \tag{5}$$

where $S_{2,0}$ is the amount invested in the asset at time $t = 0$, μ_1 and σ_1 denote the instantaneous expected return and volatility of the asset price. The parameter ρ determines the instantaneous correlation between the changes in the mid price of the liquid and the illiquid asset, and $\{B_{2,t}\}$ is a standard Brownian motion, independent of $\{B_{1,t}\}$. Note that both Brownian motions $\{B_{1,t}\}$ and $\{B_{2,t}\}$ are assumed to be independent of the aggregated claim amount distribution $\{Y_t\}$. Consequentely, $\{Y_t\}$ is also independent of $\{S_{1,t}\}$ and $\{S_{2,t}\}$. This assumption is appropriate for normal market conditions,[4] which is the case we are considering in this article.

The model of the asset prices is identical to the one used by Duffie and Ziegler (2003). However, their use of geometric Brownian motions for bid-ask spreads is not consistent with empirical findings. Biais, Hillion, and Spatt (1995) find evidence for mean reversion characteristics of bid-ask spreads. We therefore propose mean reverting square root processes for modeling bid-ask spreads. Square root processes were first introduced into the field of finance by Cox, Ingersoll, and Ross (1985) (see also Longstaff and Schwartz, 1992; Duffie and Kan, 1995). They have a convenient property: having a starting value greater than zero, square root processes never take on negative values.

We model the relative mid-to-bid spread using square root processes. The stochastic differential equation defining the mid-to-bid spread process $\{X_{1,t}, t \geq 0\}$ of the liquid asset is given by

$$dX_{1,t} = \kappa_1(\mu_3 - X_{1,t})\,dt + \sigma_3\sqrt{X_{1,t}}\,d\left(\rho_1 B_{1,t} + \sqrt{1-\rho_1^2} B_{3,t}\right), \tag{6}$$

[4] See, e.g., the empirical study of Maurer (2000, p. 251).

where μ_3 and σ_3 denote the expected return and volatility of the process, the parameter $\kappa_1 > 0$ controls the speed of the mean reversion, the parameter ρ_1 determines the instantaneous correlation between the changes in the mid-price of the asset and the spread increments and $\{B_{3,t}\}$ is a standard Brownian motion independent of $\{B_{1,t}\}$, $\{B_{2,t}\}$, and $\{Y_t\}$.[5] In analogy, the relative mid-to-bid spread $\{X_{2,t}, t \geq 0\}$ of the illiquid asset is given by the stochastic differential equation

$$dX_{2,t} = \kappa_2(\mu_4 - X_{2,t})\,dt + \sigma_4\sqrt{X_{2,t}}d\left[\rho_2(\rho B_{1,t} + \sqrt{1-\rho^2}B_{2,t}) + \sqrt{1-\rho_2^2}B_{4,t}\right], \quad (7)$$

where μ_4 and σ_4 denote the expected return and volatility of the process, the parameter $\kappa_2 > 0$ controls the speed of the mean reversion, the parameter ρ_2 determines the instantaneous correlation between the changes in the mid-price of the asset and the spread increments, and $\{B_{4,t}\}$ is a standard Brownian motion independent of $\{B_{1,t}\}$, $\{B_{2,t}\}$, $\{B_{3,t}\}$, and $\{Y_t\}$. With $\rho_i < 0$, $i = 1, 2$, the spreads are expected to increase as prices fall. The actual bid price of the illiquid asset at time t is then calculated as $S_{2,t}(1 - X_{2,t})$ and the bid price of the liquid asset as $S_{1,t}(1 - X_{1,t})$.

In our model, there are no time trends to the spread movements. Cross-correlations across different assets in changes in spreads are limited to that dependent on mid-price movements. We set the starting values of the spread processes, in line with the assumption that asset 1 is more liquid than asset 2, such that $X_{2,t} > X_{1,t} > 0$.

Within this model context, securities have to be sold for the settlement of insurance claims. Assuming that the execution of trades is only possible once a day, we face a discrete model framework. Formally the 1-year period under consideration is divided into 365, subperiods $(t_{n-1}, t_n]$, $n = 1, \ldots, 365$ and trading can take place at points of time t_n, $n = 1, \ldots, 365$.[6] Using the compound Poisson process from Equation (1), the aggregate amount to be payed out for claims C_n in period $n = 1, \ldots, 365$ is given by

$$C_n = Y(t_n) - Y(t_{n-1}), \quad n = 1, \ldots, 365. \quad (8)$$

At time $t = 0$, the insurance company starts with an overall capital of I_0 invested in a portfolio of the three assets. Let q_i, $i = 0, 1, 2$ denote the fraction of capital invested in asset i. Then $\{q_0, q_1, q_2\}$ represent the initial asset allocation of the insurer. We assume that regulatory requirements do not allow short sales or borrowing for investment purposes, so $0 \leq q_i \leq 1$, $i = 0, 1, 2$ and $\sum_{i=0}^{2} q_i = 1$. These assumptions are cemented in the legislation of many countries (see, e.g., the German Versicherungsaufsichtsgesetz and the Kapitalanlageverordnung). So at time $t = 0$, the insurance company holds

[5] Building a pure liquidation model, our spread processes $\{X_{1,t}\}$ and $\{X_{2,t}\}$ do not model the difference of ask and bid, but the difference of the mid-quote and bid. So the processes represent half spreads, both the quoted and the effective ones since these are equivalent in our simple model.

[6] Insurance claims can occur on every day during the year. Therefore we model 365 one-day periods assuming that insurance companies can trade over the counter (OTC) on days where the exchange is closed.

$\alpha_{0,0} = \frac{I_0 \cdot q_0}{S_{0,0}}$ units of cash, $\alpha_{1,0} = \frac{I_0 \cdot q_1}{S_{1,0}}$ units of the liquid asset and $\alpha_{2,0} = \frac{I_0 \cdot q_2}{S_{2,0}}$ units of the illiquid asset. Since there is no trading possibility until $t = t_1$, the asset allocation stays unchanged during the period $(t_0, t_1]$. For the rest of the article we use the notation $\alpha_{i,n}$, $i = 0, 1, 2$ representing the investment holdings in period n. The total value of the investment portfolio at point of time t_n or equivalently in period n can therefore be expressed as

$$I_n = \alpha_{0,n} S_{0,n} + \alpha_{1,n} S_{1,n} + \alpha_{2,n} S_{2,n}, \quad n = 0, \ldots, 365. \tag{9}$$

The insurance company settles claims within the same period and sells exactly as many securities as needed for paying off the claims. If there are not enough assets left, the company will be bankrupt.

