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The Effect of the Real Option to Transfer on the Value of Guaranteed Minimum Death Benefits

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Abstract

Variable annuity contracts frequently have many options and option-like features embedded in the contracts. Some are obvious, such as guaranteed minimum death benefits (GMDBs), while others are less obviously option-like. In this article, we consider the effect of the real option to transfer funds between fixed and variable accounts. If a GMDB rider is considered in isolation, it is sometimes in the policyholder's interest to transfer to the fixed fund if the fixed fund earns less than the variable fund in a risk-neutral world. On the other hand, the option to transfer will not be used if the entire annuity and rider are considered together.

Introduction

There have been a number of papers in recent years dealing with the valuation of guaranteed minimum death benefits (GMDBs) embedded in annuity contracts. These riders guarantee some minimum benefit at death, which is frequently the return of premiums paid or a fixed percentage rollup on that premium. They resemble a sequence of European put options, as the annuitant cannot choose the time of his own death.

A few papers discuss option pricing from an actuarial standpoint. Gerber and Shiu (1994) discuss methods of obtaining closed-form option prices using Esscher transforms. Tiong (2000) uses this approach to value equity-indexed annuities. Milevsky and Promislow (2001) and Milevsky and Posner (2001) extend the discussion to cases more specifically related to insurance and annuity contracts by looking at the option to annuitize and the valuation of GMDBs.

There has recently been recognition of the fact that even GMDBs have features of American options. Milevsky and Salisbury (2001) note that policyholders have a real option to lapse their policy and reinstate the death benefit when their policy is out of the money and this option should be priced for. They assume that policyholders exercise their option optimally in that they would lapse and reinstate immediately if there were no surrender charges to prevent it. With this assumption, they calculate

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the surrender charge necessary to fully compensate the insurance company for this loss of income. This surrender charge is one where the individual owning the policy would be indifferent between maintaining the policy and lapsing it.

A second option that is frequently available in these contracts is the option to transfer funds between a variable account and an attached fixed account that earns a guaranteed rate. A particular worry is that individual policyholders will choose to transfer their money from the variable account to the fixed account when the GMDB is in-the-money and thereby lock in a positive GMDB for a substantial period into the future. This option to transfer would behave very much like an American option to lock in a GMDB at a time of the policyholder's own choosing. Unlike an American option, however, the decision to transfer is not irreversible. Another American-type option exists, which allows the policyholder to transfer back to variable account. Clearly, this option has value if the fixed fund were to rise to the value of the GMDB strike, as the value would be zero if the funds were in the fixed account and positive if the funds were in the variable account.

The transfer option in these contracts bears some similarity to passport options that have attracted attention recently. They are even more similar to vacation put options introduced by Shreve and Vecer (2000). These options are special cases of call options on a traded account. The individual owning the account starts with an initial wealth π_0 and is allowed to invest a value q_t in an asset that follows a geometric Brownian motion process. At maturity time T, the investor receives the maximum of 0 and the value π_T. The value q_t is constrained to be in the range $\alpha \leq q_t \leq \beta$. The case $\alpha = \beta = 1$ corresponds to a European call option, and $\alpha = \beta = -1$ corresponds to a European put option. The case $\alpha = -1$ and $\beta = 1$ corresponds to the passport option first studied by Hyer, Lipton-Lifschitz, and Pugachevsky (1997) and Andersen, Andreasen, and Brotherton-Ratcliffe (1998), and studied later by Nagayama (1999), Henderson and Hobson (2000, 2001), and Delbaen and Yor (2002). The case $\alpha = 0$ and $\beta = 1$ corresponds to the vacation call and the case $\alpha = -1$ and $\beta = 0$ corresponds to the vacation put.

In this article, we use the Black–Scholes-type differential equation derived by Milevsky and Salisbury (2001), which must be satisfied by a GMDB option. We also show analytically that the option to transfer to the fixed fund has no value and will never be used unless the fixed growth rate is less than the risk-free rate less any asset fees taken off the variable account. If the fixed growth rate is less than this, the value of the option can be calculated and the approximate location of the optimal exercise boundary can be determined. Finally, we compare the GMDB results to those obtained for vacation puts.

