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# One-Period Model of Individual Consumption, Life Insurance, and Investment Decisions

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## Abstract

This article studies individuals' optimal decisions on consumption, life insurance, and stock purchases in a one-period framework. With exponential utility functions, individuals' life insurance and stock purchases are independent of each other; life insurance purchases are affected only by individuals' future income, bequest intensity, risk attitude, survival probability, and the insurance risk premium; stock purchases are affected only by individuals' risk attitude, the risk-free rate of return, the stock return, and stock volatility. With power utility functions, life insurance and stock purchases are positively related with each other and are affected by all the factors.

## Introduction

A wage earner purchases life insurance to protect his dependents against financial hardship in the event of his sudden death. Many Americans have included life insurance in their asset portfolios. In 2001, 69 percent of Americans owned some type of life insurance. In 2003, Americans purchased $2.9 trillion of new life insurance coverage, 1 percent more than in 2002. By the end of 2003, the total life insurance coverage in the United States reached $16.8 trillion. On the other hand, many individuals terminate their existing insurance policies later for a variety of reasons. In 2003, the lapse rate was about 7.7 percent for individual policies, 17.1 percent for credit policies, and 9.0 percent for group policy (Life Insurance Fact Book, 2004).

Terminating life insurance policies is very costly for both the policyholders and the insurers. Many researchers have done research on individuals' life insurance purchase decisions. Fisher (1973) used a discrete-time framework and concentrated on mortality risk but did not consider future income when stock was added into the asset portfolio. Richard (1975) extended Merton's (1971) model and added life insurance into the asset portfolio and derived the optimal demands for life insurance, consumption, and stock purchases in a continuous-time framework. However, they both neglected

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insurance termination costs, which can be quite high in real life. Campbell (1980) used the method of Taylor Expansion to the first order and derived the demand function for life insurance in a discrete-time framework. Babbel and Ohtsuka (1989) studied an individual's decisions on whole life and term insurance purchases in a multi-period and multi-status model. Bernheim (1991) found that bequest motives drive an individual's life insurance purchase decision. But none of them included stocks in the asset portfolio.

This article contributes to this life insurance purchase decision process and helps individuals make optimal decisions on life insurance purchase decision and insurers make better product designs and marketing decisions. To this purpose, I extended Fisher's (1973) model and performed a comprehensive study on two groups of individuals' life insurance purchase decisions, along with their choices on consumption and stock purchase. The first group has exponential utility functions and constant absolute risk aversion attitudes. The second group has power utility functions and constant relative risk aversion attitudes. Besides future income and the insurance termination costs, I also studied the effects of a wide range of other factors—wealth, future income, bequest incentive, risk attitudes, survival probability, the stock return and stock volatility, the risk-free rate of return, the insurance premium, and insurance termination cost.

This article is organized as follows. The next section describes the one-period model. "The Optimal Decisions With an Exponential Utility Function" section derives the demand functions for consumption, life insurance and stock purchases, and the static results for an exponential utility function. "The Optimal Decisions With a Power Utility Function" section describes the numerical results of the optimal choices for a power utility function. The last section summarizes and concludes.

## THE ONE-PERIOD MODEL

Here I extended Fisher's (1973) discrete-time model by adding future income and insurance termination costs in studying an individual's optimal choices on consumption, life insurance, and stock purchases in a one-period framework.

At the beginning of the period, the individual is endowed with an initial wealth $W_0$ and chooses the optimal consumption, life insurance, and stock purchases to maximize his expected utility function. At the end of the period, I assume that the individual is either alive or dead. If alive, the individual surrenders his insurance policy and consumes all the proceeds from investments, future income $H$, and insurance surrender value $SV_1$. If the individual is dead, all the proceeds from his investments and insurance benefit $B$ go to his dependents. His life insurance can be a whole life, which offers some insurance surrender value $SV_1 > 0$ at time 1, or a term life, which offers no insurance surrender value at Time 1. The utility function for consumption $U(\,)$, which is associated with the "Alive" state, is assumed to be strictly concave and have at least a third derivative. The utility function for bequest $B(\,)$, which is associated with the "Dead" state, is assumed to be a linear transformation of the utility function for consumption. That is, $B(\,) = kU(\,)$, where $k$ is the intensity for bequest and measures how the individual values his dependents' welfare comparing to his own interests (the larger is the $k$, the more he values his dependents' welfare). And the model is as follows:

$$V(W_0) = \underset{\{C,B,\pi\}}{\text{Max}}\, U(C) + \frac{1}{1+\rho} E[\, p_x U(W_A) + q_x k U(W_D)] \,, \tag{1}$$

where

$V() =$ the maximum value of the expected utility function
$W_0 =$ the initial wealth at time 0
$U() =$ the utility function of consumption and wealth when alive
$C =$ consumption at time 0
$\pi =$ the amount of money used in purchasing stock
$B =$ the face value of a life insurance policy
$\rho =$ the impatience factor for future consumption
$E() =$ the expected value operator with respect to future stock returns
$x =$ the age of the individual at time 0
$p_x =$ the probability of the individual being alive at time 1
$q_x = 1 - px =$ the probability of the individual being dead by time 1
$k =$ the intensity factor for bequest
$W_A =$ wealth at time 1 when the individual is alive
$= (W_0 - C - BP - \pi)e^r + \pi R + H + SV_1$
$W_D =$ wealth at time 1 when the individual is dead
$= (W_0 - C - BP - \pi)e^r + \pi R + B$
$r =$ the risk-free rate of return per period
$R =$ the gross stock return per period
$H =$ the future income at time 1 when the individual is alive
$P =$ the insurance premium per unit of life insurance coverage
$= \dfrac{q_x(1+\lambda)e^{-r}}{1 - p_x e^{-r} l_{sv}}$
$\lambda =$ the load for insurance risk premium
$l_{sv} =$ the load for insurance surrender value
$SV_1 = l_{sv}BP =$ the insurance surrender value at time 1 when the individual is alive.

In this article, I differentiate a term life insurance policy from a whole life policy by the load for insurance surrender value $l_{sv}$—a term life policy offers no insurance surrender value ($l_{sv} = 0$) and its insurance premium $P$ is set by $P = q_x (1+\lambda) e^{-r}$; a whole life policy offers some insurance surrender value ($l_{sv} > 0$) when the policy is terminated at time 1 and its insurance premium is larger and set by $P = \frac{q_x (1+\lambda) e^{-r}}{1 - p_x e^{-r} l_{sv}}$ to cover this surrender benefit.

For the gross stock return $R$, I assume it follows a lognormal distribution with mean $\mu$ and standard deviation $\sigma$ with the following probability density function:

$$f(R) = \frac{1}{\sqrt{2\pi}\sigma R} \exp\left(-\frac{(\ln R - u)^2}{2\sigma^2}\right), \quad R > 0$$

where

$\mu = r + r_p =$ the expected value for the log stock return per period
$r =$ the risk-free rate of return per period
$r_p =$ the stock risk premium per period $= \omega \cdot \sigma$
$\omega =$ the price of risk
$\sigma =$ the stock volatility per period.

As for all the other factors, I assume that future income $H$ is given and known. The intensity for bequest factor $k$ lies between 0 and 1 and depends upon the individual's attitude toward bequest responsibility, and the needs for bequest (how many dependents he has). His survival probability $p_x$ depends upon his health status, age, gender, wealth, and lifestyle. The impatience factor for future consumption $\rho$ is set to be equal to the risk-free rate of return $r$. The load for insurance risk premium $\lambda$ is charged to cover insurers' expenses, profits, contingency reserves, and other business risks. To study how one factor affects those optimal decisions, I change it in a wide range while holding the other factors at some fixed values.

Taking partial derivatives of the expected utility function with respect to consumption $C$, insurance purchase $B$ and stock purchase $\pi$, I have the following first-order conditions (FOCs):

$$U'(C) - \frac{e^r}{1+\rho} E[p_x U'(W_A) + q_x k U'(W_D)] = 0, \tag{2}$$

$$E[(R - e^r) p_x U'(W_A) + q_x k U'(W_D)] = 0, \tag{3}$$

$$E\left[p_x(-pe^r)U'(W_A) + q_x(1 - pe^r)kU'(W_D)\right] = 0. \tag{4}$$

Solving the above equations, I have the demand functions for consumption $C$, life insurance $B$ and stock purchases $\pi$.

