

MULTIVARIATE EXPONENTIAL TILTING AND PRICING IMPLICATIONS FOR MORTALITY SECURITIZATION

Samuel H. Cox
Yijia Lin
Shaun Wang

ABSTRACT

Normalized exponential tilting is an extension of classical theories, including the Capital Asset Pricing Model (CAPM) and the Black–Merton–Scholes model, to price risks with general-shaped distributions. The need for changing multivariate probability measures arises in pricing contingent claims on multiple underlying assets or liabilities. In this article, we apply it to valuation of mortality-based securities written on mortality indices of several countries. We show how to use multivariate exponential tilting to price the first pure mortality security, the Swiss Re bond. The same technique can be applied in other mortality securitization pricing.

INTRODUCTION

Life insurance securitization has been an important life insurance financial innovation since 1988. Securitization may increase a firm's value by reducing transaction costs, agency costs, informational asymmetries, taxation, and regulation (Cowley and Cummins, 2005). In these transactions, positive future net cash flow from the policies is dedicated to pay the bondholders. Therefore, they are similar to asset securitizations (Lin and Cox, 2006).

The complexity of life insurance securitization impedes its development because underlying cash flows of life insurance securitization are determined by numerous contingencies including mortality, persistency, regulatory risk, insurer policy dividend decisions, and other factors. Cowley and Cummins (2005) conclude that "each layer of complexity increases the degree of informational asymmetries between the investor

Samuel H. Cox and Shaun Wang are with the department of risk management & insurance, Georgia State University, P.O. Box 4036, Atlanta, GA 30302-4036, USA. Yijia Lin is with the department of accounting and finance at Youngstown State University, One University Plaza, Youngstown, OH 44555, USA. The authors can be contacted via e-mail: samcox@gsu.edu. The authors thank conference participants at the eighth Bowles Symposium and the second International Longevity Risk and Capital Market Solutions Symposium for helpful comments. We also thank the editor.

and the issuer, reducing credit ratings and adding to costs." The Swiss Re bond issued in December 2003 is a breakthrough in life insurance securitization—it is the first pure mortality security. It stripped out pure mortality risks and thus increased the transparency of the deal. Moreover, pure mortality securities may provide a diversification benefit since mortality may have no or low correlation with an investor's existing portfolio (Lin and Cox, 2005). According to MorganStanley (2003), "the appetite for this security [the Swiss Re bonds] from investors was strong."

However, mortality securities will not achieve the level of success of mortgage-backed securities and other types of asset-backed securities until a substantial volume of transactions reaches the public markets (Cowley and Cummins, 2005). Among all factors, correct transaction price is an indispensable part of their success. However, like mortality securities, mortality securitization modeling is in an early stage of development. We note there are relatively few preliminary papers in this area. Developing asset pricing theory in this area is important since it will help market participants better understand these new financial instruments. Most of the existing mortality securitization pricing and modeling papers have two major shortcomings: first, they ignore mortality jumps (Lee and Carter, 1992; Renshaw, Haberman, and Hatzoupoulos, 1996; Lee, 2000; Sithole, Haberman, and Verrall, 2000; Milevsky and Promislow, 2001; Olivieri and Pitacco, 2002; Dahl, 2003; Cairns, Blake, and Dowd, 2006) and/or correlation between reference risks. Mortality jumps should not be ignored in mortality securitization modeling since the rationale behind selling or buying mortality securities is to hedge or take catastrophe mortality risks (i.e., more death than expected). Moreover, we should take into account correlation of mortality risks if the security is based on several reference risks (e.g., population mortality indices of several countries). Second, the complete market pricing methodology may not be appropriate. Pure mortality risk bonds may not be spanned with traded securities. Therefore, we propose to price mortality bonds in an incomplete market framework with the jump processes, using multivariate exponential tilting.

Change of probability measure is a common theme in pricing and valuation of risks and contingent claims. In no-arbitrage financial pricing theory, the price of a contingent claim is evaluated as the expected payoff under a risk-neutral probability measure that is different from its statistical counterpart (Harrison and Kreps, 1979). In a complete market, the risk-neutral probability measure can be readily inferred from market transaction data. In an incomplete market, we do not have sufficient market data to infer a risk-neutral distribution. However, we may have historical data which allow us to estimate statistical distribution of the potential outcomes and their respective likelihoods. The question that then arises is how to construct a risk-neutral density from the estimated statistical density, as a basis for pricing contingent claims. Madan and Unal (2004) study this problem and refer to this change of measure as "risk neutralizing" the statistical distribution.

Exponential tilting, as a general method for neutralizing the statistical distribution, has been discussed by many authors (Buhlmann, 1980; Gerber and Shiu, 1996; Madan and Unal, 2004). As Madan and Unal (2004) put it, exponential tilting is broadly consistent with much of the current literature on no-arbitrage pricing of contingent claims (Duffie, 1992; Karatzas and Shreve, 1992; Heston, 1993; Gerber and Shiu, 1996), and is potentially widely applicable in pricing risks embedded in loan defaults, mortgage

refinancing, electricity trading, weather derivatives, and catastrophic insurance. Wang (2006) shows that normalized exponential tilting of the probability density function (PDF) of X (with respect to Z) is equivalent to applying the Wang transform to the cumulative distribution of X , and is an extension of the capital asset pricing model to risks with general-shaped distributions. By using a multiperiod equilibrium model, Kijima (2006) also reaches the same conclusions as those of multivariate exponential tilting (Wang, 2006). Our article proposes to use multivariate exponential titling to price mortality securities.

This article proceeds as follows. The section "Mortality Securitization Markets" describes the current mortality securitization market and the first mortality bond—the Swiss Re bond. The bivariate exponential tilting as an incomplete market pricing method is introduced in the section "Incomplete Market Pricing Method—Multivariate Exponential Tilting". In the section "Mortality Stochastic Processes," we propose a mortality stochastic model with jumps. We show that the jump process plays an important role in mortality securitization modeling. The section "Pricing Mortality Securities by Bivariate Exponential Tilting" prices the Swiss Re bond by using bivariate exponential tilting. Our model nicely explains the market outcome of the Swiss Re bond. "Discussion and Conclusions" is a final discussion and conclusion.