In this article, we compare the two extreme liquidation strategies. First, we consider the Cash-First strategy where the insurance company liquidates cash first, then the liquid asset, and finally the illiquid asset. Second, the Cash-Last strategy is analyzed, where the company liquidates the illiquid asset first, then the liquid asset, and finally their cash holdings. However, the application of this method is not limited to these two liquidation strategies but can be applied analogously to a set of any liquidation strategies. Based on the literature, we consider these two strategies to be the extremes, leading us to believe that an optimal liquidation strategy can be explained by properties of the limiting cases.

The liquidation algorithms of the Cash-First and Cash-Last liquidation strategies are provided in full detail in the Appendix. The output of the liquidation process is the number of units of each asset liquidated in the actual period. Let $\lambda_{i,n}$ denote this number for asset $i = 0, 1, 2$ and period n. The holdings of the three assets in the following period $n + 1$ are therefore given by

$$\alpha_{i,n+1} = \alpha_{i,n} - \lambda_{i,n}, \quad n = 1, \ldots, 365. \tag{10}$$

OPTIMIZATION PROCEDURE

Our goal is to compare liquidation strategies under different real world conditions facing an insurance company. As a matter of definition, optimality can only be achieved according to an *a priori* given criterion with a set of alternatives. As explained in the previous section, we have chosen to compare the two liquidation strategies, Cash-First and Cash-Last, using as the optimality criterion the minimum required capital for guaranteeing a predetermined safety level.

In this article, ruin probability means the probability that the company's capital is eliminated before the end of the 1 year, resulting in the insurance company not meeting its subsequent financial commitments. Let us assume that by regulatory rules (or by management decision) the insurance company is required to keep its ruin probability smaller than or equal to 2 percent.[7] The insurance company then uses the

[7] Regulatory authorities as well as the management of an insurance company normally aim at much smaller ruin probabilities (e.g., 0.1 percent or 0.01 percent). However, to get enough ruin scenarios for stable results in our simulation study, we set the ruin probability to 2 percent.

minimum capital required to ensure a ruin probability smaller than or equal to 2 percent as objective function for comparing the different liquidation strategies. The strategy that needs the least capital and, hence, enables the company to provide the target level of safety at the lowest costs will be designated optimal. Such an objective function ensures a high firm profitability, and, in the presence of positive financial distress costs, such a decision criterion is consistent with the objective of maximizing the value of the firm's equity.

Our criterion for comparing liquidation strategies differs from those prevalent in standard financial liquidation models (see, e.g., Bertsimas and Lo, 1998; Almgren and Chriss, 2000), in that a particular emphasis is placed on matching the assets after a trade to the externally imposed liability structure. Standard liquidation models place a premium on minimizing trading costs, a goal which can be in conflict with the risk emphasis of an insurance company. This method considers the costs as well as the changes in risk profile of the assets, by including in the model the risk aspects of an insurance company.

First, we show that the optimal liquidation strategy depends on the basic asset allocation of the insurance company at the beginning of the year. We do that using Monte Carlo simulations. An optimization, therefore, has to take the liquidation strategy and the basic asset allocation into account simultaneously.

We wish to confirm that the optimal liquidation strategy depends on the initial asset allocation of the insurance company. It is sufficient to show that there are initial asset allocations where the Cash-First liquidation strategy is optimal and initial asset allocations where Cash-Last is the optimal strategy for the same claim distribution. We therefore compare the liquidation strategies for different asset allocations according to the insurance company's objective function.

In the following analysis the initial values of the price processes of all three assets are normalized to one unit ($S_{0,0} = S_{1,0} = S_{2,0} = 1$). The interest rate is set to $r = 0.05$. The parameter values of the geometric Brownian motion governing the mid-price process of the liquid asset are $\mu_1 = 0.08$ and $\sigma_1 = 0.2$. This implies that the long-term mean rate of return of asset 1 is around 8 percent with a volatility of 20 percent. The corresponding parameters of the illiquid asset's mid-price process are $\mu_2 = 0.09$ and $\sigma_2 = 0.2$. Note that the two assets have the same volatility. We equated them to highlight the effect of liquidity. To allow some diversification by investing in both the liquid and the illiquid asset, we set the correlation between the two asset price processes to $\rho = 0.7$. We start with a base case assuming that there is perfect liquidity or equivalently both spread processes $\{X_{1,t}\} = \{X_{2,t}\} = 0$.

For our first case, we use a property and casualty insurance company with an industrial fire insurance business line. In his actuarial text book, Mack (2002) estimates parameters for different claim size distributions using a sample data set from industrial fire insurance (see p. 87 ff.). Following this example, we suppose that the claim size measured in US\$1,000 follows a lognormal distribution $U_i \sim LN(a, b)$ with shape parameter $b = 1.96$. For the scale parameter we set the value $a = 1.16$. In our model the interoccurrence times between claims measured in days are exponentially distributed $T_i \sim \mathrm{Exp}(\lambda)$. We use the parameter value $\lambda = 8$. This implies that in the long run there are on average 8 claims per day or 2,920 per year.

Each of the two liquidation strategies is implemented as a Monte Carlo simulation procedure demanding the initial asset allocation $\{q_0, q_1, q_2\}$ as input parameters and delivering the (minimum) required capital I_0 as output. We conduct 10,000 independent Monte Carlo simulations using the parameters specified above. Each of the generated scenarios consists of asset returns and the aggregated claim amounts for every period of our discrete model. Please note that the same sets of scenarios are used for both liquidation strategies. The rationale is that, on the one hand, we wish to compare the two liquidation strategies and on the other hand, analyze the influence of different spread parameters, but do not wish the results to be affected by differences in the simulated paths. The complete algorithm is presented in the Appendix.