In this article, I study the effects of those factors with two types of utility functions—exponential and power utility functions. Next comes the exponential family, for which I derived closed-form solutions for life insurance purchases.

## THE OPTIMAL DECISIONS WITH EXPONENTIAL UTILITY FUNCTIONS

With an exponential utility function: $U(C) = -\frac{1}{\theta^e} - \theta^C$, an individual has a constant Absolute Risk Aversion attitude ($ARA = -\frac{U''(C)}{U'(C)} = \theta$). The larger is the $\theta$, the more risk averse the individual is. When $\theta$ reaches zero, the individual becomes risk neutral. The model is as follows:

$$V(W_0) = \underset{\{C,B,\pi\}}{\text{Max}} -\frac{1}{\theta}e^{-\theta C} - \frac{1}{(1+\rho)\theta} E\left[p_x e^{-\theta W_A} + q_x k e^{-\theta W_D}\right] \tag{5}$$

with $\theta =$ the coefficient of Absolute Risk Aversion and the others defined as before.

The first-order conditions (FOCs) are:

$$e^{(-\theta C)} - \frac{e^r}{1+\rho} E\left[p_x e^{(-\theta W_A)} + q_x k e^{(-\theta W_D)}\right] = 0, \tag{6}$$

$$E\left[(R - e^r) p_x e^{(-\theta W_A)} + q_x k e^{(-\theta W_D)}\right] = 0, \tag{7}$$

$$E\left[p_x(-pe^r)e^{(-\theta W_A)} + q_x(1 - pe^r)e^{(-\theta W_D)}\right] = 0. \tag{8}$$

Solving the above equations, I derived the following demand functions for life insurance, stock, and consumption.

### The Optimal Life Insurance Purchases

For the optimal insurance purchase $B$, I find:

$$B = \frac{1}{1 - l_{sv} P}\left[H - \frac{1}{\theta}\ln\frac{p_x P\left(e^r - l_{sv}\right)}{q_x k(1 - Pe^r)}\right], \tag{9}$$

where

$$\begin{aligned} P &= \frac{q_x(1+\lambda)e^{-r}}{[1 - p_x e^{-r}(1+\lambda)l_{sv}]} \\ &= \text{the insurance premium per unit of coverage.} \end{aligned}$$

Equation (9) shows that life insurance purchase is affected only by an individual's future income $H$, bequest intensity $k$, risk attitude $\theta$, survival probability $p_x$, the insurance risk premium $\lambda$, insurance surrender value $l_{sv}$, and the risk-free rate of return $r$. Particularly, life purchase $B$ is larger when future income $H$ is larger, bequest intensity $k$ is higher, risk aversion $\theta$ is more severe, survival probability $p_x$ is higher, or insurance risk premium $\lambda$ is lower.

It also says that life insurance purchase has nothing to do with wealth $W_0$, the impatience factor for future consumption $\rho$, or stock return $R$.

When $l_{sv} = 0$, the term life insurance purchase is:

$$B = \left[H - \frac{1}{\theta}\ln\frac{p_x Pe^r}{q_x k(1 - Pe^r)}\right],$$

where

$$P = q_x(1+\lambda)e^{-r}. \tag{10}$$

The partial derivatives of a term life insurance purchase $B$ to each of the factors are:

$$\frac{\partial B}{\partial H} = 1, \quad \frac{\partial B}{\partial k} = \frac{1}{\theta k}, \quad \frac{\partial B}{\partial \theta} = \frac{1}{\theta^2}\ln\frac{p_x(1+\lambda)}{(1 - q_x(1+\lambda))k},$$

$$\frac{\partial B}{\partial \lambda} = -\frac{1}{\theta}\left(\frac{1}{1+\lambda}\frac{1}{1 - q_x(1+\lambda)}, \quad \frac{\partial B}{\partial p_x} = \frac{1}{\theta}\frac{\lambda}{p_x(1 - q_x(1+\lambda))}.\right.$$

Thus, $$\frac{\partial B}{\partial H} > 0, \quad \frac{\partial B}{\partial \theta} > 0, \quad \frac{\partial B}{\partial k} > 0, \quad \frac{\partial B}{\partial p_x} > 0, \quad \frac{\partial B}{\partial \lambda} < 0.$$

For all the other factors, $\frac{\partial B}{\partial} = 0$.

Therefore, for a term life insurance policy the optimal life insurance purchase increases \$1 for \$1 increase in future income. For a fairly priced term life policy (where $l_{sv} = 0$ and $\lambda = 0$) we have the following relationship between term life insurance purchase $B$ and future income $H$:

- $B > H$ when $k > 1$.
- $B = H$ when $k = 1$.
- $B < H$ when $k < 1$.

That is, his life insurance purchase is equal to future income when he values his own interests the same as that of his dependents, bigger than future income when he cares more about his dependents' interests, and less than future income when he cares less about his dependents' welfare.

### The Optimal Stock Purchase

The optimal stock purchase $\pi$ solves the following equations:

$$E\left[(R - e^r)e^{(-\theta\pi R)}\right] = 0 \tag{11}$$

or

$$E^Q(R - e^r) = 0 \tag{12}$$

where

$$f^Q(R) = f(R)\frac{\mathrm{e}^{(-\theta\pi R)}}{E\left[\mathrm{e}^{(-\theta\pi R)}\right]}.$$

From Equation (12), we see that the optimal stock purchases make the expected value of the gross stock return $R$ equal to the risk-free rate of return $e^r$ under the new risk neutral probability density function $f^Q(R)$. It shows that the optimal stock purchase is affected only by the stock return's distribution function $f(R)$ (including its mean value and standard deviation), the risk-free rate of return $r$, and the risk aversion parameter $\theta$. It also shows that stock purchase is not affected by all the other factors and independent of life insurance purchase $B$.

Like Campbell (1980), applying the Taylor expansion method to the second order on the above equations, I find the following demand function for the optimal stock purchase:[1]

$$\pi = \frac{(m_2 - m_1 r) - \sqrt{(m_2 - m_1 r)^2 - 2(m_1 - r)(m_3 - m_2 r)}}{\theta\,(m_3 - m_2 r)} \tag{13}$$

where $m_1 = E(R - 1), \quad m_2 = E(R - 1)^2, \quad m_3 = E(R - 1)^3$.

[1] The other root $\pi = \frac{(m_2 - m_1 r) + \sqrt{(m_2 - m_1 r)^2 - 2(m_1 - r)(m_3 - m_2 r)}}{\theta\,(m_3 - m_2 r)}$ is ignored because $\pi\theta > 1$ and the remaining terms of Taylor expansion will explode.

Taking partial derivatives of the optimal stock purchase $\pi$ with respect to each factor on both sides of Equation (11), I find the following comparative static results:

$$\begin{aligned}
\frac{\partial \pi}{\partial \theta} &= -\frac{\pi}{\theta} \\
\frac{\partial \pi}{\partial r} &= -\left\{\theta e^r E\left[\frac{R^2 e^{(-\theta\pi R)}}{e^{2r} E(e^{-\theta\pi R})} - 1\right]\right\}^{-1} \\
\frac{\partial \pi}{\partial r_p} &= \frac{E\left[(\ln R - r - r_p)(R - e^r)e^{(-\theta\pi R)}\right]}{\theta\sigma^2 E\left[R(R - e^r)e^{(-\theta\pi R)}\right]} \\
\frac{\partial \pi}{\partial \sigma} &= \frac{E\left[(R - e^r)((\ln R - r - r_p)^2 - 4\sigma^2)e^{(-\theta\pi R)}\right]}{4\theta\sigma^2 E\left[R(R - e^r)e^{(-\theta\pi R)}\right]}.
\end{aligned}$$

From the above formulas we see: $\frac{\partial \pi}{\partial \theta} < 0$, $\frac{\partial \pi}{\partial r_p} > 0$, $\frac{\partial \pi}{\partial \sigma} < 0$, $\frac{\partial \pi}{\partial r} < 0$.

For all the other parameters, $\frac{\partial \pi}{\partial} = 0$.

From the above comparative results, we see that an individual purchases more stock when $\theta$ is smaller (the individual is less risk averse), the stock return $r_p$ is higher, stock volatility $\sigma$ is lower, or the risk free of return $r$ is lower (bonds are more expensive). The stock purchase decision is independent of the individual's wealth $W_0$, future income $H$, survival probability $p_x$, the impatience factor for consumption $\rho$, insurance risk premium $\lambda$, or insurance surrender value $l_{sv}$.