DERIVATION OF THE DIFFERENTIAL EQUATION FOR A RETURN OF PREMIUM GMDB

Assume the existence of a deferred annuity with a variable account. The annuity has a GMDB rider that guarantees at least return of premium upon the death of the annuitant. This could be modeled as the sum of a continuous sequence of European put options. The weight at a given option duration would be equal to the instantaneous probability of death at that moment. The value of the strike at that moment would be $X(0)e^{pt}$. In a small interval dt, this sequence of puts collectively (f_{tot}) must obey the Black–Scholes differential equation,

$$\frac{\partial f_{\text{tot}}}{\partial t} + (r-q)S\frac{\partial f_{\text{tot}}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_{\text{tot}}}{\partial S^2} = r f_{\text{tot}}, \tag{1}$$

because each put obeys the equation individually.

What we are really interested in is the value of the GMDB if the person is still alive, which we will represent by f_a. The value if the person dies is denoted as f_d and the value if the person lapses is denoted by f_λ. This leads to the differential equation derived by Milevsky and Salisbury (2001) with a lapse term included for the GMDB value for a living annuitant of

$$\frac{\partial f_a}{\partial t} + (r-q)S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_a}{\partial S^2} = [r + \mu_x(t) + \lambda(S,t)]\, f_a - [\mu_x(t)]\, f_d(t) - [\lambda(S,t)]\; f_\lambda(t), \tag{2}$$

where $f_\lambda = 0$ for the GMDB considered in isolation and $f_\lambda = S$ when the contract as a whole is considered.

A verbal interpretation of this Equation (2) is that the combination of f_a and the death benefit is hedged by the last two terms on the left-hand side and must earn a risk-free rate. If no death benefits were available, the first term on the right-hand side says that the "live" value of the GMDB must grow faster than the risk-free rate in order to account for the expected value of not receiving this "live" value due to death or surrender of the annuitant. The last two terms on the right-hand side are a reduction in the rate of growth of the "live" GMDB value because of the expected death benefits or surrender benefits to be received.

If the rollup GMDB is being valued independently of the contract it is attached to, this specific differential equation is

$$\frac{\partial f_a}{\partial t} + (r-q)S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_a}{\partial S^2} = [r + \mu_x(t) + \lambda(S,t)]\; f_a - [\mu_x(t)]\,\mathrm{Max}(Xe^{pt} - S, 0). \tag{3}$$

If the total contract is being valued, the equation is

$$\begin{aligned} &\frac{\partial f_a}{\partial t} + (r-q)S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_a}{\partial S^2} \\ &\quad = [r + \mu_x(t) + \lambda(S,t)]\; f_a - [\mu_x(t)]\,\mathrm{Max}(Xe^{pt}, S) - [\lambda(S,t)]S. \end{aligned} \tag{4}$$

It is not necessary in practice to solve the differential equation unless an analytic solution is needed. It is often faster to work directly from the integral $f_a(S,t) = \int_0^\infty {}_s p_{x+t}^{(\tau)} \mu_{x+t}^{(d)}(s) P_s\, ds$, where P_t is the value of a t-year put option with strike $Xe^{p(t+s)}$. The derivation of this integral is similar to Hardy (2003, pp. 136-137). It can be shown that this integral does in fact satisfy Equation (3).

In the next several sections, we will derive some results for specific cases of Equations (3) and (4), which are easily evaluated from the differential equations but would be difficult or impossible to evaluate from the integral formulation. Situations that are more easily evaluated using the differential equation include those with surrender rates that are dependent on S and those with features of American options.

Solution to Equation (2) Given $p = 0\%$ and Constant Forces of Decrement

A very simple immediate application of the partial differential equation is the solution for the return of premium GMDB with a constant force of mortality and constant lapse rate. These assumptions are clearly unrealistic, but we will begin with them in order to eventually obtain an analytic result for the boundary. This solution will suggest other, more general, requirements for the existence of a boundary at all. For the same reason, we will assume geometric Brownian motion for the stock process.