Using a grid of possible asset allocations, we see that there are asset allocations $\{q_0, q_1, q_2\}$ for which the Cash-First liquidation strategy requires less capital and others for which the Cash-Last strategy is optimal (see Figure 1). Therefore the optimal liquidation strategy depends on the initial asset allocation of the insurance company, and an optimization has to focus on both the liquidation strategy and the basic asset allocation and optimize them simultaneously. So this article analyzes not only the liquidation strategies of insurance companies but also their asset allocation decision in a simple and very intuitive simulation model.

Our analysis technique for the simultaneous optimization of the asset allocation and liquidation strategy is straightforward. As we compare only two liquidation strategies, we can optimize the initial asset allocation for each strategy independently. We then select the better of the two solutions. Our overall result is a solution set with an optimal asset allocation and liquidation strategy.

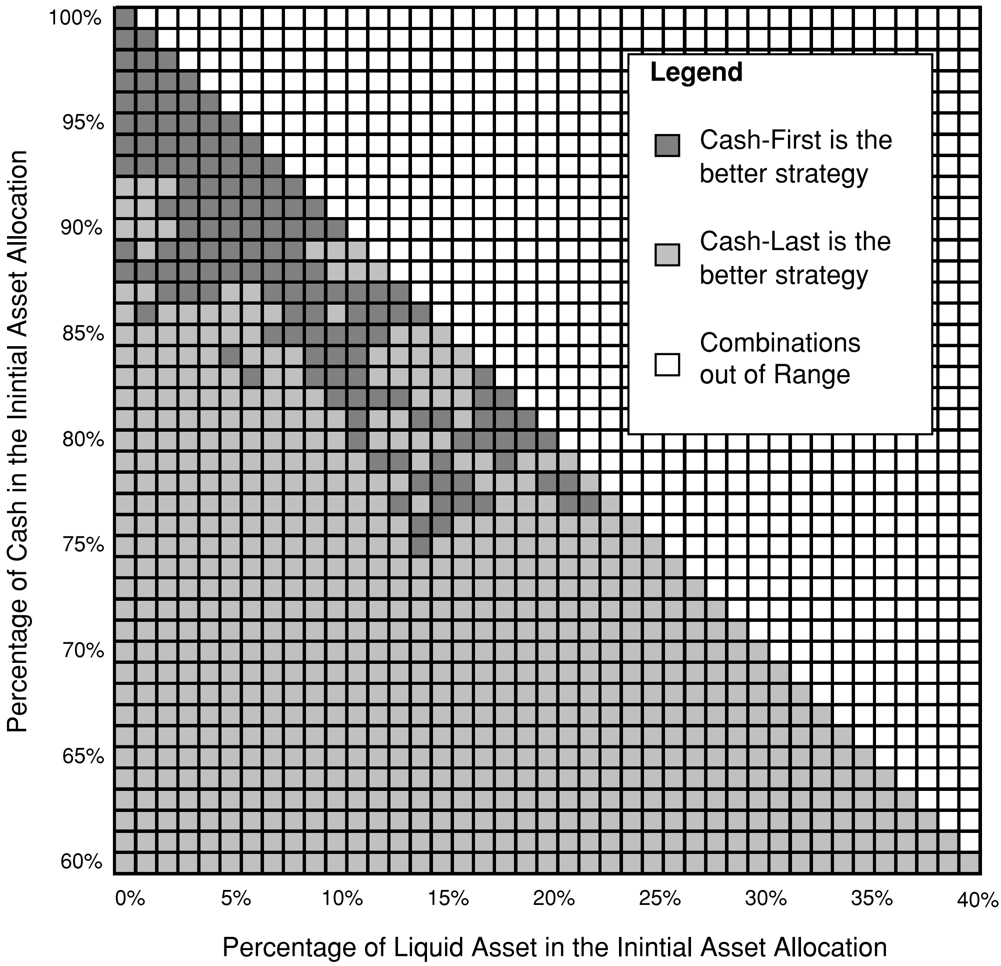
Technically speaking, the simulation algorithms of both, the Cash-First and the Cash-Last liquidation strategies, are functions $f_l(q_0, q_1, q_2)$, where the index l represents the liquidation strategy. Thus, for each liquidation strategy l the insurance company wants to find the optimal asset allocation $\{q_0^*, q_1^*, q_2^*\}$ minimizing $f_l(q_0, q_1, q_2)$. Since q_0, q_1, and q_2 are percentage numbers representing the holdings of cash, the liquid asset and the illiquid asset, respectively, the sum of these numbers has to be one. Short sales are not allowed in our model, thus $0 \leq q_i \leq 1$, $i = 0, 1, 2$. This leads to the following constraint optimization problem:

$$A(l)\begin{cases} f(q_0, q_1, q_2) \to \min \\ \sum_{i=0}^{2} q_i = 1 \\ 0 \leq q_i \leq 1, i = 0, 1, 2 \end{cases}$$

Using the equivalent formulation $|q_0| + |q_1| + |q_2| \leq 1$ and $q_2 = 1 - q_0 - q_1$ for the constraints, we transform the constraint optimization programm $A(l)$ into the two-dimensional unconstrained one given by

$$B(l)\begin{cases} f(q_0, q_1, 1 - q_0 - q_1) + c(q_0, q_1, 1 - q_0 - q_1) \to \min \\ q_0, q_1 \in \mathbb{R} \end{cases}$$

FIGURE 1
Comparison of Liquidation Strategies for Different Asset Allocations



Note: Each square represents an asset allocation $\{q_0, q_1, q_2\}$. The percentage of cash q_0 is on the y-axis, the percentage of the liquid asset q_1 is on the x-axis and the fraction of the illiquid asset is determined by $q_2 = 1 - q_0 - q_1$.

with the penalty cost function

$$c(q_0, q_1, q_2) = \begin{cases} \eta(|q_0| + |q_1| + |q_2| - 1) & \text{if } |q_0| + |q_1| + |q_2| > 1 \\ 0 & \text{else} \end{cases}$$

where the positive constant $\eta > 0$ reflects the intensity of the penalty costs.