### The Optimal Consumption

For the optimal consumption $C$, I find:

$$C = \frac{1}{\theta(1+e^r)}[\ln(1+\rho) - \ln A - r] \tag{14}$$

where

$$A = \left(p_x e^{-\theta(H + l_{sv}BP)} + q_x k e^{-\theta B}\right) e^{-\theta(W_0 - BP - \pi)e^r} E(e^{-\theta\pi R})$$

With a term life insurance policy $l_{sv} = 0$, the optimal consumption becomes:

$$\begin{aligned}
C &= \frac{1}{\theta(1+e^r)}\Big\{\ln(1+\rho) + \theta\left[(1 - q_x(1+\lambda))H + W_0 e^r\right] + q_x(1+\lambda)[\ln(1+\lambda) - \ln k] \\
&\qquad - [1 - q_x(1+\lambda)][\ln p_x - \ln(1 - q_x(1+\lambda))] - \ln[E(e^{-\theta\pi(R-e^r)})] - r\Big\} \\
&= (1+e^r)^{-1}[(1 - q_x(1+\lambda))e^{-r}H + W_0] + C_0,
\end{aligned}$$

$C_0$ = the amount of consumption defined implicitly by the above equation when there are no future income and no wealth ($H = 0$, $W_0 = 0$).

Thus for an individual with a term life policy ($l_{sv} = 0$), we find that:

- The optimal consumption is a linear function of wealth and future income.
- Future income has a smaller effect on consumption than wealth does.

- The relative effects of future income to that of wealth is $(1 - q_x(1 + \lambda))e^{-r} < 1$, which is the same as that derived by Fisher (1973) for a power utility function.

Taking partial derivatives of the optimal consumption with respect to each factor, we find the following comparative static results (when $l_{sv} = 0$):

$$\frac{\partial C}{\partial H} = \frac{1 - q_x(1+\lambda)}{(1+e^r)}, \quad \frac{\partial C}{\partial W_0} = \frac{e^r}{(1+e^r)}$$
$$\frac{\partial C}{\partial \lambda} = -\frac{q_x B}{(1+e^r)}, \quad \frac{\partial C}{\partial p_x} = -\frac{\lambda + \theta(1+\lambda)p_x B}{\theta(1+e^r)A[1-(1+\lambda)q_x]}$$
$$\frac{\partial C}{\partial k} = -\frac{(1+\lambda)q_x}{\theta(1+e^r)k}, \quad \frac{\partial C}{\partial \theta} = \frac{-C + E^Q(W_A)}{\theta(1+e^r)} \quad \left(C > E^Q(W_A) \text{ when } \rho > 0\right)$$

where

$W_A$ = the amount of wealth at the end of the period if the individual is alive.

and

$$f^Q(W_A) = f(W_A)\frac{e^{(-\theta W_A)}}{E[e^{-(\theta W_A)}]}.$$

From the above formulas, we have:

$$\frac{\partial C}{\partial W_0} > 0, \quad \frac{\partial C}{\partial H} > 0, \quad \frac{\partial C}{\partial \rho} > 0, \quad \frac{\partial C}{\partial rp} > 0,$$
$$\frac{\partial C}{\partial k} < 0, \quad \frac{\partial C}{\partial \theta} < 0, \quad \frac{\partial C}{\partial \lambda} < 0[2]$$

whereas

$$\frac{\partial C}{\partial \sigma}, \quad \frac{\partial C}{\partial r}, \quad \text{and} \quad \frac{\partial C}{\partial p_x}[3]$$

can go either way.

Thus we see that an individual consumes more when he has more wealth, a larger future income, a larger impatience factor for future consumption, or a larger stock return. He consumes less when he has a larger bequest intensity, is more risk averse, or a higher insurance premium. On the other hand, the survival probability, the risk-free rate of return, and the stock volatility have mixed effects on consumption.

To summarize, for individuals with exponential utility functions, their life insurance purchases are independent of stock purchases and are affected only by future income, risk aversion attitude, bequest intensity, survival probability, insurance premium, and insurance surrender values. Their stock purchases are affected only by their risk

[2] If the insurance purchase is nonnegative ($B \geq 0$).
[3] $\frac{\partial C}{\partial p_x} > 0$ when the insurance purchase is nonnegative ($B \geq 0$).

aversion attitude, the risk-free rate of return, the stock return, and stock volatility. Their consumption is affected by all the factors.

## THE OPTIMAL DECISIONS WITH POWER UTILITY FUNCTIONS

For individuals with power utility functions $U(C) = \frac{C^\gamma}{\gamma},\ \gamma < 1,\ \gamma \neq 0$, they have constant Relative Risk Aversion attitudes ($RRA = -C\frac{U''(C)}{U'(C)} = 1 - \gamma$). The larger is the $\gamma$, the less risk averse they are. When $\gamma = 1$, the individuals are risk neutral and indifferent to taking on a fair game; when $\gamma$ reaches 0, the utility function becomes a logarithmic form.

With power utility functions, the model for the optimal control problem becomes:

$$V(W_0) = \underset{\{C,B,\pi\}}{\text{Max}} \frac{C^\gamma}{\gamma} + \frac{1}{1+\rho}\frac{1}{\gamma}E\left[p_x(W_A)^\gamma + q_x k(W_D)^\gamma\right]. \tag{15}$$

The optimal consumption, life insurance, and stock purchases solve the following FOC equations:

$$C^{\gamma-1} = \frac{e^r}{1+\rho}E\left[p_x(W_A)^{\gamma-1} + q_x k(W_D)^{\gamma-1}\right] = E\left[(vW_D)^{\gamma-1}\right] \tag{16}$$

$$E\left(W_A^{\gamma-1}\right] = E\left[(\alpha W_D)^{\gamma-1}\right] \tag{17}$$

$$E^Q(R - e^r) = 0 \tag{18}$$

where

$$v = \left[\frac{ke^r}{(1+\rho)(1+\lambda)}\right]^{\frac{1}{\gamma-1}}, \quad \alpha = \left[\frac{[(1-(1+\lambda)q_x]q_x k}{p_x}\right]^{\frac{1}{\gamma-1}}$$

and

$$f^Q(R) = f(R)\frac{p_x(W_A)^{\gamma-1} + q_x k(W_D)^{\gamma-1}}{E\left[p_x(W_A)^{\gamma-1} + q_x k(W_D)^{\gamma-1}\right]}.$$

The above equations are similar to those derived by Fisher (1973) when he included bonds, not stocks, in his asset portfolio. Equation (17) shows that wealth at the "Alive"state $W_A$ is "proportional" to wealth at the "Dead" state $W_D$. Equation (16) shows that the optimal consumption is "proportional" to the wealth in each state. Equation (18) shows that the optimal consumption, life insurance, and stock purchases make the expected value of the stock return equal to the risk-free rate of return under the new risk neutral probability.

With power utility functions, there are no closed-form solutions so I use a numerical method in studying the effects of all the factors.

### The Numerical Model

With a power utility function, the numerical model for an individual's decisions on consumption, life insurance, and stock purchases is as follows:

$$V(W_0) = \underset{\{C,B,\pi\}}{\text{Max}} \frac{C^\gamma}{\gamma} + \frac{1}{1+\rho}\frac{1}{\gamma}\int_a^b \left[p_x(W_A)^\gamma + q_x k(W_D)^\gamma\right] \cdot f(R) \cdot dR. \tag{19}$$

Subject to the following first-order conditions (FOCS):

$$C^{\gamma-1} - \frac{e^r}{1+\rho}\int_a^b \left[p_x(W_A)^\gamma + q_x k(W_D)^\gamma\right]\cdot f(R)\cdot dR = 0 \tag{20}$$

$$\int_a^b (R-e^r)\left[p_x(W_A)^\gamma + q_x k(W_D)^\gamma\right]\cdot f(R)\cdot dR = 0 \tag{21}$$

$$\int_a^b \left[p_x(-Pe^r)(W_A)^\gamma + q_x k(1-Pe^r)(W_D)^\gamma\right]\cdot f(R)\cdot dR = 0 \tag{22}$$

where

$$f(R) = \frac{1}{\sqrt{2\pi}\sigma R}\exp\left[-\frac{(\ln R-\mu)^2}{2\sigma^2}\right], \quad R > 0.$$

I set $a = 0.01$ and $b = 20$ so that the probability of the stock return $R$ lying between $a$ and $b$ is close to 1.

