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A Shortcut Way of Pricing Default Risk Through Zero-Utility Principle

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Abstract

Pricing the default risk is a hot challenge for every risk manager. The problem is tackled in the framework of the zero-utility principle. According to Pratt (1964), an approximation of the risk premium should be proportional to the Arrow–Pratt absolute risk aversion coefficient and the variance. Is that still true as a default risk is concerned? The answer appears to be negative, because the variance does not look to be an appropriate tool for asymmetrical risk. On the other hand, fear of ruin coefficient and probability of default are proved to be well-tailored tools for a preliminary pricing. Bid and ask price approximations are both elicited and a necessary condition for risk exchange set out.

Introduction

Pricing the default risk premium is a pressing challenge in risk management. This short article is aimed at giving a practice-oriented hint for pricing defaultable assets in the framework of the zero-utility principle. According to a classical result, the greater the individual risk aversion and the perception of the "magnitude of riskiness," the greater the premium. More precisely, the former attitude is captured by the Arrow–Pratt absolute risk aversion coefficient $(-\frac{u''}{u'})$ of the agent endowed with the utility function u, and the latter by the variance σ^2, if the risk is small (see Pratt, 1964). A questionable point arises: are these two "measures" still well-fitted tools for pricing the risk of default? The answer appears to be negative. Although, the premium always increases with the Arrow–Pratt coefficient (see Gerber, 1982), the variance is a statistical index of "stability" of data around the mean value. Therefore, it does not seem to be well-tailored to control what happens on the "extreme left tail" of the distribution. As already pointed out in Tibiletti (2004), our perplexities are well grounded. Under minimal assumptions, we show that a first-order approximation of the pricing is proportional to:

- the *individual "fear of ruin" coefficient* $\frac{u}{u'}$;

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- the *utility variation in percent* if the default occurs; and
- the *subjective probability of default;*

where the "fear of ruin" is a notion coined by Aumann and Kurz (1977, pp. 1147-1148) in their seminal paper on taxation. In conclusion, the first two components capture the sensitivity in utility "in the small and in the large" and the last one is an inverse measure of the risk. It is worthwhile noting that the variance is not directly involved. The Arrow–Pratt coefficient has direct influence in higher-order pricing approximations.

Another question deserves our attention. It is well known that premium pricing depends on the subjective point of view of the counterparts: on one hand, there is an individual (the insured) willing to get rid of a risk against payment of a premium and, on the other hand, there is an agent (the insurer) accepting to endorse the risk against collection of a fair compensation. Understanding the gears that move the bid and ask prices may cast light on conditions for a successful risk exchange. A necessary condition for bargaining is worked out.

The article is organized as follows. In the "Pricing the Default Risk" section the concept of default premium in the zero-utility framework is posed. Premium approximations from both the counterparts are discussed. In the "A Shortcut Bargaining Condition" section a necessary condition for the risk bargaining is set out. A conclusion in the last section ends the article.

PRICING THE DEFAULT RISK

Basic material on utility theory and insurance goes back to Borch (1968), using the utility concept of von Neumann and Morgenstern (1944). The fundamentals of premium principles were laid by Buhlmann (1970) who introduced the zero-utility premium (see also Gerber, 1979 and comprehensively Goovaerts, Vylder, and Haezendonck, 1984). This principle states that the premium for a risk should be calculated such that the expected utility is (at least) equal to the zero utility.[1] Consider an individual (the insured or the "*seller*" from now on) would be willing to pay to get rid of an infinitesimal chance p of undergoing a default risk

$$X = \begin{cases} -w, & p \\ 0, & (1-p), \end{cases} \tag{1}$$

where $w > 0$. On the other hand, the counterpart (the insurer or the "*buyer*" from now on) would be willing to accept the risk of undergoing a possible huge loss against collecting a certain amount. Clearly, perception of risk as well as the initial wealth differ for the two counterparts. Let us restate the classical definitions (in line with the seminal paper of La Valle, 1968).