This alternative formulation allows us to use the multidimensional downhill simplex method[8] according to Nelder and Mead (1965) for solving the optimization

[8] A sample implementation of the downhill simplex method can be found in Press et al. (2002).

numerically.[9] The downhill simplex method has one big advantage, it requires only function evaluations and no derivatives. This property makes it very stable and suitable for optimizing a simulation procedure with three input parameters.[10]

After optimizing the initial asset allocation for each liquidation strategy independently, the insurance company then selects the best of the solutions. Formally, it selects the liquidation strategy l^* which minimizes $\min_{l \in L} B(l)$. The overall solution set includes the optimal liquidation strategy l^* as well as the corresponding optimal asset allocation $\{q_0^*, q_1^*, q_2^*\}$.

In summary, this section consists of three parts. Initially, we determine the optimization criterion. Then we show that the liquidation strategy depends on the initial asset allocation, defining the basic parameters and completing the computations for the capital required to meet the risk requirements. Finally, we formulate a simultaneous optimization procedure for a solution set.

LIQUIDATION STRATEGIES IN ILLIQUID MARKETS

In this section we analyze the asset allocation and liquidation strategy of an insurance company in illiquid markets. Using the model framework introduced in the last two sections, we focus on the influence of bid-ask spread behavior on relevant variables for the management of insurance companies. More precisely, we study different parameter settings for the spread magnitudes and correlations between spreads and asset returns, and analyze their influence on the (minimum) required capital and the optimal solution set of asset allocation and liquidation strategy.

Concerning the parameter values of the spread processes (see Equations (6) and (7)) we follow Duffie and Ziegler (2003) and analyze four cases, each with four different settings. The four cases differ in the (expected) bid-ask spread magnitude. The base case has no spreads, so the expected values of the processes as well as their starting values are zero ($\mu_3 = \mu_4 = X_{1,0} = X_{2,0} = 0$). Case 2 sets the expected values of the spread processes to $\mu_3 = 0.1$ percent for the liquid asset and $\mu_4 = 0.5$ percent for the illiquid asset, reflecting low spreads. Case 3 uses $\mu_3 = 0.2$ percent and $\mu_4 = 1$ percent, representing medium size spreads, and case 4 uses $\mu_3 = 0.5$ percent and $\mu_4 = 2.5$ percent, indicating high spreads. The starting values of the spread processes are assumed to equal their expected values, and the two mean reversion parameters are set to $\kappa_1 = \kappa_2 = 5$ in all four cases. Reflecting the idea that the maximum value of the

[9] For solving this optimization problem, we first perform a grid search, providing us with possible starting points for an optimization routine. We then use the downhill simplex method to pin down the solution. The grid search indicates that the objective function has multiple local minimums. We perform the minimization procedure with different sets of starting values including these locally minimal grid points. In cases where there are multiple local minimaums, we simply pick the smallest. Even though we cannot test directly whether we are obtaining a local or a global minimum, we can perform some robustness checks. We use a narrower grid with ten times as many points per dimension resulting in 100 times as many grid points. Taking the locally minimal points from this narrower grid as starting values for the optimization procedure leads to the same global minimum. Therefore our obtained results seem to be stable.

[10] Note that in exchange for this stability one has to accept some inefficiency concerning the number of function evaluations.

TABLE 1
Optimal Asset Allocations given the Cash-Last Liquidation Strategy

Spread Behavior	Required Capital (in US$1,000)	Asset Allocation (in %) q_0	q_1	q_2
A. No Spreads: $\mu_3 = \mu_4 = 0$				
Constant spreads	80,256	80.117	5.705	14.179
B. Low Spreads: $\mu_3 = 0.1\%$ *and* $\mu_4 = 0.5\%$				
Constant spreads	80,317	80.055	5.706	14.239
Variable spreads				
$\rho_1 = \rho_2 = 0$	80,336	80.224	7.302	12.474
$\rho_1 = \rho_2 = -0.5$	80,335	79.814	5.984	14.202
$\rho_1 = \rho_2 = -0.8$	80,325	79.969	5.798	14.233
C. Medium Spreads: $\mu_3 = 0.2\%$ *and* $\mu_4 = 1\%$				
Constant spreads	80,361	86.386	10.075	3.539
Variable spreads				
$\rho_1 = \rho_2 = 0$	80,363	86.400	9.986	3.614
$\rho_1 = \rho_2 = -0.5$	80,363	86.455	10.014	3.531
$\rho_1 = \rho_2 = -0.8$	80,362	86.481	10.047	3.472
D. High Spreads: $\mu_3 = 0.5\%$ *and* $\mu_4 = 2.5\%$				
Constant spreads	80,423	85.466	14.534	0.000
Variable spreads				
$\rho_1 = \rho_2 = 0$	80,426	85.529	14.471	0.000
$\rho_1 = \rho_2 = -0.5$	80,423	85.636	14.364	0.000
$\rho_1 = \rho_2 = -0.8$	80,421	85.696	14.304	0.000

spread processes over all trading times t_n should not exceed two times its expected value in most of the scenarios, we set different variances for the three cases. For the high spread case, we set $\sigma_3 = 0.05$ and $\sigma_4 = 0.125$, for the medium spread case we set $\sigma_3 = 0.032$ and $\sigma_4 = 0.079$, and in the low spread case we set $\sigma_3 = 0.0227$ and $\sigma_4 = 0.0559$. Then $\{X_{1,t}\}$ violates our criterion in approximately 0.6 percent and $\{X_{2,t}\}$ in approximately 1.6 percent of the scenarios.[11]

For each of the above cases, we study four settings. First we consider constant spreads, using these as a reference case. The three stochastic settings vary in the level of correlation between the spread and asset price processes. Negative correlations were chosen to show the phenomenon of worsening sell conditions correlating with falling market value for an asset. We look at stochastic spreads uncorrelated with asset prices ($\rho_1 = \rho_2 = 0$), stochastic spreads moderately negatively correlated with asset prices ($\rho_1 = \rho_2 = -0.5$), and stochastic spreads highly negatively correlated with asset prices ($\rho_1 = \rho_2 = -0.8$).

Following the optimization procedure outlined in the last section, we analyze the two liquidation strategies separately. The results regarding the Cash-Last strategy are presented in Table 1, and those relating to the Cash-First strategy in Table 2.