## The Numerical Output With Term Life Insurance

To study the effects of a factor, I change it in a wide range of possible values while holding the other factors at their default values. For a factor that is related with another factor, such as the stock volatility, which is related to the stock risk premium, I study its effects under two scenarios: with the other factor fixed, and with the other factor changing with it.

First, I study the effects of those factors that are related to the individual's own circumstances—his wealth and future income, bequest intensity, risk aversion parameter, the impatience factor for consumption, and survival probability; then the factors that are related to the financial market—the risk-free rate of return, the stock return, and stock volatility; and then the factors that are related to the insurance market—the load for life insurance premium.

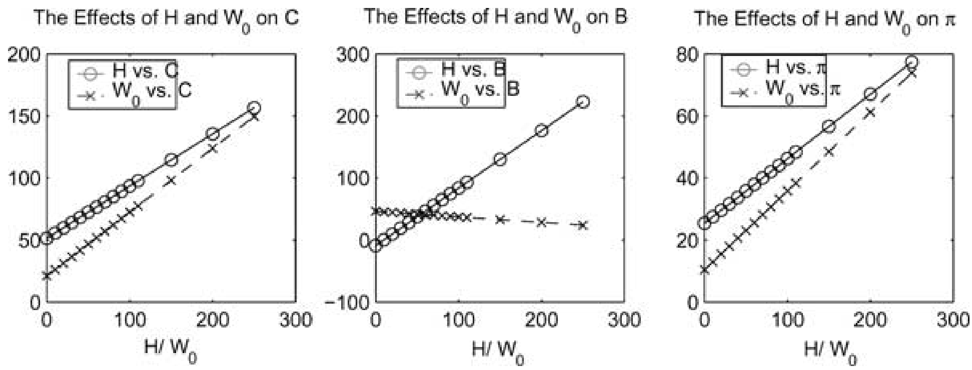
With a power utility function, the default values for all the factors are set as follows:

- Factors related to the individual: $W_0 = 100$, $H = 50$, $k = 1$, $\gamma = -1$, $p_x = 0.9$.
- Factors related to the insurance market: $\lambda = 0.4$, $l_{sv} = 0$.
- Factors related to the financial market: $r = 0.05$, $\sigma = 0.2$, $r_p = 0.022$.

*The Effects of Wealth and Future Income.* According to Fisher (1973), wealth and future income have linear effects on consumption and life insurance purchases (when only bonds in the asset portfolio).

$$C = a_c W_0 + b_c H$$
$$B = a_B W_0 + b_B H.$$

**FIGURE 1**
The Effects of Wealth and Income When $k = 1$




He gave the following relative effects of future income and wealth on consumption and insurance purchases:

$$\frac{b_c}{a_c} = \frac{b_{\text{Bond}}}{a_{\text{Bond}}} = e^{-r}(p_x - q_x\lambda)$$

$$\frac{b_B}{a_B} = \frac{e^{-r} + v}{1-\alpha}, \quad \text{where} \quad v = \left[\frac{ke^r}{(1+\rho)(1+\lambda)}\right]^{\frac{1}{\gamma-1}}.$$

Now with stock added into the asset portfolio, I find the following numerical results on the effects of wealth and future income under two scenarios—with the intensity for bequest $k$ equal to 1 and 0.5.

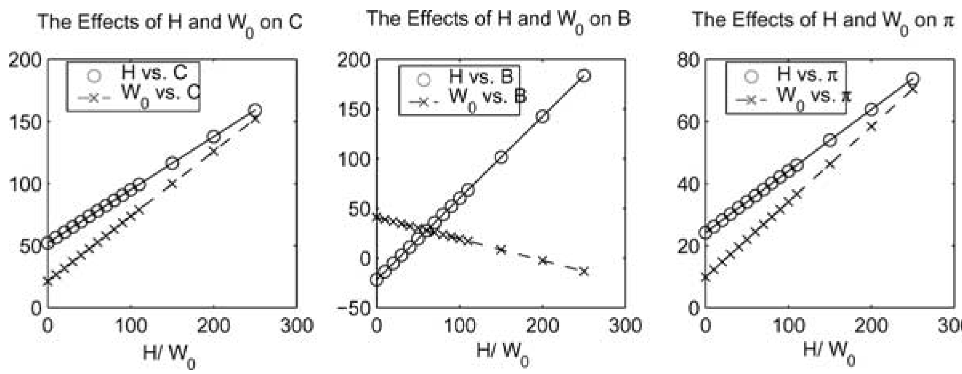
$$\begin{array}{ll} \text{when } k = 1, & \text{when } k = 0.5, \\ C = 0.5142\, W_0 + 0.4206\, H & C = 0.523\, W_0 + 0.4278\, H \\ B = -0.0899\, W_0 + 0.9265\, H & B = -0.2176\, W_0 + 0.822\, H \\ \pi = 0.254\, W_0 + 0.208\, H; & \pi = 0.2424\, W_0 + 0.1984\, H. \end{array}$$

Under both cases $\frac{b_c}{a_c} = \frac{b_s}{a_s} = 0.818 = e^{-r}(p_x - q_x\,\lambda)$, but $\frac{b_B}{a_B} \neq \frac{e^{-r}+v}{1-\alpha}$.

That is, my findings agree with Fisher's (1973) on the relative effects of the two on consumption and stock/bond purchases, but we disagree on their relative effects on insurance purchases.

Figures 1 and 2 show that wealth and future income have linear effects on all these three decisions. Particularly, future income and wealth have significant and positive effects on consumption and stock purchases and wealth has steeper slopes and larger effects on both than future income does. However, they have opposite effects on life insurance purchases—future income has a larger and positive effect on insurance purchases, whereas wealth has a smaller and negative effect on life insurance purchases.

**FIGURE 2**
The Effects of Wealth and Income When $k = 0.5$




According to Fisher (1973), when there is no stock in the asset portfolio, the demand functions for consumption and life insurance purchase are:[4]

$$\begin{array}{ll} \text{when } k = 1, & \text{when } k = 0.5, \\ C_F = 0.5132\, W_0 + 0.4195\, H & C_F = 0.5216\, W_0 + 0.4267\, H \\ B_F = -0.0916\, W_0 + 0.9251\, H; & B_F = -0.2223\, W_0 + 0.8185\, H. \end{array}$$

Comparing our numerical demand functions (when stock is included in the asset portfolio) with those of Fisher's, I find the following relationship:

$$\begin{array}{ll} \text{when } k = 1, & \text{when } k = 0.5, \\ C = 1.002\, C_F & C = 1.003\, C_F \\ B = 1.0015\, B_F + 0.0018\, W_0 & B = 1.0043\, B_F + 0.0057\, W_0 \\ \pi = 0.254\, W_0 + 0.208\, H; & \pi = 0.2424\, W_0 + 0.1984\, H. \end{array}$$

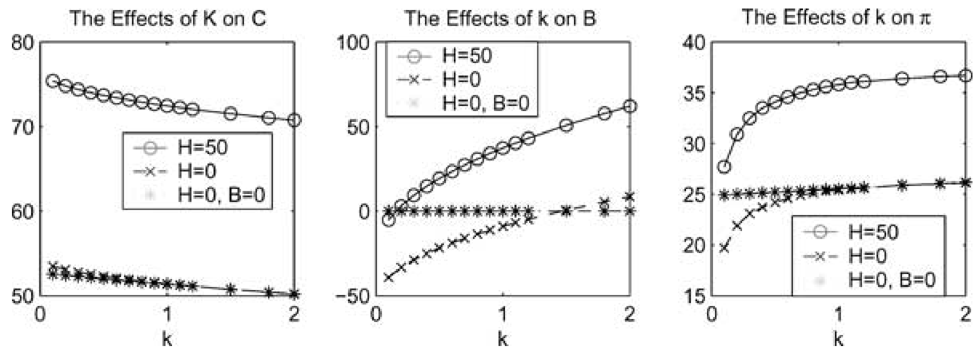
From the above equations, we see when stock is added into the asset portfolio:

- An individual consumes more and purchases more life insurance.
- The relative effects of future income and wealth on consumption stay the same as Fisher's $\frac{b_c}{a_c} = e^{-r}(p_x - q_x\, \lambda)$.
- But future income affects life insurance purchases more when stock is added in the asset portfolio because

$$\overbrace{\left|\frac{b_B}{a_B}\right|}^{\text{ours with stock}} > \overbrace{\left|\frac{e^{-r} + v}{1 - \alpha}\right|}^{\text{Fisher's without stock}} .$$

[4] Subscript $F$ denotes demand function according to Fisher's (1973) model.