Definition 1: Let u be the seller utility function, w_0 (with $w_0 > w$) the initial wealth, and p the evaluation of the default probability. The real number $z_s = z_s(p)$ is called the *seller default risk price* if

[1] For an application of zero-utility rule for pricing defaultable bonds, see Chunchi (1991).

$$u(w_0 - z_s) = (1 - p)\, u(w_0) + p u(w_0 - w)\,. \tag{2}$$

Definition 2: Let v be the buyer utility function, x_0 (with $x_0 > w$) the initial wealth, and q the evaluation of the default probability. The real number $z_b = z_b(q)$ is called the *buyer default risk price* if

$$v(x_0) = v(x_0 + z_b)(1 - q) + v(x_0 + z_b - w)\, q\,. \tag{3}$$

Note that $z_b = z_b(q)$ and $z_s = z_s(p)$ are the (positive) bid and ask price of X, respectively. Obviously, the bargaining may succeed if

$$z_b \leq z_s\,. \tag{4}$$

In order to keep the main ideas in mind, in what follows u and v are assumed to be once-differentiable.

Seller's Default Risk Price

Let us differentiate (2) with reference to p, so that

$$z_s'(p) = \frac{u(w_0) - u(w_0 - w)}{u'(w_0 - z_s(p))}.$$

Suppose that p is small enough so that a first-order Taylor approximation of the default premium can be used

$$z_s(p) = z_s(0) + z_s'(0)p + o(p) \approx z_s'(0)p.$$

Thus,

$$z_s(p) \approx \frac{u(w_0) - u(w_0 - w)}{u'(w_0)} p = \frac{u(w_0)}{u'(w_0)} \frac{u(w_0) - u(w_0 - w)}{u(w_0)} p. \tag{5}$$

Therefore, the default price is proportional to:

- the *"fear of ruin" coefficient* $\frac{u}{u'}$ at level $w_{0;}$
- the *utility variation in percent* if the default occurs $\frac{u(w_0) - u(w_0 - w)}{u(w_0)}$; and
- the estimate *probability of default p*.

The first two components express the sensitivity in utility "in the small and in the large." The former gives the utility variation in marginal terms, the latter in percentage terms. The fear of ruin coefficient is the fitting measure of the "attitude toward risking all fortune." To understand why, let us consider its reciprocal $\frac{u'}{u}$, the so-called *boldness coefficient* (see Aumann and Kurz, 1977). Boldness represents the individual's willingness to risk financial ruin in exchange for a marginal increase in wealth. Aumann and Kurz (1977) proved that the marginal tax rate is directly dependent on the fear of ruin. This concept is also of interest in an entirely different context, namely that of the Nash bargaining problem (see Nash, 1950). Consider two players bargaining over the division of a fixed amount. The Nash solution calls for that compromise which makes the two players equally fearful of ruin, where ruin is here taken to mean disagreement. Moreover, in the value of life literature (see Viscusi, 1993 and references therein) the fear of ruin has been recognized as a crucial factor for explaining how risk preferences affect the willingness to pay for reductions in mortality risks (see Eeckhoudt and Hammitt, 2001).

Let us turn back to (5). The last component confirms the groundedness of our conjecture: the "real gear" in pricing is just the estimate probability p of default, instead of the variance.

A final question may arise. Why does our premium approximation no longer depend on the Arrow–Pratt coefficient $-\frac{u''}{u'}$ and the variance, as in Pratt (1964)? Let us go a step further in approximation:

$$z_s(p) \approx z_s'(0)p + \frac{1}{2}z_s''(0)p^2.$$

Since $z_s''(0) = -[\frac{u(w_0)}{u'(w_0)}\frac{u(w_0)-u(w_0-w)}{u(w_0)}]^2(-\frac{u''(w_0)}{u'(w_0)})$, the Arrow–Pratt coefficient is involved in a second-order approximation. Vice versa, the lack of direct dependence on the variance is due to a different reason. We approximate $z_s = z_s(p)$ for a small change in probability p, not for a small change in the variation final wealth w_0, as in Pratt (1964). A consequence is that we can derive a first-order approximation of the price by simply examining the rate of increase of the premium with respect to p. It is not so in Pratt (1964), because the equivalent first-order effect for a variation in final wealth is zero, so the second-order effect becomes the crucial one. (See Gollier, 2001, pp. 21-24 for an analysis of Pratt's "in the small" approximation.) Anyway, the variance partially influences the premium, because $\sigma^2 = w_0^2 p(1-p) \approx w_0^2 p$ so p can be seen as a linear approximation of σ^2.