Following Milevsky and Salisbury, these assumptions reduce the partial differential equation to an ordinary differential equation

$$-[r + \mu_x + \lambda(S)]\, f_a + (r - q)S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_a}{\partial S^2} = -\mu_x \,\mathrm{Max}(X - S, 0). \tag{6}$$

We will repeat here the result of Milevsky and Salisbury for the usual case where λ is independent of S, because the values of the constants will be used later in the solution where variable to fixed transfers are allowed. The general solution is

$$f_a(S, t) = a_1 S^{m_1} + a_2 S^{m_2}, \tag{7}$$

where

$$m_1 = \frac{-\left(r - q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(r + \mu_x + \lambda)}}{\sigma^2}, \tag{7a}$$

$$m_2 = \frac{-\left(r - q - \frac{\sigma^2}{2}\right) - \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(r + \mu_x + \lambda)}}{\sigma^2}, \tag{7b}$$

and particular solutions are

$$f_a(S, t) = 0, \quad S > X \tag{8a}$$

$$f_a(S, t) = \frac{\mu_x}{r + \mu_x + \lambda} X - \frac{\mu_x}{q + \mu_x + \lambda} S, \quad S < X. \tag{8b}$$

We now impose the boundary conditions that $f_a(0, t)$ is finite, $f_a(\infty, t)$ is 0, and the function and first derivative is continuous across the boundary $S = X$. These boundary conditions are justified in Appendix B and give a final result of

$$f_a(S,t) = \left\{ \frac{\mu_x X}{2(r+\mu_x+\lambda)} \left[1 - \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r+\mu_x+\lambda)}} \right] - \frac{\mu_x X}{2(q+\mu_x+\lambda)} \left[1 - \frac{\xi_1}{\sqrt{\xi_1^2 + 2(q+\mu_x+\lambda)}} \right] \right\} \left(\frac{S}{X} \right)^{m_2}, \quad S > X; \tag{9a}$$

and

$$f_a(S,t) = \frac{\mu_x}{r+\mu_x+\lambda} X - \frac{\mu_x}{q+\mu_x+\lambda} X \left(\frac{S}{X} \right) + \left\{ \frac{\mu_x X}{2(q+\mu_x+\lambda)} \left[1 + \frac{\xi_1}{\sqrt{\xi_1^2 + 2(q+\mu_x+\lambda)}} \right] - \frac{\mu_x X}{2(r+\mu_x+\lambda)} \left[1 + \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r+\mu_x+\lambda)}} \right] \right\} \left(\frac{S}{X} \right)^{m_1}, \quad S < X; \tag{9b}$$

where

$$\xi_1 = \frac{\left(r - q + \frac{\sigma^2}{2} \right)}{\sigma}, \tag{10a}$$

$$\xi_2 = \frac{\left(r - q - \frac{\sigma^2}{2} \right)}{\sigma}. \tag{10b}$$

Some specific values of the constants as well as some numerical solutions are given in Table 1.

Solution to Equation (2) When Transfers to a Fixed Fund Are Allowed

Many insurance companies sell variable annuities with a fixed or guaranteed option earning a fixed interest rate. This means that the policyholder has at least two options for the location of his funds. One option is to leave the funds in the variable account, where they would be subject to fluctuations in the market. A second would be to move the funds to the fixed account and forgo the market swings. One might worry about the effect such transfers would have on the value of the GMDB. For instance, assume the GMDB is currently far in the money. If the policyholder leaves his funds in the variable account, there is a reasonable possibility that the stock market will rise far enough fast enough to move the GMDB out of the money. On the other hand, transferring the money in the fixed account could "lock in" the death benefit. The policyholder could even achieve this without any loss of stock exposure if he has a