**FIGURE 3**
The Effects of Bequest Intensity




Comparing the two numerical demand functions when $k = 1$ and $k = 0.5$, I have:[5]

$$
\begin{aligned}
C_1 &= 0.983\, C_{0.5} \\
B_1 &= 1.127 B_{0.5} + 0.155 W_0 \\
\pi_1 &= 1.084\, \pi_{0.5}.
\end{aligned}
$$

The above equations show when $k$ increases from 0.5 to 1, the individual consumes less but purchases more life insurance and more stock. Next comes the direct study of the bequest intensity's effects.

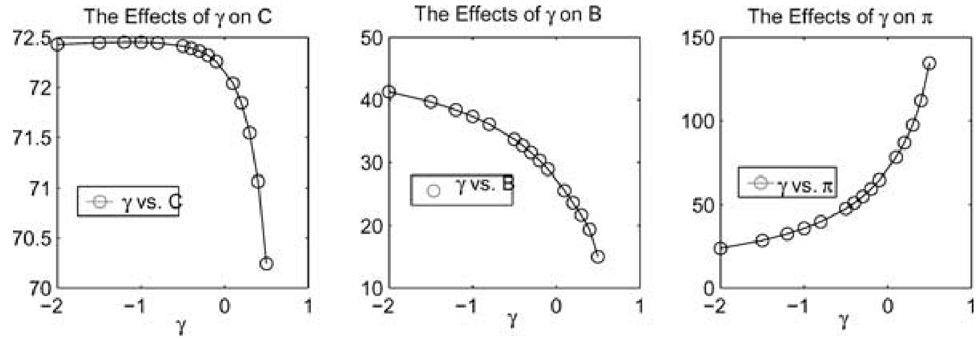
*The Effects of the Intensity for Bequest.* The intensity for bequest $k$ measures an individual's bequest motives—a caring parent has a larger intensity for bequest. In Fisher (1973), he suggested that the bequest intensity had negative effects on stock purchases when there is no future income. To be comparable with his findings, I study the intensity's effects under two conditions—with and without future income ($H = 50$ and $H = 0$). In addition, when future income is set to zero, I study the individual's behavior under two scenarios—allow negative insurance purchases and force life insurance purchase to be nonnegative. The intensity for bequest is set between 0.05 and 2, representing individuals from being very insensitive to very sensitive to their dependents' welfare.

Figure 3 shows that with or without future income when the bequest intensity increases the individual consumes less, purchases more insurance and more stock, which is opposite to Fisher's (1973) suggestion that an individual would buy less stock when the intensity for bequest increases.

The second graph shows that when bequest intensity is very small, the individual sells, not buys, life insurance ($B < 0$) when he has no income. Further, when there is no future income and the negative life insurance purchases are forced to be zero

[5] Subscript "1" denotes the optimal values when $k = 1$, subscript "0.5" denotes the optimal values when $k = 0.5$.

**FIGURE 4**
The Effects of Risk Aversion Factor




(life insurance purchases increase from negative to zero), the individual purchases more stock as shown in the third graph. Therefore, we see that when an individual purchases more life insurance he also purchases more stock when he has a power utility function.

Overall, we see that with or without future income the bequest intensity has a very significant positive effects on life insurance purchases, some secondary positive effects on stock purchases, and very modest negative effects on consumption. Also, by comparing the slopes we see that the intensity has a more noticeable effect on those decisions when he has future income ($H = 50$) than when he has no future income ($H = 0$).

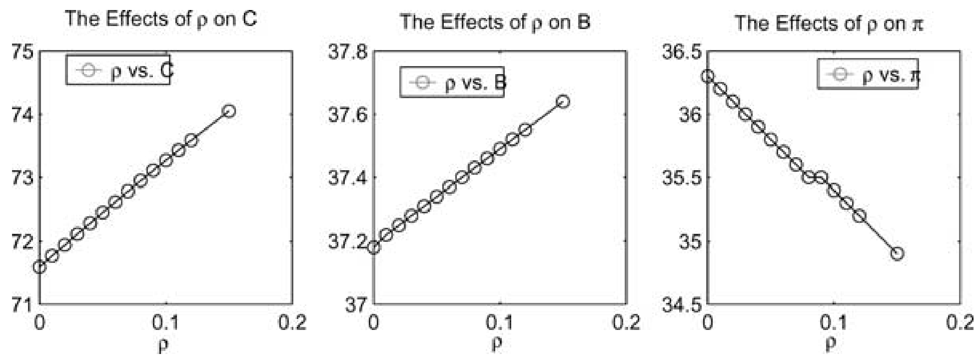
*The Effects of the Risk Aversion Parameter.* When the risk aversion parameter $\gamma$ increases, the individual becomes less risk averse. I set it between $-2$ and 0.5, representing individuals from very risk averse to less risk averse.

Figure 4 shows that when the individual becomes less risk averse ($\gamma$ increases), he consumes less, purchases less life insurance, but purchases more stock. Comparing the slopes from the three graphs, we see the risk aversion factor has a significant effect on stock purchases, a very moderate effect on life insurance purchases, and a very minor effect on consumption. Comparing with findings from the bequest intensity section, we see life insurance purchases are very sensitive to the bequest intensity factor whereas stock purchases are very sensitive to the risk aversion factor.

*The Effects of the Impatience Factor for Consumption.* The impatience factor for consumption $\rho$ measures an individual's consumption preference over time—when it increases, an individual values current consumption more. I set $\rho$ between 0 and 0.15, which is 0.05 below and 0.10 above the default value of the risk-free rate of return ($r = 0.05$).

Figure 5 shows that the impatience factor has near linear effects on all these three decisions. When it increases the individual does consume more now. It also shows that the individual purchases more life insurance but less stock when he has a stronger

**FIGURE 5**
The Effects of the Impatience Factor




preference for current consumption. But judging from the slopes we see that this factor's effects on life insurance and stock purchases are very limited.

*The Effects of Survival Probability.* Fisher (1973) suggested that survival probability had negative effects on life insurance purchases. Here I study the survival probability's effects under two conditions—with and without future income ($H = 50$ and $H = 0$). I find that when the individual has no future income he sells insurance (insurance purchases are negative). Since in reality individuals could not sell life insurance so in one scenario I set the actual insurance purchases as $\max(B, 0)$—the negative insurance purchases are forced to be zero. By increasing life insurance purchases from negative to zero, we can also see how the individual's insurance purchase decision affects his stock purchase decision. The survival probability $p_x$ is set between 0.4 and 1, representing individuals from being subhealthy to super healthy.

When there is no future income and no life insurance purchase ($H = 0$, $B = 0$), wealth at the "Alive" state is the same as that at the "Dead" state. As a result, the optimal consumption and stock purchases are constant and independent of survival probability. So we have $H = 0$, $B = 0$, $C = 51.372$, $\pi = 25.54$ at all levels of survival probabilities, which corresponds to the horizontal line at each of the graphs in Figure 6.

The first graph of Figure 6 shows that when the individual has future income ($H = 50$) he consumes more when survival probability is larger. On the other hand, when he has no income he consumes a little bit less when survival probability is larger.

The second graph of Figure 6 shows that with or without future income survival probability has positive effects on life insurance purchases—the individual purchases more life insurance when he has a larger chance of survival, which is opposite to Fisher's (1973) suggestion that life insurance purchase was negatively related to survival probability. The difference may be due to the fact that insurance premium in my model is related to the individual's survival probability whereas it may not be so in his article.