MORTALITY SECURITIZATION MARKETS

Lane and Beckwith (2005) describe recent activity in the insurance securitization market. Capital market investors search for uncorrelated risk for diversification and the risk-adjusted excess return " α ." Insurance-linked securities have low or no correlation with financial markets, providing diversification. Moreover, the existing insurance-linked securities provide high risk-adjusted excess returns. They attract more and more investors. For example, according to Lane and Beckwith (2005), hedge funds have increased their investment in insurance-linked securities. At the same time, insurers are looking for new sources of risk financing in the capital markets. Table 1 shows the property and life insurance securitization issues in dollar terms and by number of transactions through March 2005. In general, insurance securitization has increased in both dollar amounts and the number of issues, especially in the last

TABLE 1
Property and Life Insurance Securitization Issuance

Period	Issuance in Millions	Number of Deals
Pre 3/1998	\$886.1	7
4/1998–3/1999	\$1,366.9	7
4/1999–3/2000	\$1,219.4	11
4/2000–3/2001	\$1,126.0	10
4/2001–3/2002	\$826.9	6
4/2002–3/2003	\$832.20	25
4/2003–3/2004	\$1,894.60	27
4/2004–3/2005	\$1,803.30	21

Source: Lane Financial L.L.C., April, 2005.

two years. The life securitization includes three "pure" mortality bonds in the US and one in Europe since December 2003.

The first publicly known pure mortality security is the Swiss Re bond issued in December 2003 (MorganStanley, 2003; Swiss Re, 2003; The Actuary, 2004). The Swiss Re bonds sold very well. We develop a mortality securitization model to help explain why investors found its price attractive.

Mortality Shocks

The financial capacity of the life insurance industry to pay catastrophic death losses from hurricanes, epidemics, earthquakes, and other natural or man-made disasters is limited. Although mortality among the general and insured populations has witnessed remarkable improvements over the last several decades, mortality could continue to improve or it could worsen. Some events (for example, the 1918 worldwide flu) act like a common shock to the entire mortality curve and decrease the expected lifetimes of a large number of the population. A more recent example of unanticipated catastrophe death losses is the December 26, 2004 earthquake and tsunami. It caused massive devastation across southern Asia and eastern Africa (Guy Carpenter, 2005). The earthquake damage in Indonesia was obscured by the subsequent tsunami that hit 15–20 minutes later. The death and missing count has now exceeded 300,000 (Table 2). The last column in Table 2 shows that the 2004 Indonesian population death index increased by 16.58 percent relative to the 2003 level. The excess population mortality

TABLE 2

Death Toll and Percentage Excess Death Rates by Country on March 4, 2005

Country	Confirmed Deaths ^a	Missing ^a	% Excess Death Rate ^b
Indonesia	127,420	116,368	16.58%
Sri Lanka	38,195	4,924	33.81%
India	10,779	5,614	0.18%
Thailand	5,395	2,991	1.90%
Somalia	298	-	0.21%
Myanmar	90	-	-
Maldives	82	-	3.25%
Malaysia	68	-	0.06%
Tanzania	10	-	-
Bangladesh	2	-	-
Kenya	1	-	-
Total	182,340	129,897	-

^aSource: Associated Press.

^bBased on the authors' calculation. The percentage excess death rate is equal to the excess death rate from tsunami and missing divided by the normal population death rate.

Notes: The normal population death rate is obtained from <http://www.education.yahoo.com>.

death rate is even higher for Sri Lanka, about 34 percent. It raises a realistic question for life insurers, "Are you well prepared to handle such disasters in the future?"

Design of the Swiss Re Bond

To expand its capacity to pay catastrophic mortality losses, the Swiss Reinsurance Company, one of the world's largest reinsurance companies, obtained \$400 million in coverage from institutional investors from the first pure mortality security. Swiss Re issued the bond in late December 2003. It matures on January 1, 2007. So it is a three-year deal. Only the principal is exposed to mortality risk. The mortality risk is defined in terms of an index q based on the weighted average annual population death rates in the US, UK, France, Italy, and Switzerland. If the index q exceeds 130 percent of the actual 2002 level, q_0 , then the investors will have a reduced principal payment. The following equation describes the principal repayment mathematically.

$$\text{Principal repayment} = \begin{cases} 400,000,000 & \text{if } q \leq 1.3q_0 \\ 400,000,000 \frac{1.5q_0 - q}{0.2q_0} & \text{if } 1.3q_0 < q \leq 1.5q_0 \\ 0 & \text{if } q > 1.5q_0 \end{cases}$$

where q = weighted average population mortality in the US, UK, France, Italy, and Switzerland, q_0 = Year 2002 level, q_i = Year (2002 + i) level, and $q = \max(q_1, q_2, q_3)$. This is based on documents provided by MorganStanley (2003). The definition in the actual indenture is more complex (Blake, Cairns, and Dowd, 2006) but similar.

If the maturity value of the Swiss Re deal were based on Sri Lanka's population index, the excess death rate of 33.81 percent caused by the earthquake and tsunami in 2004 (Table 2) would exceed the trigger level of 30 percent. Then the maturity value would be lower than the face value.

INCOMPLETE MARKET PRICING METHOD—MULTIVARIATE EXPONENTIAL TILTING

Pricing derivative securities in complete markets involves replicating portfolios. For example, if we have a traded bond and stock index, then options on the stock index can be replicated by holding bonds and the index, which are priced. The analogy for the Swiss Re bond does not work. The bond is a mortality derivative, but we have no efficiently traded mortality index with which to create a replicating hedge. Situations like this are called incomplete markets. Pricing in this situation must rely on some other assumption—there is no traded underlying security. With multivariate exponential tilting, we can use the observed price of the Swiss Re bond to derive implicitly the value of the market price of risk. The multivariate exponential tilting model applies to other insurance risks, so it gives a method of comparing price parameters of mortality deals and catastrophe property deals (and potentially other risks as well).

This section introduces the concept of normalized exponential tilting and establishes an important link with probability distortions for the multivariate case. Normalized exponential tilting provides a general framework for pricing risks with respect to given reference risks and for valuing contingent claims on given underlying risks. We apply the multivariate exponential tilting to price the Swiss Re deal in Section 5.

Normalized Multivariate Exponential Tilting

Consider n variables $X_1, X_2, \dots, X_n$ and k references $Y_1, Y_2, \dots, Y_k$ on a probability space (Ω, P) .

Definition 1: For each scenario ω in the probability space (Ω, P) , the exponential tilting of $X_1, X_2, \dots, X_n$ with respect to references $Y_1, Y_2, \dots, Y_k$ is defined by the following PDF:

$$\frac{f^*(x_1(\omega), x_2(\omega), \dots, x_n(\omega))}{f(x_1(\omega), x_2(\omega), \dots, x_n(\omega))} = c [\exp(\sum_{j=1}^k \lambda_j Y_j(\omega))], \quad (1)$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are real-valued parameters that control the magnitude of risk-adjustments, and c is a normalizing coefficient.