Buyer's Default Risk Price

Following the same idea, analogous results can be found for the buyer risk premium. Deriving (3) with reference to q, we get

$$z_b'(q) = \frac{v(x_0+z_b) - v(x_0+z_b-w)}{v'(x_0+z_b)(1-q) + v'(x_0+z_b-w)q}.$$

If q is small enough, the first-order Taylor approximation can be used, so

$$z_b(q) = z_b(0) + z_b(0)q + o(q) \approx z_b(0)q.$$

Thus,

$$z_b(q) \approx \frac{v(x_0) - v(x_0-w)}{v'(x_0)}q = \frac{v(x_0)}{v'(x_0)}\frac{v(x_0)-v(x_0-w)}{v(x_0)}q. \tag{6}$$

Again premium approximation is proportional to:

- the *fear of ruin coefficient* at *wealth* x_0, i.e., the utility sensitivity "in the small";

- the *relative variation in utility* in the case of default $\frac{v(x_0) - v(x_0 - w)}{v(x_0)}$, i.e., the utility sensitivity "in the large"; and
- the estimate *probability of default* q.

A SHORTCUT BARGAINING CONDITION

Clearly, as the bargaining can successfully come off, the bid and ask prices should be related as: $z_b \leq z_s$. Substituting we get

$$\frac{v\,(x_0)}{v'\,(x_0)} \frac{v\,(x_0) - v\,(x_0 - w)}{v\,(x_0)} q \leq \frac{u\,(w_0)}{u'\,(w_0)} \frac{u\,(w_0) - u\,(w_0 - w)}{u\,(w_0)} p$$

or

$$\frac{\dfrac{v(x_0)}{v'(x_0)} \dfrac{v(x_0) - v(x_0 - w)}{v(x_0)}}{\dfrac{u(w_0)}{u'(w_0)} \dfrac{u(w_0) - u(w_0 - w)}{u(w_0)}} \frac{q}{p} \leq 1. \tag{7}$$

An easy-to-read bargaining condition can be summarized in this statement: The *greater* the sensitivity in utility "in the small and in the large" and the perception of the risk p of the seller with respect to those of the buyer, the *easier* the bargaining.

CONCLUSION

"The *greater* the Arrow–Pratt coefficient and the variance, the *greater* the risk premium" states a classical result based on the zero-utility principle (Pratt, 1964). Is this statement still true when pricing defaultable assets is involved? The answer is proved to be negative. A first-order approximation shows that the default risk pricing is proportional to: (1) the sensitivity "in the small and in the large" in risk aversion, where the sensitivity "in the small" is measured by the fear of ruin coefficient and (2) the individual evaluation of the probability of default. No direct influence of the variance occurs. Moreover, since the bid and ask prices differ, a necessary condition for risk exchange is also worked out.

REFERENCES

Aumann, R. J., and M. Kurz, 1977, Power and Taxes, *Econometrica*, 45: 1137-1161.

Borch, K., 1968, *The Economics of Uncertainty* (Princeton, NJ: Princeton University Press).

Buhlmann, H., 1970, *Mathematical Methods in Risk Theory* (Berlin: Springer-Verlag).

Chunchi, W., 1991, A Certainty Equivalent Approach to Municipal Bond Default Risk Estimation, *Journal of Financial Research*, Fall, 14(3): 241-247.

Eeckhoudt, L., and J. K. Hammitt, 2001, Background Risks and the Value of a Statistical Life, *Journal of Risk and Uncertainty*, 23(3): 261-279.

Gerber, H. U., 1979, *An Introduction to Mathematical Risk Theory* (Philadelphia: Homewood).

Gerber, H. U., 1982, A Remark on the Principle of Zero Utility, *ASTIN Bulletin*, 13(2): 133-134.

Gollier, C., 2001, *The Economics of Risk and Time* (Cambridge: MIT Press).

Goovaerts, M. J., F. De Vylder, and J. Haezendonck, 1984, *Insurance Premiums, Theory and Applications* (Amsterdam: North-Holland).

La Valle, I., 1968, On Cash Equivalents and Information Evaluation in Decision Under Uncertainty, *Journal of The American Statistical Association*, 63, parts I-II-III: 252-290.

Nash, J. F., 1950, The Bargaining Problem, *Econometrica*, 18: 155-162.

Pratt, J. W., 1964, Risk Aversion in the Small and in the Large, *Econometrica*, 32: 122-136.

Tibiletti, L., 2004, Pricing the Default Risk Premium with "the Fear of Ruin," *Atlantic Economic Journal*, 32(4): 356.

Viscusi, W. K., 1993, The Value of Risks to Life and Health, *Journal of Economic Literature*, Dec, 31: 1912-1946.

von Neumann, J., and O. Morgenstern, 1944, *Theory of Games and Economic Behavior* (Princeton, NJ: Princeton University Press).