TABLE 1
GMDB Values for Various Combinations of Parameters

$r(\%)$	5	10	5	5	5	5
q(bp)	100	100	200	100	100	100
$\sigma(\%)$	20	20	20	30	20	20
$\mu_x(\%)$	2	2	2	2	12	2
$\lambda(\%)$	0	0	0	0	0	10
ξ_1	0.300000	0.550000	0.250000	0.283333	0.300000	0.300000
ξ_2	0.100000	0.350000	0.050000	−0.016667	0.100000	0.100000
m_1	1.436492	1.260399	1.637459	1.304011	2.458040	2.458040
m_2	−2.436492	−4.760399	−2.137459	−1.192900	−3.458040	−3.458040
a_2	0.030837	0.006057	0.039502	0.068044	0.065787	0.010965
b_1	0.411790	0.506057	0.253788	0.448997	0.282982	0.047164
S/X	Values	Values	Values	Values	Values	Values
2.0	0.005697	0.000223	0.008978	0.029764	0.005986	0.000998
1.9	0.006455	0.000285	0.010018	0.031642	0.007148	0.001191
1.8	0.007364	0.000369	0.011246	0.033750	0.008618	0.001436
1.7	0.008464	0.000484	0.012707	0.036132	0.010501	0.001750
1.6	0.009812	0.000646	0.014465	0.038842	0.012951	0.002158
1.5	0.011482	0.000879	0.016605	0.041950	0.016189	0.002698
1.4	0.013584	0.001221	0.019243	0.045549	0.020551	0.003425
1.3	0.016272	0.001737	0.022546	0.049759	0.026554	0.004426
1.2	0.019777	0.002543	0.026753	0.054744	0.035021	0.005837
1.1	0.024447	0.003848	0.032221	0.060731	0.047316	0.007886
1.0	0.030837	0.006057	0.039502	0.068044	0.065787	0.010965
0.9	0.039667	0.009792	0.049286	0.077073	0.093529	0.015588
0.8	0.051239	0.015325	0.061824	0.088019	0.130933	0.021822
0.7	0.065742	0.022821	0.077237	0.101048	0.177490	0.029582
0.6	0.083407	0.032483	0.095666	0.116362	0.232657	0.038776
0.5	0.104523	0.044576	0.117287	0.134224	0.295845	0.049307
0.4	0.129465	0.059454	0.142320	0.154981	0.366410	0.061068
0.3	0.158755	0.077626	0.171055	0.179126	0.443632	0.073939
0.2	0.193176	0.099894	0.203909	0.207434	0.526683	0.087780
0.1	0.234120	0.127784	0.241562	0.241344	0.614560	0.102427
0.0	0.285714	0.166667	0.285714	0.285714	0.705882	0.117647

different external fund that he could transfer from a fixed-like account to a variable account.

The option to transfer funds is like an American option that could be exercised at any time the policyholder believes it is in his interest. Unlike a standard American option, however, it is actually reversible. A standard American option expires when it is exercised. However, funds can be transferred from the fixed account back to the variable account at a later date if the policyholder is still alive and believes it to be in his interest. This makes the situation somewhat more complicated than the usual American option case.

Fortunately, Equation (2) can still be used in these situations. It is well known that the standard Black–Scholes equation still holds for an American option while it is live (see, for example, Wilmott, Howison, and DeWynne, 1995). Therefore, Equation (2) ought to hold in those situations, where the funds are in the variable account. In addition, a modified version of Equation (2) with a different growth rate and zero volatility ought to hold while the funds are in the fixed account. In the case of a guaranteed fund with growth rate g and zero volatility, the Equation (2) becomes

$$\frac{\partial f_a}{\partial t} + gS\frac{\partial f_a}{\partial S} = [r + \mu_x(t) + \lambda(S,t)]\, f_a - [\mu_x(t)]\,\mathrm{Max}(Xe^{pt} - S, 0), \tag{11}$$

for the GMDB alone, or

$$\frac{\partial f_a}{\partial t} + gS\frac{\partial f_a}{\partial S} = [r + \mu_x(t) + \lambda(S,t)]\, f_a - [\mu_x(t)]\,\mathrm{Max}(Xe^{pt}, S) - [\lambda(S,t)]S, \tag{12}$$

for the value of the entire contract.

We will again consider the simplest case of constant mortality and surrender rates. As we go along, some of the results will generalize. In particular, the idea that the growth rate of the fixed fund, g, must be less than $r - q$ in order for the transfer option to be valuable in the case of the isolated GMDB is proved in a more general case in Appendix A.

In this case, we now need to consider solutions to the differential equations in three separate regions. Let us define C to be the boundary between the area where all money is invested in the variable account and the area where all money is invested in the fixed account. If $S > X$, the GMDB will have value if the money is in the variable account, but no value if the money is in the fixed account. Therefore, C must be less than X, and the three equations and ranges are

$$-[r + \mu_x + \lambda]\, f_a + (r - q)S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_a}{\partial S^2} = 0, \quad S > X; \tag{13a}$$

$$-[r + \mu_x + \lambda]\, f_a + (r - q)S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_a}{\partial S^2} = -\mu_x(X - S), \quad C < S < X; \tag{13b}$$

$$-[r + \mu_x + \lambda]\, f_a + gS\frac{\partial f_a}{\partial S} = -\mu_x(X - S), \quad S < C. \tag{13c}$$

The solutions for the three ranges are given below, assuming again that $f_a(0, t)$ is finite and $f_a(\infty, t)$ is 0.