[11] The parameter values of the spread variances were determined applying 100,000 independent Monte Carlo simulations. These scenarios are not the one used for the optimization procedure.

TABLE 2
Optimal Asset Allocations Given the Cash-First Liquidation Strategy

Spread Behavior	Required Capital (in US$1,000)	Asset Allocation (in %)		
		q_0	q_1	q_2
A. No Spreads: $\mu_3 = \mu_4 = 0$				
Constant spreads	80,125	94.191	2.941	2.869
B. Low Spreads: $\mu_3 = 0.1\%$ *and* $\mu_4 = 0.5\%$				
Constant spreads	80,139	94.174	2.943	2.882
Variable spreads				
$\rho_1 = \rho_2 = 0$	80,140	94.183	2.947	2.870
$\rho_1 = \rho_2 = -0.5$	80,141	94.190	2.942	2.868
$\rho_1 = \rho_2 = -0.8$	80,141	94.192	2.938	2.869
C. Medium Spreads: $\mu_3 = 0.2\%$ *and* $\mu_4 = 1\%$				
Constant spreads	80,153	94.158	2.946	2.897
Variable spreads				
$\rho_1 = \rho_2 = 0$	80,155	94.175	2.953	2.872
$\rho_1 = \rho_2 = -0.5$	80,157	94.189	2.943	2.868
$\rho_1 = \rho_2 = -0.8$	80,157	94.194	2.935	2.870
D. High Spreads: $\mu_3 = 0.5\%$ *and* $\mu_4 = 2.5\%$				
Constant spreads	80,188	95.208	4.792	0.000
Variable spreads				
$\rho_1 = \rho_2 = 0$	80,184	95.206	4.794	0.000
$\rho_1 = \rho_2 = -0.5$	80,184	95.218	4.781	0.000
$\rho_1 = \rho_2 = -0.8$	80,185	95.228	4.772	0.000

Let us take a closer look at the minimum required capital first. With both liquidation strategies the expected values of the spread processes have a strong influence on the minimum required capital, no matter whether the spreads are constant or stochastic. If they are stochastic, however, the degree of correlation between the spread processes and the asset price processes does not have a significant influence on the amount of capital required. The minimum required capital for the different correlation settings of the same spread magnitude lies approximately at the same level in all examined cases. There are only minor differences and these follow no systematic pattern. These findings can be explained using the following argumentation. The times of trade executions and the amounts liquidated by the insurance company do only depend on the insurance claims, which are assumed to be stochastically independent of the asset returns and spread dynamics. As the insurance company settles the claims immediately, it has to liquidate financial assets nearly every day paying the bid spreads. Over the 1-year time horizon the average bid spread payed is approximately the same, no matter which stochastic model is used. So the only parameter that counts is the expected spread width. It is evident that spread width does have an influence because it can be interpreted as costs which reduce the expected asset returns.

The same argumentation holds for the optimal, initial asset allocation. Let us focus first on the optimal asset allocation given the Cash-Last liquidation strategy (see Table 1). We find three blocks of different asset allocations. The reference case without spreads and the case with the lowest spreads ($\mu_3 = 0.1$ percent and $\mu_4 = 0.5$ percent) have about the same asset allocation with approximately 80 percent invested in cash, 6 percent in the liquid asset, and 14 percent in the illiquid asset. The expected return of the illiquid asset is 1 percent higher than the one of the illiquid asset. But even without spreads both assets are included in the asset allocation. This can be explained with the diversification effect. Remember, the two asset returns are correlated with $\rho = 0.7$. The increase of spread width to the first level does not change the optimal asset allocation. The next step in spread magnitude up to expected values of $\mu_3 = 0.2$ percent for the liquid asset and $\mu_4 = 1$ percent for the illiquid one does. The optimal asset allocation is now approximately 86 percent investments in cash, 10 percent in the liquid asset, and 4 percent in illiquid asset. We see a shift of invested capital from the illiquid asset towards the liquid one and cash. The bigger spread of the illiquid asset starts to offset the higher expected return of this asset. The shift of capital toward more liquid forms of investments proceeds for the biggest spread level. In the latter case, the optimal asset allocation only includes investments in cash and the liquid asset ($q_0 \approx 86$ percent, $q_1 \approx 14$ percent, and $q_2 \approx 0$).

Comparing the optimal asset allocation of the two liquidation strategies (see Tables 1 and 2), the first thing we notice is the large percentage of assets that are in cash. This can be partly explained through the risk-limiting restrictions in the model. Our insurance company cannot risk investing too much in the stock market. The value of the stocks might have decreased when the money is needed to pay the claims. Three more comments on this result. First, only those assets necessary to cover outstanding obligations are simulated in the model. The surplus, if any, does not fall under the legislative and / or strategic dictates pertaining to ruin possibility, and might be invested less restrictively therefore increasing the total stock exposure of the company. Second, it is important to understand that this high cash percentage is a fraction of the total yearly income. If we would like to compare the model numbers with an insurance companies balance sheet, we would have to sum the companies total yearly premium and investment income (as well as its total assets) and calculate the asset holdings as a percentage of this total yearly income. As much of this income is expected to be distributed to claimants, it is evident that it must be kept as cash at hand, and cannot be invested in long-term investments like stocks. Third, in this model, a single product, was modeled. An insurance company will have any number of products, allowing it to take advantage of the lower aggregate risk of a well diversified portfolio. This would *ceteris paribus* result in a lower required capital. Compared with data from the German insurance industry, these low stock percentages are standard. Thus, the large cash percentage is reasonable both in the model world and the real world.

The next point is an absolute difference in the cash positions for the different strategies. For all sets of parameters the optimal asset allocation given the Cash-Last strategy has a much smaller portion invested in cash und a bigger portion in the two risky assets. These financial assets are exposed to market risk as well as liquidity risk. Since the Cash-Last strategy liquidates the illiquid asset first and the liquid asset second, it reduces the risk exposure of the insurance company due to investments relatively soon. Therefore, the initial asset allocation can include a bigger portion of these assets without undermining the safety level of the whole company.