In terms of the joint PDF, we can reformulate Equation (1) as follows:

$$\frac{f^*(x_1, x_2, \dots, x_n)}{f(x_1, x_2, \dots, x_n)} = c E [\exp(\sum_{j=1}^k \lambda_j Y_j) \mid X_1 = x_1, X_2 = x_2, \dots, X_n = x_n]. \quad (2)$$

We leave much flexibility in the choice of the references $Y_1, Y_2, \dots, Y_k$. For instance, one can choose the references $Y_1, Y_2, \dots, Y_k$ to be the risks $X_1, X_2, \dots, X_n$ themselves, the company aggregate, or some industry indices. In the following discussion, $Y_1, Y_2, \dots, Y_k$ refer to population mortality indices.

With the exponential tilting in Equation (1), we do not have a consistent interpretation of the λ parameter, except for in the special case when Y is a normal (Gaussian) variable. Indeed, if we keep the value of λ fixed, the scale and shape of the reference variable Y can significantly impact the result of exponential tilting. In order to get a meaningful interpretation of the parameters $\lambda_1, \lambda_2, \dots, \lambda_k$, we need to apply a normalization procedure to all references $Y_1, Y_2, \dots, Y_k$. Wang (2006) proposes a normalization procedure of the reference variable Y through percentile-matching to a standard normal variable Z . In other words, $Y = F_Y^{-1}(\Phi(Z))$, with Φ being the cumulative density function of Z , and $F_Y^{-1}(p) = \inf\{y \mid F_Y(y) \geq p\}$. Z is the normalized variable corresponding to Y and replaces Y in the exponential tilting.

We assume that there are standard normal variables $Z_1, Z_2, \dots, Z_k$ such that

$$Y_1 = F_{Y_1}^{-1}(\Phi(Z_1)), Y_2 = F_{Y_2}^{-1}(\Phi(Z_2)), \dots, Y_k = F_{Y_k}^{-1}(\Phi(Z_k)), \quad (3)$$

obtained by percentile mapping. We define the normalized multivariate exponential tilting of $X_1, X_2, \dots, X_n$ with respect to references $Y_1, Y_2, \dots, Y_k$ as follows:

$$\frac{f^*(x_1, x_2, \dots, x_n)}{f(x_1, x_2, \dots, x_n)} = c E [\exp(\sum_{j=1}^k \lambda_j Z_j) \mid X_1 = x_1, X_2 = x_2, \dots, X_n = x_n]. \quad (4)$$

Wang (2006) proves that the normalized multivariate exponential tilting of $X_1, X_2, \dots, X_n$ with respect to references $Y_1, Y_2, \dots, Y_k$ shown in Equation (4) is equivalent to the multivariate Wang transform. Here is the precise statement.

Theorem 1: Assume that $\{X_1, X_2, \dots, X_n; Y_1, Y_2, \dots, Y_k\}$ follow a normal copula with correlation matrix

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy} & \Sigma_{yy} \end{pmatrix}. \quad (5)$$

The normalized multivariate exponential titling in Equation (4) of $X_1, X_2, \dots, X_n$ w.r.t. $Y_1, Y_2, \dots, Y_k$ is equivalent to applying Wang transforms to X_i with

$$F_{X_i}^*(x_i) = \Phi[\Phi^{-1}(F_{X_i}(x_i)) + \beta_i], \text{ and } \beta_i = \sum_{j=1}^n \rho_{X_i, Y_j} \cdot \lambda_j, \quad (\text{for } i = 1, 2, \dots, n). \quad (6)$$

The correlation matrix between $X_1, X_2, \dots, X_n$ is unchanged after the normalized multivariate exponential tilting, $\Sigma_{xx}^* = \Sigma_{xx}$. Kijima (2006) reaches the same conclusion as Equation (6) by using a multiperiod equilibrium argument.

MORTALITY STOCHASTIC PROCESSES

The Swiss Re bond and some proposed mortality securities link their payoffs to population mortality indices. This provides transparency since the indices are available to the public. The Swiss Re bond is based on mortality experience of five countries. Our discussion shows how to describe the dynamics of the US and UK population mortality indices. Our method can be extended to five countries in principle but it requires more data than are available.

Model

Our approach combines a Brownian motion and a compound Poisson process. The probability of a jump (e.g. a big change in mortality) occurring during a time interval of $(t, t + h)$ (where h is as small as you like) can be written as

$$\begin{aligned} \Pr[\text{No event occurs in } (t, t + h)] &= 1 - \Lambda h + o(h) \\ \Pr[\text{One event occurs in } (t, t + h)] &= \Lambda h + o(h) \\ \Pr[\text{More than one event occurs in } (t, t + h)] &= o(h), \end{aligned} \quad (7)$$

where Λ is the mean number of arrivals per unit time, where $o(h)/h$ tends to 0 as h tends to 0. N_t represents the total number of jumps during a time interval of $(0, t)$. The stochastic process N_t and the standard Brownian motion W_t (described below) are independent.

We describe the US population mortality index q_t^{us} dynamics at time t as the combination of a Brownian motion and a compound Poisson process as follows:

$$\frac{dq_t^{\text{us}}}{q_t^{\text{us}}} = \begin{cases} (\alpha^{\text{us}} - \Lambda^{\text{us}}k^{\text{us}} - \Lambda^{\text{intl}}k^{\text{intl}})dt + \sigma^{\text{us}}dW_t^{\text{us}}, & \text{if the Poisson event does not occur at time } t; \\ (\alpha^{\text{us}} - \Lambda^{\text{us}}k^{\text{us}} - \Lambda^{\text{intl}}k^{\text{intl}})dt + \sigma^{\text{us}}dW_t^{\text{us}} + (Y^{\text{us}} - 1) + (Y^{\text{intl}} - 1), & \text{if the Poisson event occurs at time } t, \end{cases} \quad (8)$$

where α^{us} is the instantaneous expected force of the US population mortality index; σ^{us} is the instantaneous volatility of the mortality index, conditional on no jumps. W_t^{us} is a standard Brownian motion with mean 0 and variance t .

The quantity $(Y^{\text{us}} - 1)$ is an impulse function producing a finite intra-US jump in q_t^{us} to $q_t^{\text{us}} Y^{\text{us}}$. We can get $k^{\text{us}} \equiv E(Y^{\text{us}} - 1)$ where $E(Y^{\text{us}} - 1)$ is the expected percentage change in the mortality index if a Poisson event occurs. Λ^{us} is the intra-US Poisson jump parameter. Our model also captures the common mortality jumps to all countries, for example, the 1918 worldwide flu. Λ^{intl} is the inter-country (or international) Poisson jump parameter. The quantity $(Y^{\text{intl}} - 1)$ is an impulse function producing a finite international common jump in the US population mortality index q_t^{us} to $q_t^{\text{us}} Y^{\text{intl}}$ with $k^{\text{intl}} \equiv E(Y^{\text{intl}} - 1)$.