The third graph of Figure 6 shows that with or without future income the individual purchases more stock when survival probability is larger. The middle line in the graph

**FIGURE 6**
The Effects of Survival Probability

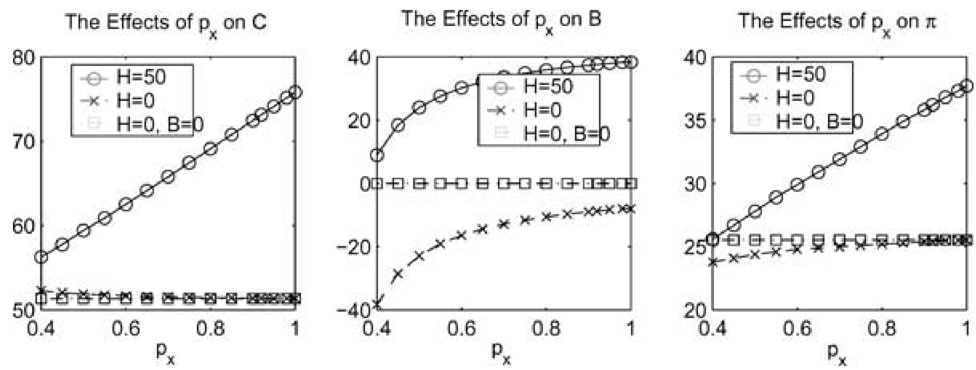
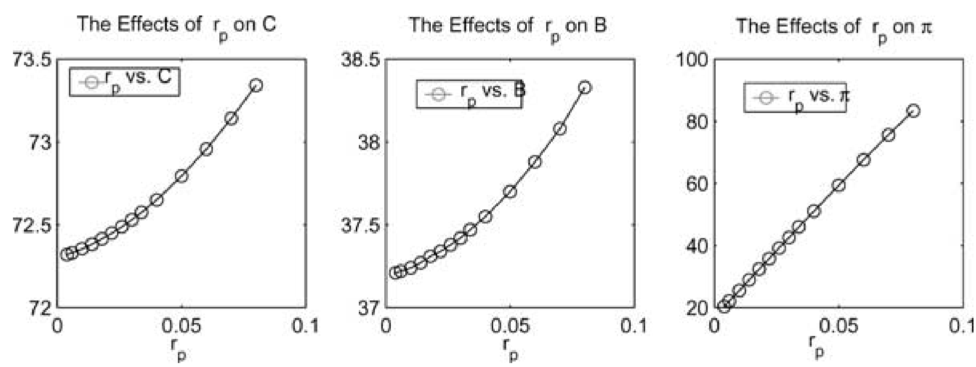



**FIGURE 7**
The Effects of Stock Risk Premium

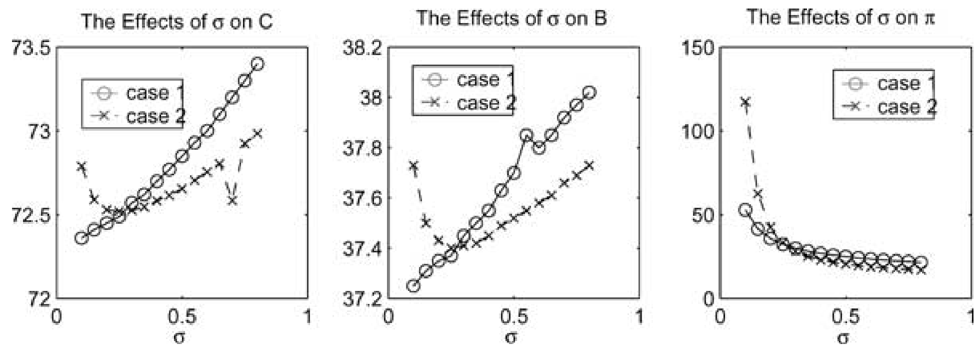



corresponds to the case when $H = 0$ and $B = 0$, and the bottom line is for $H = 0$ and $B < 0$. So we see when the individual purchases more life insurance (from $B < 0$, $\rightarrow B = 0$), he also purchases more stock. Therefore, with a power utility function, life insurance purchase has a positive effect on stock purchase.

*The Effects of Stock Risk Premium.* The stock risk premium $r_p$ is set to be a linear function of stock volatility $\sigma$ with $r_p = \omega\sigma$, with $\omega$ as the market price for risk. In this section, the stock volatility is fixed at its default level ($\sigma = 0.2$) whereas the market price for risk $\omega$ is set from 0.02 to 0.5, and the stock risk premium varies from 0.004 to 0.1.

Figure 7 shows when the stock risk premium increases (with the stock volatility held constant), the individual consumes more, purchases more life insurance and even more stock. Judging from the slopes we see that stock risk premium has a very significant effect on stock purchases and very minor effects on consumption and life insurance purchases.

**FIGURE 8**
The Effects of Stock Volatility




*The Effects of Stock Volatility.* I study stock volatility's effects under two conditions: Case 1—the stock risk premium changes with the stock volatility as $r_p = 0.11\sigma$; and Case 2—the stock risk premium is fixed at $r_p = 0.08$. The stock volatility varies from 0.1 to 0.8.

The first graph in Figure 8 shows that the individual consistently consumes more under Case 1 when stock risk premium and stock volatility increase together. The individual consumes less first and then starts to consume more under Case 2 when stock volatility increases while the stock risk premium is fixed.

The second graph in Figure 8 shows the same pattern as the first graph. That is, the individual purchases more life insurance under Case 1, but has a U-shape demand function for life insurance under Case 2.

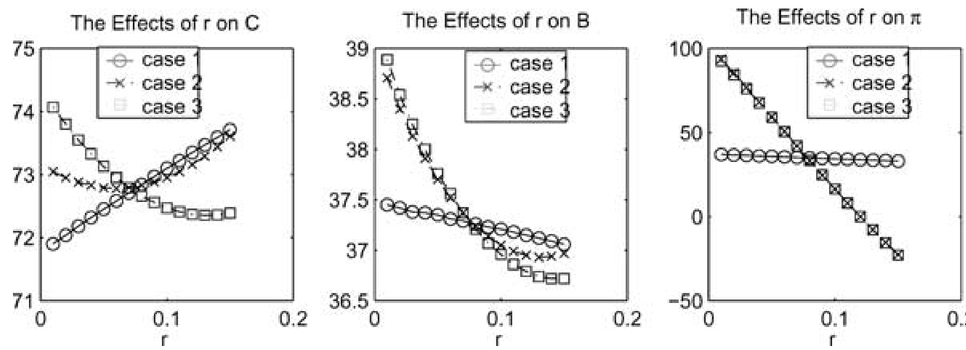
On the other hand, the third graph in Figure 8 shows that the individual purchases less stock when stock volatility increases under both cases. But the stock purchase does decrease more slowly under Case 2 when the stock risk premium is fixed at a very high level.

Comparing the slopes we see that stock volatility has a significant effect on stock purchases and very modest effects on consumption and life insurance purchases.

*The Effects of Risk-Free Rate of Return.* I study the effects of the the risk-free rate of return under three conditions: Case 1—both the stock return and the impatience factor change with it, Case 2—the impatience factor changes with it while the stock return is fixed at its default value, and Case 3—both the stock return and the impatience factor are fixed at their default values. The risk-free rate of return varies from 0.01 to 0.15.

The first graph of Figure 9 shows that under Case 1 the individual consistently consumes more when the risk-free rate of return, the stock return, and the impatience factor increase together; under Case 3, the individual consistently consumes less when the risk-free rate of return increases whereas the other two are fixed at their default values; under Case 2 the individual consumes less first and then starts to consume

**FIGURE 9**
The Effects of Risk Free Rate of Return




more when the risk-free rate of return and the impatience factor increase together whereas the stock return is fixed at its default value.

The second graph of Figure 9 shows that under all the three conditions when the risk-free rate of return increases the individual reduces life insurance purchases, though at different speeds—faster under Case 2 (when the stock return is fixed ) than under Case 1 (when stock return increases with the risk free rate of return). Under Case 3, life insurance purchases almost overlap with those under Case 2, which suggests that the impatience factor does not have much effects on life insurance.

The third graph of Figure 9 shows similar patterns as the second graph—the individual purchases less stock under all the three conditions, where stock purchases decrease faster under Case 2 and Case 3 than under Case 1.

Judging from the slopes we see that the risk-free rate of return has a significant effect on stock purchases and very minor effects on consumption and life insurance purchases.

*The Effects of Life Insurance Premium.* The load for life insurance premium $\lambda$ is used to cover an insurer's expenses, profit and contingency margins. When $\lambda$ increases, insurance premium also increases. Here I set it from 0 to 2.