Examining the optimal asset allocation given the Cash-First strategy, we can distinguish two blocks of different asset allocations. The cases with none, low, and medium spreads have about the same asset allocation with approximately 94 percent invested in cash and 3 percent invested in the liquid and the illiquid asset. In the case with the widest spreads the optimal asset allocation consists of approximately 95 percent investments in cash, 5 percent in the liquid asset, and none in the illiquid asset. So for the Cash-First strategy, only a big increase of spreads has an influence on the optimal asset allocation. This phenomenon can be explained by the characteristics of the type of insurance business assumed. The aggregated claim amount distribution of an industrial fire insurance business line has an essential part of its probability mass on the tail. This means that high aggregated claim amounts occur with a (relatively) high probability, which leads to a (relatively) high required capital to ensure a specific safety level of the insurance company. But in a lot of scenarios, a significant part of the invested capital is not needed to pay of the claims. Because the Cash-First strategy sells the liquid and the illiquid assets last, these positions are not liquidated in the mentioned cases. Therefore, the spreads have to increase substantially before they have an influence on the optimal asset allocation.

Let us now turn to the overall result. Comparing two asset allocations given a liquidation strategy for each parameter setting, and using the minimum required capital as criterion, we see that the required capital applying the Cash-First strategy is always lower than the one applying the Cash-Last strategy. So the solution set of Cash-First liquidation strategy and the corresponding optimal initial asset allocation can be considered optimal.

In this section, we have analyzed the case of industrial fire insurance. We applied the optimization procedure introduced in the previous section and varied the spread parameter values to show the influence of the spread parameter values on the optimal solution set.

LIQUIDATION STRATEGIES FOR DIFFERENT BUSINESS LINES

The previous analysis considered the case of a property and casualty insurance company with an industrial fire insurance business line. We now compare this type of business modeled with a lognormal claim amount distribution with a standard high frequency, low severity business line, for example, automobile insurance. Assuming that the claim amounts follow a gamma distribution our second line of business is much less "dangerous."

Following the example of Kaufmann, Gadmer, and Klett (2001, p. 241), we suppose that the claim size measured in U.S. dollars follows a gamma distribution $U_i \sim \Gamma(a, b)$ with shape parameters $a = 9.091$ and scale parameter $b = 242$. In our model the inter-occurrence times between claims measured in days are exponentially distributed $T_i \sim \text{Exp}(\lambda)$. We now use the parameter value $\lambda = 16$. This implies that in the long run there are on average 16 claims per day or 5,840 per year. This results in smaller and more frequent claims. All other parameters take values as specified before. Our goal is the comparison of the two different business lines. Therefore, we apply exactly the same procedure to the automobile case as we did for the fire insurance case and compare the two results.

TABLE 3
Cash-Last Results for a High-Frequency, Low-Severity Business Line

Spread Behavior	Required Capital (in US$1,000)	Asset Allocation (in %)		
		q_0	q_1	q_2
A. No Spreads: $\mu_3 = \mu_4 = 0$				
Constant spreads	12,883.52	97.505	0.663	1.833
B. Low Spreads: $\mu_3 = 0.1\%$ *and* $\mu_4 = 0.5\%$				
Constant spreads	12,884.78	97.480	0.734	1.786
Variable spreads				
$\rho_1 = \rho_2 = 0$	12,884.78	97.458	0.757	1.785
$\rho_1 = \rho_2 = -0.5$	12,884.77	97.466	0.749	1.785
$\rho_1 = \rho_2 = -0.8$	12,884.77	97.475	0.740	1.785
C. Medium Spreads: $\mu_3 = 0.2\%$ *and* $\mu_4 = 1\%$				
Constant spreads	12,885.13	100.000	0.000	0.000
Variable spreads				
$\rho_1 = \rho_2 = 0$	12,885.13	100.000	0.000	0.000
$\rho_1 = \rho_2 = -0.5$	12,885.13	100.000	0.000	0.000
$\rho_1 = \rho_2 = -0.8$	12,885.13	100.000	0.000	0.000
D. High Spreads: $\mu_3 = 0.5\%$ *and* $\mu_4 = 2.5\%$				
Constant spreads	12,885.13	100.000	0.000	0.000
Variable spreads				
$\rho_1 = \rho_2 = 0$	12,885.13	100.000	0.000	0.000
$\rho_1 = \rho_2 = -0.5$	12,885.13	100.000	0.000	0.000
$\rho_1 = \rho_2 = -0.8$	12,885.13	100.000	0.000	0.000

The optimal asset allocation and minimum required capital given the Cash-Last liquidation strategy are presented in Table 3, and the optima given the Cash-First strategy in Table 4. These results show the same characteristics with respect to the different spread parameter settings as in the lognormal case. So we abridge the analysis and focus on differences to the latter case only.

Let us focus first on the optimal asset allocation given the Cash-Last liquidation strategy (see Table 3). We find two blocks of different asset allocations. The reference case without spreads and the case with the lowest spreads ($\mu_3 = 0.1$ percent and $\mu_4 = 0.5$ percent) have the same asset allocation with approximately 97.5 percent invested in cash, 0.7 percent in the liquid asset, and 1.8 percent in the illiquid asset. Remember, the two asset returns are correlated with $\rho = 0.7$. The increase of spread width to the first level does not change the optimal asset allocation. The next step in spread magnitude up to expected values of $\mu_3 = 0.2$ percent for the liquid asset and $\mu_4 = 1$ percent for the illiquid one does. For this case, it is optimal to invest the whole capital in cash. The disadvantages of wider spreads for assets 1 and 2 outweigh the advantages of higher expected returns, so none of them is included in the optimal asset allocation. This leads *de facto* to a Cash-First liquidation strategy because there is only cash there to liquidate.