The " $\sigma^{\text{us}} dW_t^{\text{us}}$ " part describes the instantaneous part of the unanticipated "normal" mortality index change, and " $Y^{\text{us}} - 1$ " and " $Y^{\text{intl}} - 1$ " describe the part due to the "abnormal" mortality shocks. If $\Lambda^{\text{us}} = 0$, then $Y^{\text{us}} - 1 = 0$. Similarly, $Y^{\text{intl}} - 1 = 0$ if $\Lambda^{\text{intl}} = 0$. When $\Lambda^{\text{us}} = 0$ and $\Lambda^{\text{intl}} = 0$, it is the same as the standard stochastic model without jumps.

The mortality index, q_t^{us} , will be continuous most of the time with finite jumps of differing signs and amplitudes occurring at discrete points of time. If α^{us} , Λ^{us} , k^{us} , σ^{us} , Λ^{intl} , and k^{intl} are constants, we can solve the differential Equation (8) as

$$\frac{q_t^{\text{us}}}{q_0^{\text{us}}} = \exp \left[\left(\alpha^{\text{us}} - \frac{1}{2} \sigma^{\text{us}2} - \Lambda^{\text{us}} k^{\text{us}} - \Lambda^{\text{intl}} k^{\text{intl}} \right) t + \sigma^{\text{us}} W_t^{\text{us}} \right] Y(N_t^{\text{us+intl}}), \quad (9)$$

where $N_t^{\text{us+intl}}$ is the total number of mortality jumps with parameter $\Lambda^{\text{us}} t$ and $\Lambda^{\text{intl}} t$ during a time interval of length t . And it follows the independent Poisson process described in Equation (7). The cumulative jump size $Y(N_t^{\text{us+intl}}) = 1$ if $N_t^{\text{us+intl}} = 0$ and $Y(N_t^{\text{us+intl}}) = \prod_{j=1}^{N_t^{\text{us}} + N_t^{\text{intl}}} Y_j$ for $N_t^{\text{us+intl}} \geq 1$ where the size of the j th jump, Y_j , is independently and identically distributed.

From Equation (9), we can derive the US index value q_{t+h}^{us} , given as q_t^{us} resulting in

$$\begin{aligned} q_{t+h}^{\text{us}} | \mathcal{F}_t &= q_t^{\text{us}} \exp \left[\left(\alpha^{\text{us}} - \frac{1}{2} \sigma^{\text{us}2} - \Lambda^{\text{us}} k^{\text{us}} - \Lambda^{\text{intl}} k^{\text{intl}} \right) h + \sigma^{\text{us}} \Delta W_t^{\text{us}} \right] \\ &\times \prod_{j > N_t^{\text{us}}}^{N_{t+h}^{\text{us}}} Y_j^{\text{us}} \prod_{i > N_t^{\text{intl}}}^{N_{t+h}^{\text{intl}}} Y_i^{\text{intl}}, \end{aligned} \quad (10)$$

where $\mathcal{F}_t$ is the information set up to time t .

Similarly, the UK index value q_{t+h}^{uk} given as q_t^{uk} is similarly defined as follows:

$$\begin{aligned} q_{t+h}^{\text{uk}} | \mathcal{F}_t &= q_t^{\text{uk}} \exp \left[\left(\alpha^{\text{uk}} - \frac{1}{2} \sigma^{\text{uk}2} - \Lambda^{\text{uk}} k^{\text{uk}} - \Lambda^{\text{intl}} k^{\text{intl}} \right) h + \sigma^{\text{uk}} \Delta W_t^{\text{uk}} \right] \\ &\times \prod_{j > N_t^{\text{uk}}}^{N_{t+h}^{\text{uk}}} Y_j^{\text{uk}} \prod_{i > N_t^{\text{intl}}}^{N_{t+h}^{\text{intl}}} Y_i^{\text{intl}}, \end{aligned} \quad (11)$$

where α^{uk} is the instantaneous expected force of the UK population mortality index; σ^{uk} is the instantaneous volatility of the UK mortality index, conditional on no jumps. W_t^{uk} is a standard Brownian motion with mean 0 and variance t . The covariance of W_t^{us} and W_t^{uk} is equal to

$$\text{Cov}(W_t^{\text{us}}, W_t^{\text{uk}}) = \rho \sigma^{\text{us}} \sigma^{\text{uk}}, \quad (12)$$

where ρ is the correlation coefficient.

We assume Y^{us} , Y^{uk} , and Y^{intl} are log-normally and independently distributed, that is,

$$\begin{aligned} Y_j^{\text{us}} &= e^{m^{\text{us}} + s^{\text{us}} u}, & \text{where } u &\sim N(0, 1) \\ Y_j^{\text{uk}} &= e^{m^{\text{uk}} + s^{\text{uk}} v}, & \text{where } v &\sim N(0, 1) \\ Y_j^{\text{intl}} &= e^{m^{\text{intl}} + s^{\text{intl}} y}, & \text{where } y &\sim N(0, 1). \end{aligned} \quad (13)$$

If Y^c where $c = \text{us}, \text{uk}, \text{or intl}$ are log-normally distributed, then the distribution of $\frac{q_{t+h}^c}{q_t^c}$ will be log-normal too. The Appendix shows the derivation of our maximum likelihood function from Equations (10) and (13).

Data

Our US data are obtained from the Vital Statistics of the United States (VSUS).¹ The VSUS reports the United States age-adjusted death rates per 100,000 standard million population (2000 standard) for selected causes of death. Age-adjusted death rates are used to compare relative mortality risks across groups and over time; they are the indexes rather than the direct measures. The UK data are obtained from the Human Mortality Database.² The Human Mortality Database reports the death and population size of the England and Wales for different ages. We divide the total number of deaths in different ages by the total population to get the UK population mortality index. We plot our data from 1900 to 1998 in Figure 1.

Figure 1 shows that the mortality stochastic process does not follow a mean-reverting process.³ Moreover, there are several jumps in the US and UK population mortality evolution which should be captured by a good mortality stochastic model. Interestingly, the UK population mortality index was more volatile than that of the US up to the 1970s and did not improve afterward. Also, the US and UK had some common big shocks, like the 1918 worldwide flu. Mortality shocks may cause financial distress or bankruptcy of insurers and they are also the risks underlying the mortality securities.