Figure 10 shows that insurance premium has near linear effects on all the three decisions. When insurance premium increases the individual consumes less, purchases less life insurance and less stock. Judging from the slopes we see that the insurance premium has a very significant effect on life insurance purchases, a modest effect on stock purchases, and a very minor effect on consumption.

This section covers a term life insurance that offers no insurance surrender value ($l_{sv} = 0$) and entails a lower insurance premium. Next I describe the numerical output with a whole life insurance policy that offers some positive insurance surrender value ($l_{sv} > 0$) and entails a higher insurance premium.

## The Numerical Output With Whole Life Insurance

As before, I study the effects of each factor by changing it in a wide range while holding the other factors at their default values, which are:

**FIGURE 10**
The Effects of Insurance Risk Premium

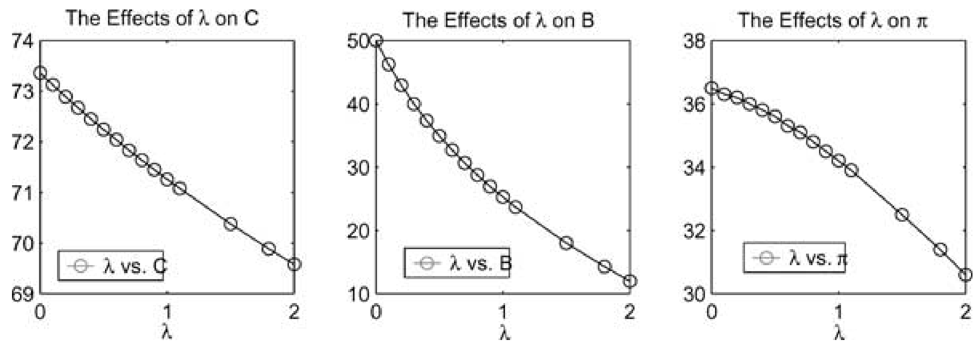
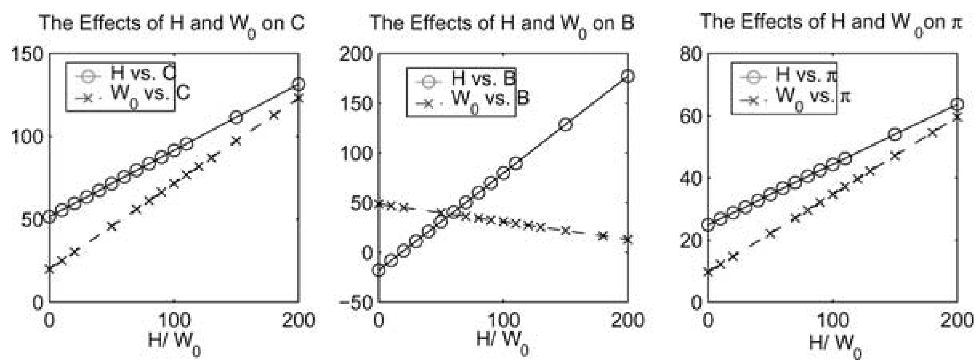



**FIGURE 11**
The Effects of Wealth and Future Income




- Factors related to individuals: $W_0 = 100, H = 50, k = 1, \gamma = -1, p_x = 0.9$.
- Factors related to the insurance market: $\lambda = 0.4, l_{sv} = 0.4$.
- Factors related to the financial market: $r = 0.05, \sigma = 0.2, r_p = 0.022$.

The numerical results are as shown in Figures 11 to 19. From the output we see all the factors affect consumption, whole life insurance, and stock purchases in a very similar way as they do with a term life insurance.

## CONCLUSION

In conclusion, I find that all individuals (with exponential or power utility functions) purchase life insurance mainly to protect their dependents against financial hardship at their sudden deaths. Specifically, they purchase more life insurance when they have larger future income, stronger bequest needs, severer risk aversion attitude, or cheaper insurance premium. They purchase more stock when stocks have higher returns and/or lower volatilities, bonds offer lower risk-free rate of returns, or when they themselves are less risk averse.