TABLE 4
Cash-First Results for a High-Frequency, Low-Severity Business Line

Spread Behavior	Required Capital (in US$1,000)	Asset Allocation (in %)		
		q_0	q_1	q_2
A. No Spreads: $\mu_3 = \mu_4 = 0$				
Constant spreads	12,877.82	99.050	0.603	0.348
B. Low Spreads: $\mu_3 = 0.1\%$ *and* $\mu_4 = 0.5\%$				
Constant spreads	12,878.02	99.118	0.724	0.157
Variable spreads				
$\rho_1 = \rho_2 = 0$	12,877.93	99.103	0.690	0.207
$\rho_1 = \rho_2 = -0.5$	12,878.03	99.120	0.722	0.157
$\rho_1 = \rho_2 = -0.8$	12,878.05	99.120	0.722	0.158
C. Medium Spreads: $\mu_3 = 0.2\%$ *and* $\mu_4 = 1\%$				
Constant spreads	12,878.22	99.117	0.725	0.158
Variable spreads				
$\rho_1 = \rho_2 = 0$	12,878.17	99.101	0.691	0.207
$\rho_1 = \rho_2 = -0.5$	12,878.25	99.104	0.692	0.204
$\rho_1 = \rho_2 = -0.8$	12,878.29	99.106	0.693	0.201
D. High Spreads: $\mu_3 = 0.5\%$ *and* $\mu_4 = 2.5\%$				
Constant spreads	12,878.81	99.112	0.727	0.161
Variable spreads				
$\rho_1 = \rho_2 = 0$	12,878.82	99.111	0.721	0.167
$\rho_1 = \rho_2 = -0.5$	12,878.92	99.117	0.719	0.163
$\rho_1 = \rho_2 = -0.8$	12,878.99	99.123	0.723	0.155

Examining the optimal asset allocation given the Cash-First strategy (see Table 4), we can distinguish two blocks of asset allocations, which differ only slightly from each other. The base cases without spreads invests approximately 99.1 percent in cash, 0.6 percent in the liquid asset, and 0.3 percent in the illiquid asset. In all other cases with positive spreads the optimal asset allocation consists of approximately 99.1 percent investments in cash, 0.7 percent in the liquid asset, and 0.2 percent in the illiquid one.

When comparing the optimal asset allocation of the two liquidation strategies, we have to distinguish between the base case without spreads and the one with the lowest positive spreads on the on side, and the cases with higher spreads on the other side. For the first two cases the optimal asset allocation given the Cash-Last strategy has a smaller portion invested in cash und a bigger portion in the two risky assets. We saw the same result in the analysis with lognormal claim sizes and can use the same argumentation explaining it here. Since the Cash-Last strategy liquidates the two risky assets first, it reduces the risk exposure of the insurance company due to investments faster than the Cash-First strategy. So the initial asset allocation can include a bigger portion of these assets without undermining the safety level of the insurance company. On the other side, liquidating these asset also means paying the spreads (almost) surely. If the spreads are so high that they outweigh the advantages of higher expected returns, the company should not invest in the corresponding assets.

Applying the Cash-First strategy the two financial assets are sold last, and in many scenarios not all of the invested capital is needed to pay off the claims. Therefore, the spreads have less influence on the optimal asset allocation. This explains why for high spreads the optimal asset allocation given the Cash-Last strategy has a bigger portion invested in cash than the one given the Cash-First strategy.

For all parameter settings, the overall result is the solution set of Cash-First liquidation strategy and the corresponding optimal, initial asset allocation, as the minimum required capital given the Cash-First strategy is always lower as in the Cash-Last case. This gives us the same result as in the model with lognormal claim sizes. Let us now focus on the optimal asset allocations. In the previously examined lognormal case, the insurance company invests approximately 94 percent to 95 percent in cash and the remaining 5 percent to 6 percent in the two financial assets, whereas in the gamma distribution case the insurance company invests over 99 percent in cash and only less than 1 percent in the two risky assets. The discussion of the high cash percentage can be found in the previous section.

Summarizing this result in other words: More dangerous business lines, for example, industrial fire insurance, can invest a bigger portion in the stock market than high-frequency, low-severity business lines, for example, automobile insurance. An insurance line with frequent claim payments and little deviation in the severity of the claims cannot risk to invest a lot of money in volatile stocks. The stocks might have lost some of its value, when the money is needed. And the money will be needed with a very high probability. But insurance lines with low-frequency, high-severity risks do not need a large portion of the invested money most of the time. In this case the expected additional profit from investing in stocks is considerably higher than the expected loss from selling stocks in bad market conditions. So these companies should invest more money in the stock market than the ones with low-risk business lines.

The optimal solution set of Cash-First liquidation strategy and corresponding asset allocation is in line with the literature. It can be interpreted as the realization of a cash-flow matching strategy, which Elten and Gruber (1992) proved to be ideal for hedging the liabilities of an institutional investor. Our simple model helps to develop an intuitive understanding of asset allocation decisions for insurance companies considering their liabilities.

CONCLUSION

In this article, we study different liquidation strategies and asset allocation decisions of property and casualty insurance companies. We combine the dynamic liquidation models from the finance literature with the standard actuarial risk model and propose a simple and very intuitive cash-flow model of an insurance company. The objective function is defined not just as an accumulation of transaction costs but includes the limitations on asset risk set by legislative and strategic decisions. Within this framework we show that the best liquidation strategy depends on the initial asset allocation, and strategy and asset allocation should be optimized simultaneously.

Comparing the two extreme liquidation strategies, Cash-First and Cash-Last, we find that in all examined cases the Cash-First strategy is dominant. The corresponding asset allocation in the solution set consists of a huge cash holding ($>$ 94 percent) and only minor holdings of the two securities. This result can be interpreted as the realization of the cash-flow matching strategy.

We then analyze the influence of different bid-ask spread magnitudes and negative return-spread correlations on the minimum capital requirement and the optimal asset allocation and liquidation strategy. We present evidence that the stochastic characteristics of the spread processes do not influence the model output, whereas the expected spread width does. This result is due to the fact that the aggregated claim amount process and the asset price processes are assumed to be stochastically independent. This is an appropriate assumption for normal market conditions. Extreme events are not covered in this article.

Furthermore, we compare the optimal asset allocation and liquidation strategy for different types of insurance business. We find that more dangerous business lines, example, industrial fire insurance, can invest a bigger portion in the stock market, than can high-frequency, low-severity business lines, for example, automobile insurance.

Overall, the cash-flow-based liquidation model of a property and casualty insurance company proposed in this article allows economic interpretations of the optimal asset allocation and liquidation decisions. However, the model only covers normal market conditions. Extreme events, where one would expect tail dependencies between the aggregated claim process and asset price processes, are left for future research.