Estimation Results. Based on the US and UK population mortality indices q_t^{us} and q_t^{uk} from 1900 to 1998 shown in Figure 1, Table 3 reports our maximum likelihood estimation results. The US instantaneous expected force of mortality index α^{us} is equal

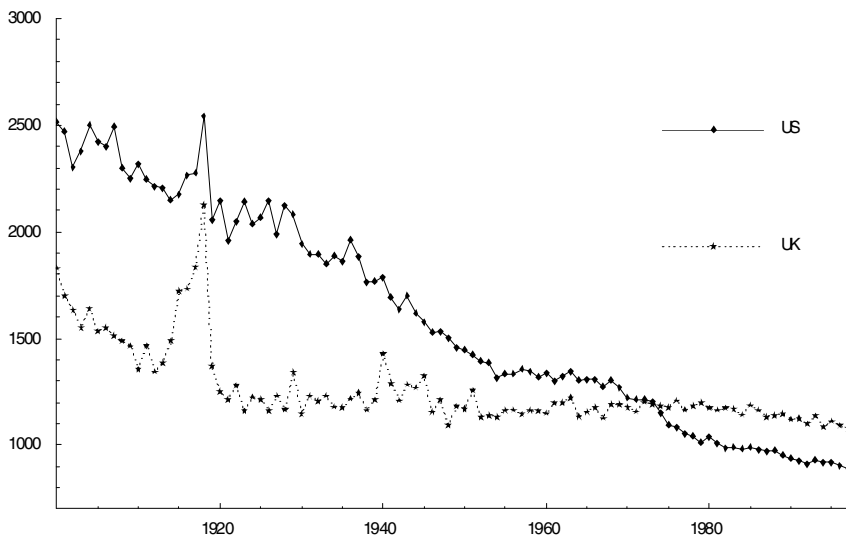
¹ Source: <http://www.cdc.gov>.

² Source: <http://www.mortality.org>.

³ But it could be a process that is mean reverting around a trend.

FIGURE 1

1900–1998 US and UK Total Population Death Rate Per 100,000 ($=100,000q_t^{us}$ or $100,000q_t^{uk}$ where $t = 1900, 1901, \dots, 1998$)

**TABLE 3**

Maximum Likelihood Parameter Estimates Based on the US and UK Population Mortality Index 1900–1998. The Model without Jumps Is Rejected at the Significance Level of 0.1 Percent.

Parameter	Estimate	Parameter	Estimate	Parameter	Estimate
α^{us}	-0.0100	α^{uk}	-0.0033	ρ	0.5299
σ^{us}	0.0308	σ^{uk}	0.0237	Λ^{intl}	0.0309
Λ^{us}	10^{-6}	Λ^{uk}	0.8533	m^{intl}	-0.0295
m^{us}	-0.0050	m^{uk}	-0.0114	s^{intl}	0.1412
s^{us}	10^{-6}	s^{uk}	0.0600	k^{intl}	-0.0193
k^{us}	-0.0050	k^{uk}	-0.0096		

to -0.0100 which is lower than that of the UK $\alpha^{uk} = -0.0033$. The negative sign of α^{us} and α^{uk} suggests the population mortality improves over time. However, on average, the UK mortality index improves less than that of the US. The US instantaneous volatility of the mortality index, conditional on no jumps, σ^{us} is equal to 0.0302 which is a little bit larger than that of the UK $\sigma^{uk} = 0.0237$. However, the internal jump size volatility parameter $s^{uk} = 0.0600$ is much larger than that of the US $s^{us} = 10^{-6}$. It coincides with what we observe from Figure 1. The jump size of the UK mortality index, in most cases, is larger than that of the US mortality index. The estimate of the Poisson parameter Λ^{intl} implies that the worldwide mortality jump is approximately a one-in-thirty-years ($1/\Lambda \approx 30$) event. Most of the US mortality jumps arise from the worldwide events since the US internal jump parameter Λ^{us} is almost zero. Moreover,

the correlation of the US and UK instantaneous expected force of the population mortality indice is equal to $\rho = 0.5299$.⁴ Our likelihood ratio test rejects the model without jumps at the significance level of 0.1 percent.

PRICING MORTALITY SECURITIES BY BIVARIATE EXPONENTIAL TILTING

We show how to describe the US and UK population mortality dynamics by using Brownian motions W^{us} , W^{uk} , and jump sizes Y^{us} , Y^{uk} , Y^{intl} in the section "Mortality Stochastic Processes." The correlation between W^{us} and W^{uk} is ρ . The jump sizes Y^{us} , Y^{uk} , and Y^{intl} are independent of each other and of W^{us} and W^{uk} . Assume that $\{W^{us}, W^{uk}, Y^{us}, Y^{uk}, Y^{intl}\}$ use themselves as references, that is,

$$W^{us}, W^{uk}, Y^{us}, Y^{uk}, Y^{intl}; W^{us}, W^{uk}, Y^{us}, Y^{uk}, Y^{intl}. \quad (14)$$

Therefore, the correlation matrix of these five factors is as follows:

$$\Sigma = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.5299 & 0 & 0 & 0 \\ 0.5299 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (15)$$

According to Theorem 1, the risk-neutralized joint distribution for $\{W^{us}, W^{uk}, Y^{us}, Y^{uk}, Y^{intl}\}$ is also bivariate normal with correlation coefficients:

$$\Sigma^* = \Sigma = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.5299 & 0 & 0 & 0 \\ 0.5299 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

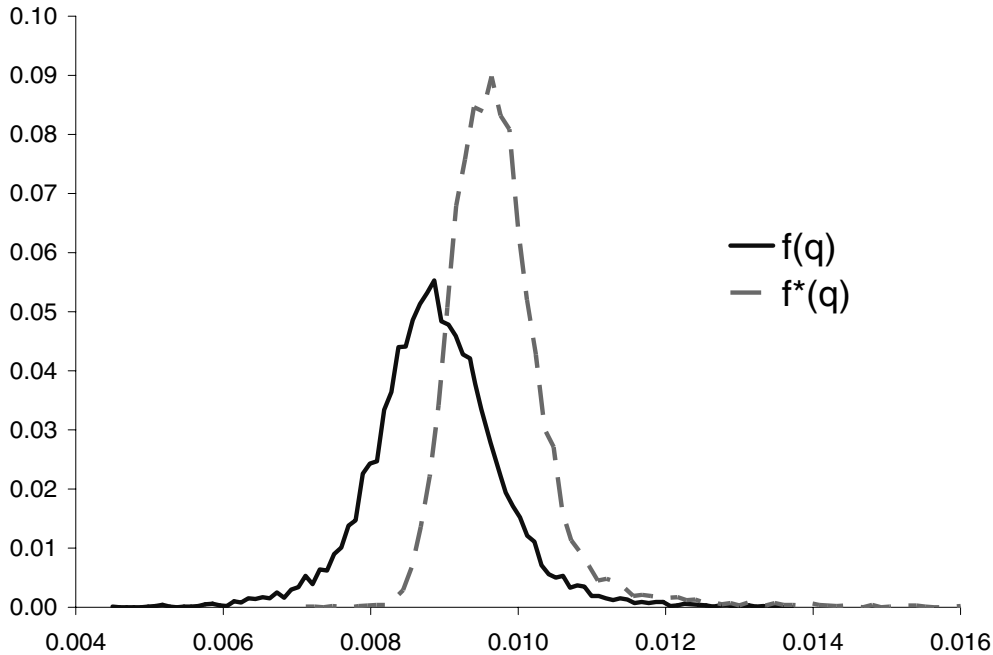
The variables $\lambda_{W^{us}}, \lambda_{W^{uk}}, \lambda_{Y^{us}}, \lambda_{Y^{uk}}$, and $\lambda_{Y^{intl}}$ represent the risk-adjustment parameters for the US and the UK physical mortality distribution of Brownian motions and jump sizes. We assume $\lambda_{W^{us}} = \lambda_{W^{uk}} = \lambda_{Y^{us}} = \lambda_{Y^{uk}} = \lambda_{Y^{intl}} = \lambda$.⁵ Based on Equation (6), the risk-neutralized distributions are equivalent to changing measure on the Brownian motion and jump sizes. That is, $W_t^{i*} = W_t^i + \beta_{W^i} t$ for $i = us$ or uk and $m^{j*} = m^j + \beta_{Y^j} \sigma^j$ for $j = us, uk, \text{ or } intl$, with