**FIGURE 12**
The Effects of Bequest Intensity

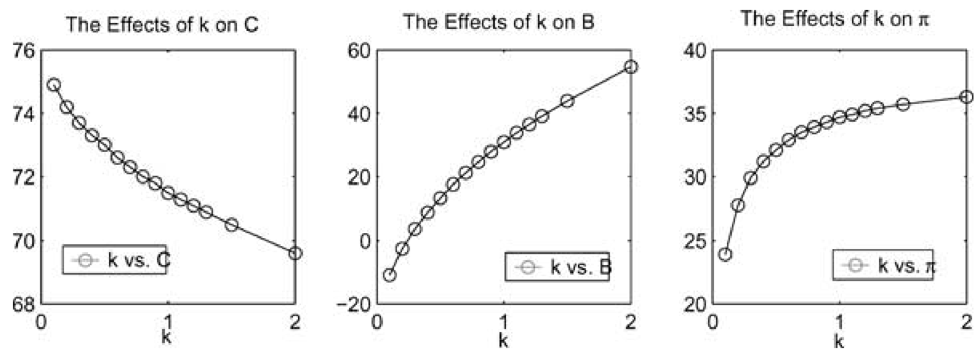



**FIGURE 13**
The Effects of Risk Aversion Factor

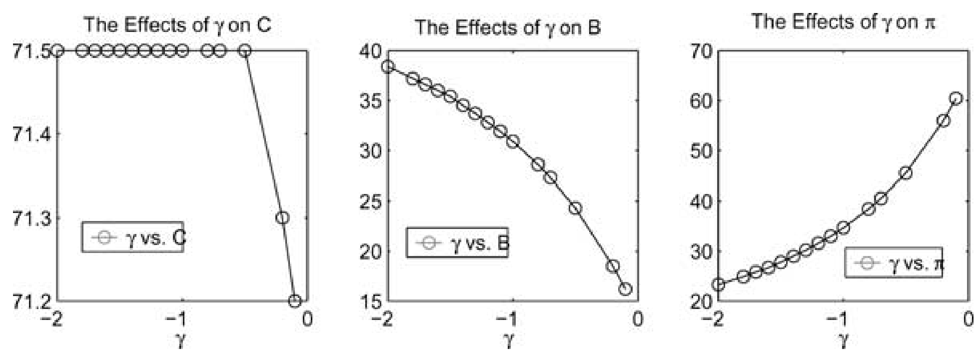



**FIGURE 14**
The Effects of Survival Probability

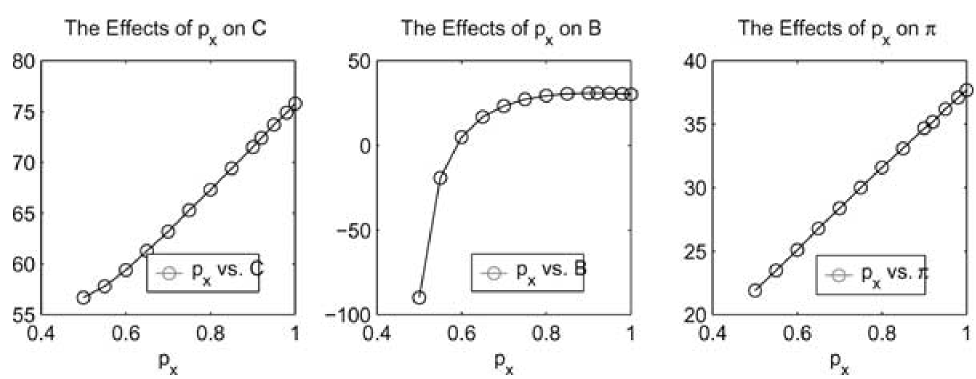


**FIGURE 15**
The Effects of Stock Risk Premium

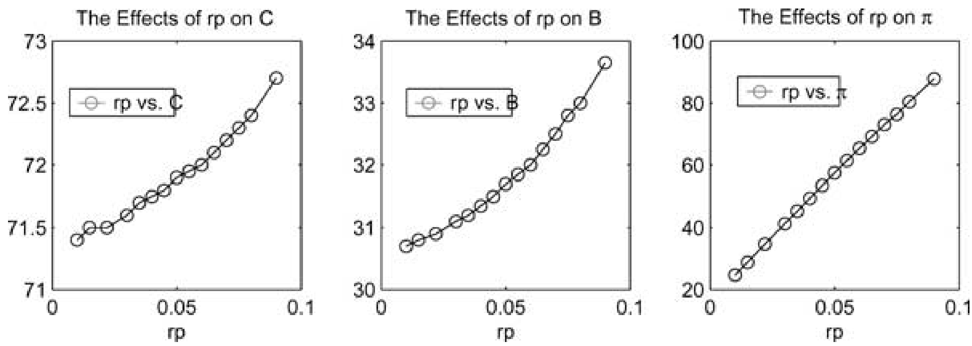



**FIGURE 16**
The Effects of Stock Volatility

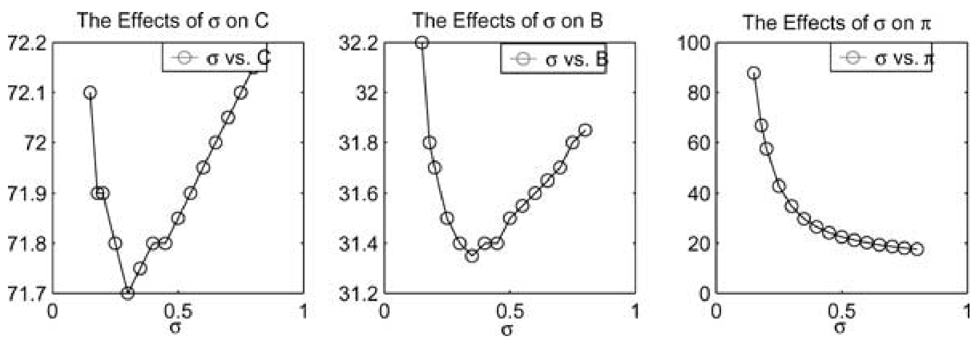



**FIGURE 17**
The Effects of Risk-Free Rate of Return

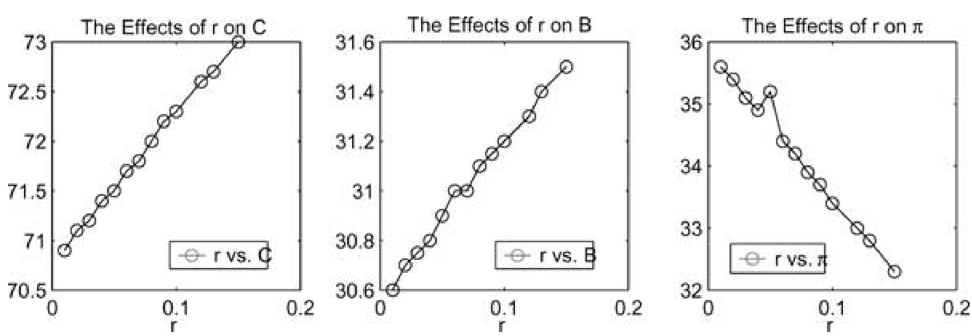


**Figure 18**
The Effects of Insurance Risk Premium

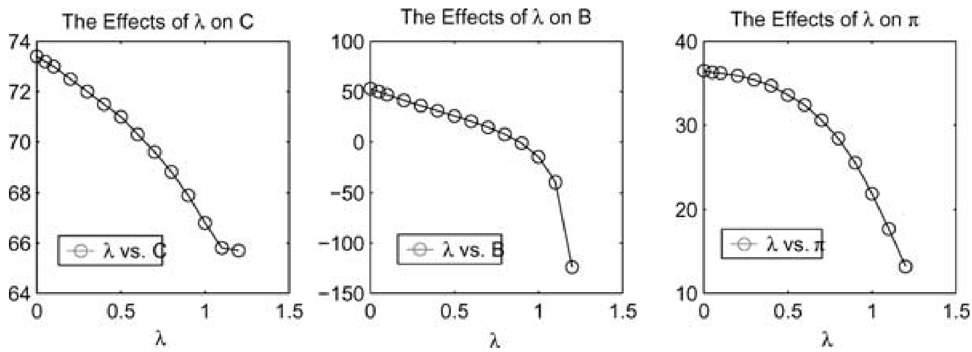
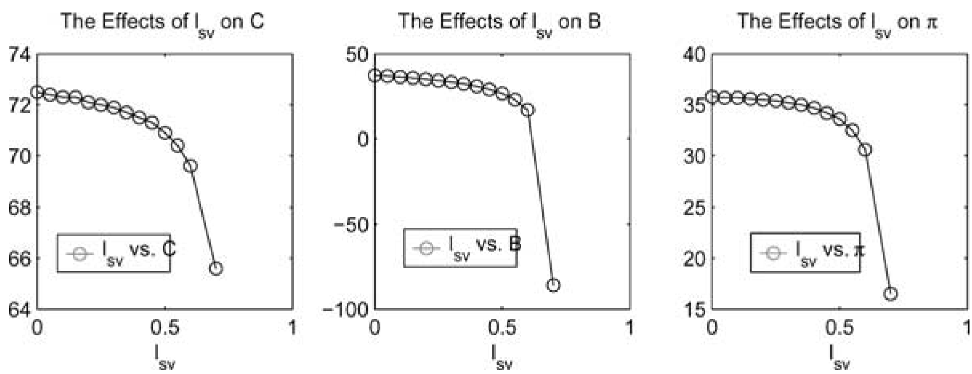



**Figure 19**
The Effects of Insurance Surrender Value




However, individuals with different forms of utility functions do behave quite differently. For individuals with exponential utility functions their life insurance and stock purchases are independent of each other. And their life insurance purchases are affected only by their future income, bequest intensity, risk aversion attitude, insurance premium, and insurance surrender value. Their stock purchases are affected only by their risk aversion attitude, the risk-free rate of return, the stock return, and stock volatility.

For individuals with power utility functions their life insurance and stock purchases are positively related to each other and are affected by all the factors. Particularly, we see that:

- Life insurance purchases are affected mainly by future income, intensity for bequest, survival probability, risk aversion factor, insurance premium, and surrender value.
- Stock purchases are affected mainly by wealth, future income, risk aversion factor, bequest intensity, stock risk premium and volatility, insurance premium, and surrender value.

- Consumption is mostly affected by wealth and future income, somewhat by insurance premium, and the intensity for bequest.
- The impatience factor for consumption and the risk-free rate of return do not affect consumption, insurance, or stock purchases much.

Especially, with a power utility function I find that:

- Bequest intensity has positive effects on stock purchases.
- Survival probability has positive effects on life insurance purchases.

These two findings are different from those suggested by Fisher (1973).

For a concise summary see Table A1 in the Appendix.

For future research, we can relate future income to an individual's health status and/or the financial market conditions. Also, we can extend this one-period model into a multi-period framework and combine an individual's insurance purchase and lapse decisions. Furthermore, it will be very rewarding if we can collect data on individual level and perform empirical studies to validify/check the above findings.

## APPENDIX

**TABLE A1**
Comparison of Exponential and Power Utility Functions

| Choice | | Consumption (C) Exponential/(*Power*) | Insurance (B) Exponential/(*Power*) | Stock ($\pi$) Exponential/(*Power*) |
|---|---|---|---|---|
| Individual conditions | $k$ | Negative/(*Negative*) | Positive/(*Positive*) | No/(*Positive*) |
| | $\rho$ | Positive/(*Positive*) | No/(*Positive*) | No/(*Negative*) |
| | $\theta/\gamma$ | Negative/(*Negative*) | Positive/(*Negative*) | Negative/(*Positive*) |
| | $p_x$ | Mixed/(*Mixed*) | Positive/(*Mixed*) | No/(*Mixed*) |
| | $H$ | Positive/(*Positive*) | Positive/(*Positive*) | No/(*Positive*) |
| | $W_0$ | Positive/(*Positive*) | No/(*Negative*) | No/(*Positive*) |
| Financial market | $r$ | Mixed/(*Mixed*) | No/(*Negative*) | Negative/(*Negative*) |
| | $\sigma$ | Mixed/(*Mixed*) | No/(*Mixed*) | Negative/(*Negative*) |
| | $r_p$ | Positive/(*Positive*) | No/(*Positive*) | Positive/(*Positive*) |
| Insurance market | $\lambda$ | Negative/(*Negative*) | Negative/(*Negative*) | No/(*Negative*) |
| | $l_{sv}$ | Negative/(*Negative*) | Negative/(*Negative*) | No/(*Negative*) |

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