APPENDIX

CASH-FIRST LIQUIDATION ALGORITHM

Liquidation of assets can take place at the end of every period $(t_{n-1}, t_n]$, $n = 1, \ldots, 365$. For simplicity, we refer to points of time t_n using indices n. The following algorithm calculates the number of units of each asset liquidated in period n and is applied to every period.

If $\alpha_{0,n}\, S_{0,n} \geq C_n$, the cash position will be sufficient to cover the claim payments. The cash holding is reduced by $\lambda_{0,n}$ and the two financial assets stay untouched:

$$\begin{aligned}\lambda_{0,n} &= \tfrac{C_n}{S_{0,n}}\\ \lambda_{1,n} &= 0\\ \lambda_{2,n} &= 0.\end{aligned}$$

Else, if $\alpha_{0,n}\, S_{0,n} + \alpha_{1,n}\, S_{1,n}(1 - X_{1,n}) \geq C_n$, the cash position and the holding of the liquid asset together will be sufficient to cover the claims. The cash position is liquidated completely and the holding of the liquid asset will be reduced by $\lambda_{1,n}$:

$$\begin{aligned}\lambda_{0,n} &= \alpha_{0,n}\\ \lambda_{1,n} &= \tfrac{C_n - \alpha_{0,n} S_{0,n}}{S_{1,n}(1 - X_{1,n})}\\ \lambda_{2,n} &= 0.\end{aligned}$$

Else, all three positions are needed to pay for outstanding claims. The cash position and the holding of the liquid asset are liquidated completely and the position in the illiquid asset is reduced by $\lambda_{2,n}$:

$$\begin{aligned}\lambda_{0,n} &= \alpha_{0,n}\\ \lambda_{1,n} &= \alpha_{1,n}\\ \lambda_{2,n} &= \tfrac{C_n - \alpha_{0,n} S_{0,n} - \alpha_{1,n} S_{1,n}(1-X_{1,n})}{S_{2,n}(1-X_{2,n})}.\end{aligned}$$

If $\lambda_{2,n} > \alpha_{2,n}$, there will not be enough assets left for liquidation. The insurance company cannot settle all outstanding claims and therefore is bankrupt.

CASH-LAST LIQUIDATION ALGORITHM

Liquidation of assets can take place at the end of every period $(t_{n-1}, t_n]$, $n = 1, \dots, 365$. For simplicity, we refer to points of time t_n using indices n. The following algorithm calculates the number of units of each asset liquidated in period n and is applied to every period.

If $\alpha_{2,n}\ S_{2,n}(1 - X_{2,n}) \geq C_n$, the position in the illiquid asset will be sufficient to cover the claim payments. This position is reduced by $\lambda_{2,n}$ and the two holdings of cash and the liquid asset stay unchanged:

$$\begin{aligned}\lambda_{0,n} &= 0\\ \lambda_{1,n} &= 0\\ \lambda_{2,n} &= \tfrac{C_n}{S_{2,n}(1-X_{2,n})}.\end{aligned}$$

Else, if $\alpha_{2,n}\ S_{2,n}(1 - X_{2,n}) + \alpha_{1,n}\ S_{1,n}(1 - X_{1,n}) \geq C_n$, the holding of the illiquid and the liquid asset together will be sufficient to cover the claims. The position in the illiquid asset is liquidated completely and the holding of the liquid asset will be reduced by $\lambda_{1,n}$:

$$\begin{aligned}\lambda_{0,n} &= 0\\ \lambda_{1,n} &= \tfrac{C_n - \alpha_{2,n} S_{2,n}(1-X_{2,n})}{S_{1,n}(1-X_{1,n})}\\ \lambda_{2,n} &= \alpha_{2,n}.\end{aligned}$$

Else, all three positions are needed to pay for outstanding claims. The positions in the illiquid and the liquid asset are liquidated completely, and the cash position is reduced by $\lambda_{2,n}$:

$$\begin{aligned}\lambda_{0,n} &= \tfrac{C_n - \alpha_{2,n} S_{2,n}(1-X_{2,n}) - \alpha_{1,n} S_{1,n}(1-X_{1,n})}{S_{0,n}}\\ \lambda_{1,n} &= \alpha_{1,n}\\ \lambda_{2,n} &= \alpha_{2,n}.\end{aligned}$$

If $\lambda_{0,n} > \alpha_{2,n}$, there will not be enough assets left for liquidation. The insurance company cannot settle all outstanding claims and therefore is bankrupt.

SIMULATION ALGORITHM

The simulation procedure requires the liquidation strategy, the spread parameter values and an initial asset allocation as input parameters. The output is the (minimum) required capital to assure a ruin probability less than or equal to 2 percent.

1. Generate 10,000 independent scenarios, each consisting of asset returns, spread dynamics, and the aggregated claim amounts for every period of our discrete model.
   a. Generate independent realizations of the standard Brownian motions $\{B_{1,t}\}$, . . ., $\{B_{4,t}\}$ and derive the price processes from Equations (4) and (5), and the spread processes from Equations (6) and (7) using the Euler approximation method (see, e.g., Kloeden and Platen, 1992).
   b. Generate sequences of exponentially distributed inter-occurrence times for the one year time horizon, and for every claim occurrence a realization of the claim size distribution. Aggregate the claim sizes for every 1-day period using Equation (8).
2. Using a prespecified value as starting capital, we have all we need for applying the Cash-First (Cash-Last) liquidation algorithm described in the previous sections of the Appendix. Do this for all 10,000 scenarios and save the scenarios, in which ruin occurred. The starting capital is calibrated, so that there are slightly more than 200 ruin scenarios. In the following steps, we will work with these possible ruin scenarios only.
3. We now work toward our goal using a systematic trying procedure. Starting with the first digit of the input parameter "capital," we increase this number unit by unit and apply the Cash-First (Cash-Last) algorithm every time, until the number of ruins occurring is smaller than 200.
4. Decrease the next digit of the "capital" number unit by unit and apply the Cash-First (Cash-Last) algorithm every time, until the number of ruins occurring is greater than or equal to 200.
5. Repeat Steps 3 and 4, until the desired accuracy is reached. End with Step 3 and add one unit to the varied digit, getting the minimum required capital resulting in a ruin probability less than or equal to 2 percent.

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