⁴ ρ measures the correlation of two Brownian motions, not jump processes. Shocks affecting both countries are modeled as an inter-country (or international) Poisson jump process. Therefore, high ρ is not simply driven by the 1918 worldwide flu. To show this, we get rid of the 1918 data point and then re-estimate all parameters. The re-estimated ρ equals to 0.5726, very close to 0.5299 estimated with the 1918 data.

⁵ If we have other mortality bond prices, we can relax this assumption and solve for $\lambda_{W^{us}}, \lambda_{W^{uk}}, \lambda_{Y^{us}}, \lambda_{Y^{uk}}$, and $\lambda_{Y^{intl}}$, respectively.

FIGURE 2

Bivariate Exponential Tilting Transformed Probability Distribution of q with $\lambda = 0.83$ and Physical Probability Distribution of q (Shown as Solid Line). The Horizontal Axis is the 1-Year Weighted Mortality Index (80 Percent on the US Population Death Rate and 20 Percent on the UK) and the Vertical Axis Stands for the Probability.



$$\begin{pmatrix} \beta_{W_{us}} \\ \beta_{W_{uk}} \\ \beta_{Y_{us}} \\ \beta_{Y_{uk}} \\ \beta_{Y_{intl}} \end{pmatrix} = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{W_{us}} \\ \lambda_{W_{uk}} \\ \lambda_{Y_{us}} \\ \lambda_{Y_{uk}} \\ \lambda_{Y_{intl}} \end{pmatrix} = \begin{pmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda \\ \lambda \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda + \lambda\rho \\ \lambda + \lambda\rho \\ \lambda \\ \lambda \\ \lambda \end{pmatrix}. \quad (17)$$

We assume the Swiss Re mortality index is a weighted index of which 80 percent is based on the US population mortality index and 20 percent on the UK population mortality index.⁶ Based on equations (6) and (10) and the US and UK population mortality index from 1900 to 1998, our estimated market price of risk λ for the references of the Swiss Re deal is 0.83. Figure 2 shows that the transformed PDF $f^*(q)$ with $\lambda = 0.83$ by applying the bivariate exponential tilting lies on the right of the PDF of the simulated US population mortality index $f(q)$. After transforming the data, we put

⁶ In reality, 70 percent of the Swiss Re bond is based on the US mortality index, 15 percent on the UK, 7.5 percent on France, 5 percent on Switzerland, and 2.5 percent on Italy. Although our example is only based on the US and UK population mortality experience, our multivariate tilting model can be extended to more countries.

more weight on the right tail. It implies that the market expects a higher probability of having a big loss than the actual probability suggests.

Wang (2000) reports that the average market price of risk of property catastrophe bonds is about 0.45 based on the univariate two-factor Wang transform with 6 degrees of freedom.⁷ Bantwal and Kunreuther (1999) found the spread of property catastrophe bonds are too high to be explained by standard financial theory. The market price of risk of the Swiss Re bond for three jump processes $\beta_{Y^{us}}$, $\beta_{Y^{uk}}$, and $\beta_{Y^{intl}}$ is all equal to 0.83 and 1.27 for two Brownian motions $\beta_{W^{us}}$ and $\beta_{W^{uk}}$. They are even higher than that of the property catastrophe bonds, 0.45.⁸ Although the high market price of risk of the Swiss Re deal may suggest high transaction costs of the first mortality security, it may also be interpreted as the Swiss Re overcompensating the investors for their taking its mortality risks. The risk premium of the Swiss Re bond is much higher than our model suggests. So, it explains why the “appetite” for the Swiss Re bond was strong.

Why did the Swiss Re company pay such a high risk premium to the investors? The Swiss Re Company’s life reinsurance business accounted for 43 percent of its group revenues in 2002, up from 38 percent in 2001 (MorganStanley, 2003). Although capital is crucial for a firm to absorb mortality shocks, the true economic capital requirements of life reinsurance business is not straightforward. Moreover, the costs of potential financial distress are high. Minton, Sanders, and Strahan (2004) conclude that securitization of financial institutions is a contracting innovation aimed at lowering financial distress costs. In addition, MorganStanley (2003) concludes that Swiss Re must be taking a view that the cost of capital that is relieved via this transaction exceeds the effective net cost of servicing the bond. Finally, the high price may include a premium to develop the mortality securitization market. If catastrophes deplete traditional reinsurance risk-taking capacity, the insurers can turn to the mortality security market for protection. In all, the Swiss Re mortality bond is a good deal to the investors.

DISCUSSION AND CONCLUSIONS

We are learning every day how important mortality forecasts are in the management of life insurers and private pension plans. Securitization and development of mortality bonds can be an important part of a capital market solution to these problems. Before the Swiss Re bond issued at the end of 2003, life insurance securitization was not designed to manage mortality risk; rather they were exactly like asset securitization. In these cases, the insurers convert future life insurance profits into cash to increase liquidity rather than manage mortality risk. The new securitization we study in this article focuses on the other side of an insurer’s balance sheet—liabilities on future

⁷ Applying the bivariate exponential tilting with $\lambda = 0.45$, our calculated par spread for the Swiss Re bond is 0.45 percent which is lower than that of the Swiss Re bond 1.35 percent. The difference may arise from the fact that we use a weighted index based on the US and UK population indices as the benchmark while the Swiss Re deal is based on the weighted average of mortality experiences of five developed countries. If we use the weighted indices on five countries, we expect that our calculated par spread will be even lower because of the diversification effect of mortality risks among these five countries.

⁸ Admittedly, there exists differences between the methods to get univariate and multivariate estimates which make comparison more difficult.

mortality payments. The introduction of mortality securities to transfer the insurer's risks in the liabilities side increases its capacity and maintains its competitiveness.

A market for mortality-based securities will develop if the prices and contracting features make the securities attractive to potential buyers and sellers. The Swiss Re bond sold well. We explain its market outcomes by looking at its risk premiums. To calculate the risk premium we need models, analogous to the term structure on interest rate models. The mortality bond market will be richer in that, in addition to default-free zero coupon bonds, it will have bonds which will be redeemed at face value only if a specific mortality event occurs. We find only a few preliminary papers on this topic (Dahl, 2003; Blake, Cairns, and Dowd, 2006; Cairns, Blake, and Dowd, 2006). Development of the theory in this direction is important as an extension of traditional bond market models, and it would be very useful in explaining mortality market risk to potential market participants.

Multivariate exponential tilting can be applied to price the correlated risks in a very general market framework. The market price of risk obtained from the multivariate exponential tilting reflects the correlation between different risks. This article is the first attempt to price the mortality securities by using the multivariate exponential tilting. We first discussed methods of risk neutralizing statistical distributions by applying exponential tilting of the probability density of mortality risks, with respect to a reference risk—mortality indices. Then, we used the market data to price the Swiss Re bond.

Our model shows that the Swiss Re mortality bond offers a higher risk premium to investors than the existing property-linked catastrophe bonds. It seems that the Swiss Re overcompensated the investors for them to take its catastrophic mortality risks and therefore it sold well.

Some may argue that the index-linked mortality securities are subject to unacceptable levels of basis risk. The basis risk is low for the Swiss Re bond because the Swiss Re Company is a global insurance leader with a 25 percent market share of all global reinsurance business. It is appropriate for it to link its mortality bond payments to the population indices. Moreover, using population indices provides transparency to investors, thus reducing the moral hazard problem. Lastly, insurers may not be willing to disclose their underwriting experiences to the public. If they used an index linked to their business, they would be forced to do so.

Our article shows how to price mortality securities by transforming Brownian motion and the jump sizes. Future research may include transformed Poisson frequency into our model in the incomplete market framework. In a complete market, expected values from a transformed process which is a martingale lead to arbitrage-free pricing. But in an incomplete market, there is no unique transformation. In other words, there is no perfect hedging strategy available which is associated with a single specific martingale transform. The minimum martingale transform and the minimum entropy martingale transform can be applied to find the hedging strategy that will minimize the variance of the payoff risk. Moller (2003) shows their application in insurance pricing. Venter, Barnett, and Owen (2006) combine the minimum martingale transform and the minimum entropy martingale to price catastrophe reinsurance programs.

In summary, we contribute to the mortality securitization literature by proposing a mortality stochastic model with jumps and pricing mortality securities in a general

market framework. Our model nicely explains the market outcomes of the Swiss Re deal. In addition, we comment on the basis risk problem of these two bonds. It shows the basis risk may not be a big issue to the Swiss Re deal. Future work may focus on how to include the jump frequency transformation into mortality securitization pricing.

APPENDIX: MAXIMUM LIKELIHOOD ESTIMATION OF MORTALITY STOCHASTIC MODEL WITH JUMPS

After taking logarithm on both sides of Equations (10) and (13), we obtain the vector $Z(h)$

$$\begin{aligned} Z(h) &= \begin{pmatrix} \log q_{t+h}^{\text{us}} - \log q_t^{\text{us}} \\ \log q_{t+h}^{\text{uk}} - \log q_t^{\text{uk}} \end{pmatrix} \\ &= \begin{pmatrix} \left(\alpha^{\text{us}} - \frac{1}{2} \sigma^{\text{us}2} - J^{\text{us}} \right) h + \sigma^{\text{us}} \Delta W_t^{\text{us}} + S J_h^{\text{us}} \\ \left(\alpha^{\text{uk}} - \frac{1}{2} \sigma^{\text{uk}2} - J^{\text{uk}} \right) h + \sigma^{\text{uk}} \Delta W_t^{\text{uk}} + S J_h^{\text{uk}} \end{pmatrix}, \end{aligned} \quad (\text{A1})$$

where $J^{\text{us}} = \Lambda^{\text{us}} k^{\text{us}} + \Lambda^{\text{intl}} k^{\text{intl}}$ and $J^{\text{uk}} = \Lambda^{\text{uk}} k^{\text{uk}} + \Lambda^{\text{intl}} k^{\text{intl}}$. Moreover,

$$\begin{aligned} S J_h^{\text{us}} &= \sum_{j > N_t^{\text{us}}}^{N_{t+h}^{\text{us}}} \log(Y_j^{\text{us}}) + \sum_{i > N_t^{\text{intl}}}^{N_{t+h}^{\text{intl}}} \log(Y_i^{\text{intl}}) \\ S J_h^{\text{uk}} &= \sum_{j > N_t^{\text{uk}}}^{N_{t+h}^{\text{uk}}} \log(Y_j^{\text{uk}}) + \sum_{i > N_t^{\text{intl}}}^{N_{t+h}^{\text{intl}}} \log(Y_i^{\text{intl}}). \end{aligned} \quad (\text{A2})$$

If the variable $\Delta N_h^c = N_{t+h}^c - N_t^c$ where $c = \text{us}, \text{uk}, \text{or intl}$ is the number of events in the time interval h , the variable $Z(h) \mid (\Delta N_h^{\text{us}} = n_1, \Delta N_h^{\text{uk}} = n_2 \text{ and } \Delta N_h^{\text{intl}} = n_3)$ will be normally distributed with mean

$$M_{n_1, n_2, n_3} = \begin{pmatrix} \alpha^{\text{us}} h - \frac{1}{2} \sigma^{\text{us}2} h - J^{\text{us}} h + n_1 m^{\text{us}} + n_3 m^{\text{intl}} \\ \alpha^{\text{uk}} h - \frac{1}{2} \sigma^{\text{uk}2} h - J^{\text{uk}} h + n_2 m^{\text{uk}} + n_3 m^{\text{intl}} \end{pmatrix}, \quad (\text{A3})$$

and covariance matrix

$$\Sigma_{n_1, n_2, n_3} = \begin{pmatrix} \sigma^{\text{us}2} h + n_1 s^{\text{us}2} + n_3 s^{\text{intl}2} & \sigma^{\text{us}} \sigma^{\text{uk}} \rho h + n_3 s^{\text{intl}2} \\ \sigma^{\text{us}} \sigma^{\text{uk}} \rho h + n_3 s^{\text{intl}2} & \sigma^{\text{uk}2} h + n_2 s^{\text{uk}2} + n_3 s^{\text{intl}2} \end{pmatrix}. \quad (\text{A4})$$

From $E[Y_j^c] = \exp(m^c + s^c/2)$, we get $k^c \equiv \exp(m^c + s^c/2) - 1$ since the expected value of the mortality index percentage change $k^c \equiv E[Y_j^c] - 1$ if the Poisson event occurs.

The conditional density function of $Z(h)$, $f_{Z(h)}(z | n_1, n_2, n_3)$, where

$$z = \begin{pmatrix} z^{\text{us}} \\ z^{\text{uk}} \end{pmatrix},$$

can be written in terms of the conditional density of $Z(h) | (\Delta N_h^{\text{us}} = n_1, \Delta N_h^{\text{uk}} = n_2, \text{ and } \Delta N_h^{\text{intl}} = n_3)$,

$$\begin{aligned} f_{Z(h)}(z | n_1, n_2, n_3) &= f_{Z(h)}(z | \Delta N_h^{\text{us}} = n_1, \Delta N_h^{\text{uk}} = n_2 \text{ and } \Delta N_h^{\text{intl}} = n_3) \\ &= \frac{\exp\left[-\frac{1}{2}(z - M_{n_1, n_2, n_3})' \Sigma_{n_1, n_2, n_3}^{-1} (z - M_{n_1, n_2, n_3})\right]}{\sqrt{(2\pi)^2 |\Sigma_{n_1, n_2, n_3}|}}, \end{aligned} \quad (\text{A5})$$

which is a normal distribution. Therefore, the density function of $f_{Z(h)}(z)$ is expressed as

$$\begin{aligned} f_{Z(h)}(z) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} f_{Z(h)}(z | n_1, n_2, n_3) \Pr(\Delta N_h^{\text{us}} = n_1) \Pr(\Delta N_h^{\text{uk}} = n_2) \Pr(\Delta N_h^{\text{intl}} = n_3) \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} f_{Z(h)}(z | n_1, n_2, n_3) \frac{e^{-\Lambda^{\text{us}} h} (\Lambda^{\text{us}} h)^{n_1}}{n_1!} \cdot \frac{e^{-\Lambda^{\text{uk}} h} (\Lambda^{\text{uk}} h)^{n_2}}{n_2!} \cdot \frac{e^{-\Lambda^{\text{intl}} h} (\Lambda^{\text{intl}} h)^{n_3}}{n_3!} \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} f_{Z(h)}(z | n_1, n_2, n_3) \frac{e^{-(\Lambda^{\text{us}} + \Lambda^{\text{uk}} + \Lambda^{\text{intl}})h} (\Lambda^{\text{us}})^{n_1} (\Lambda^{\text{uk}})^{n_2} (\Lambda^{\text{intl}})^{n_3} h^{n_1+n_2+n_3}}{n_1! n_2! n_3!}. \end{aligned} \quad (\text{A6})$$

If we have a time series of K observations of q_t where $t = 0, 1, 2, \dots, K-1$, there will be $K-1$ observations of z s with time interval equal to $h = 1$. In each time interval of length $h = 1$, we assume that the probability of an event from time t to $t+h$ is Λ and the probability of more than one event during such a time interval is negligible. We can estimate the parameters $\Lambda, \alpha, \sigma, m$, and s by maximizing the following loglikelihood function (A7) based on observations $z_1, z_2, \dots, z_{K-1}$:

$$\begin{aligned} &\sum_{i=1}^{K-1} \log f_{Z(1)}(z_i) \\ &= \sum_{i=1}^{K-1} \log \left(\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} f_{Z(1)}(z_i | n_1, n_2, n_3) \Pr(\Delta N_1^{\text{us}} = n_1) \Pr(\Delta N_1^{\text{uk}} = n_2) \Pr(\Delta N_1^{\text{intl}} = n_3) \right) \\ &= \sum_{i=1}^{K-1} \log \left(\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} f_{Z(1)}(z_i | n_1, n_2, n_3) \frac{e^{-\Lambda^{\text{us}} + \Lambda^{\text{uk}} + \Lambda^{\text{intl}}} (\Lambda^{\text{us}})^2 (\Lambda^{\text{uk}})^2 (\Lambda^{\text{intl}})^2}{n_1! n_2! n_3!} \right) \\ &\approx \sum_{i=1}^{K-1} \log \left(\sum_{n_1=0}^{10} \sum_{n_2=0}^{10} \sum_{n_3=0}^{10} f_{Z(1)}(z_i | n_1, n_2, n_3) \frac{e^{-\Lambda^{\text{us}} + \Lambda^{\text{uk}} + \Lambda^{\text{intl}}} (\Lambda^{\text{us}})^2 (\Lambda^{\text{uk}})^2 (\Lambda^{\text{intl}})^2}{n_1! n_2! n_3!} \right), \end{aligned} \quad (\text{A7})$$

where

$$f_{Z(1)}(z_i | n_1, n_2, n_3) = f_{Z(1)}(z_i | \Delta N_1^{\text{us}} = n_1, \Delta N_1^{\text{uk}} = n_2 \text{ and } \Delta N_1^{\text{intl}} = n_3) \\ = \frac{\exp\left[-\frac{1}{2}(z_i - M_{n_1, n_2, n_3})' \Sigma_{n_1, n_2, n_3}^{-1} (z_i - M_{n_1, n_2, n_3})\right]}{\sqrt{(2\pi)^2 |\Sigma_{n_1, n_2, n_3}|}}. \quad (\text{A8})$$

When $h = 1$, we obtain

$$M_{n_1, n_2, n_3} = \begin{pmatrix} \alpha^{\text{us}} - \frac{1}{2}\sigma^{\text{us}^2} - J^{\text{us}} + n_1 m^{\text{us}} + n_3 m^{\text{intl}} \\ \alpha^{\text{uk}} - \frac{1}{2}\sigma^{\text{uk}^2} - J^{\text{uk}} + n_2 m^{\text{uk}} + n_3 m^{\text{intl}} \end{pmatrix}, \text{ and} \quad (\text{A9}) \\ \Sigma_{n_1, n_2, n_3} = \begin{pmatrix} \sigma^{\text{us}^2} + n_1 s^{\text{us}^2} + n_3 s^{\text{intl}^2} & \sigma^{\text{us}\text{uk}}\rho + n_3 s^{\text{intl}^2} \\ \sigma^{\text{us}\text{uk}}\rho + n_3 s^{\text{intl}^2} & \sigma^{\text{uk}^2} + n_2 s^{\text{uk}^2} + n_3 s^{\text{intl}^2} \end{pmatrix}.$$

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