$$f_a(S,t) = (a + b_2)S^{m_2}, \quad S > X; \tag{14a}$$

$$f_a(S,t) = \frac{\mu_x}{r + \mu_x + \lambda}X - \frac{\mu_x}{q + \mu_x + \lambda}S + b_1 S^{m_1} + b_2 S^{m_2}, \quad C < S < X; \tag{14b}$$

$$f_a(S,t) = \frac{\mu_x}{r + \mu_x + \lambda}X - \frac{\mu_x}{r - g + \mu_x + \lambda}S + b_f S^{m_f}, \quad S < C; \tag{14c}$$

where m_1 and m_2 are defined as before and

$$m_f = \frac{r + \mu_x + \lambda}{g}. \tag{15}$$

The function and first derivative are continuous across the $S = X$ boundary, and the function, first and second derivatives are continuous across the $S = C$ boundary. The continuity conditions at $S = C$ can be derived from two facts. First, the policyholder must be indifferent to which fund the money is in at the boundary. Second, the location of the boundary should be chosen to maximize the value of the option. The proofs can be found in Appendix B. The ultimate solution is

$$\begin{aligned} a = {} & \frac{\mu_x X^{1-m_2}}{2(r + \mu_x + \lambda)} \left[1 - \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r + \mu_x + \lambda)}} \right] \\ & - \frac{\mu_x X^{1-m_2}}{2(q + \mu_x + \lambda)} \left[1 - \frac{\xi_1}{\sqrt{\xi_1^2 + 2(q + \mu_x + \lambda)}} \right], \end{aligned} \tag{16a}$$

$$\begin{aligned} b_1 = {} & \frac{\mu_x X^{1-m_1}}{2(q + \mu_x + \lambda)} \left[1 + \frac{\xi_1}{\sqrt{\xi_1^2 + 2(q + \mu_x + \lambda)}} \right] \\ & - \frac{\mu_x X^{1-m_1}}{2(r + \mu_x + \lambda)} \left[1 + \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r + \mu_x + \lambda)}} \right], \end{aligned} \tag{16b}$$

$$C = \left(\frac{(r - q - g)\left(\xi_1 + \sqrt{\xi_1^2 + 2(q + \mu_x + \lambda)}\right)}{(q + \mu_x + \lambda)\,\sigma + (r - q - g)\left(\xi_1 + \sqrt{\xi_1^2 + 2(q + \mu_x + \lambda)}\right)} \right)^{\frac{1}{m_1 - 1}} X, \tag{16c}$$

$$b_2 = \left[\frac{\left(\frac{\mu_x}{q + \mu_x + \lambda} - \frac{\mu_x}{r - g + \mu_x + \lambda} \right)(m_f - 1)}{(m_f - m_2)m_2} \right] C^{1-m_2} - b_1 \left[\frac{(m_f - m_1)m_1}{(m_f - m_2)m_2} \right] C^{m_1 - m_2}, \tag{16d}$$

and

$$b_f = b_1 \frac{m_1(m_1 - 1)}{m_f(m_f - 1)} C^{m_1 - m_f} + b_2 \frac{m_2(m_2 - 1)}{m_f(m_f - 1)} C^{m_2 - m_f}, \tag{16e}$$

Each of the equations depends only on things known from previous equations. Some insight into the solution can be acquired by looking at Equation (16c) for the boundary. All of the terms except $r - q - g$ are positive, so the equation only has a solution if $g < r - q$, that is, the growth rate in the fixed fund must be less than the expected growth in the variable fund in a risk-neutral environment. This makes some sense, as the fixed account is only able to "lock in" the death benefit to the extent that it grows more slowly than the variable account. In addition, the boundary is simply proportional to the strike, with a percentage that increases from 0 percent at $g = r - q$ to 100 percent at $g \ll r - q$. Some specific solutions for the constants as well as numerical values are given in Table 2.

The values in Tables 1 and 2 are illustrative only. They should not be viewed as representative of the value of the GMDB if more realistic mortality, lapse, and stock market performance assumptions were to be used. However, the fact that transfers to the fixed fund only increase the value of the isolated GMDB in the case where $g < r - q$ is a more general result that is proved in the general case in Appendix A.

Solution to Equation (2) for Valuing the Entire Contract

In the preceding analysis, we considered only the effect of transfers on the value of the GMDB in isolation. In reality, an individual cannot separate the effect of his transfers on his GMDB value and the effect of his transfers on the future surrender value of the contract. A low fixed growth rate increases the value of the GMDB by transferring to fixed, but simultaneously decreases the expected future surrender value. Not surprisingly, it will turn out that the cutoff point where the fixed fund is the more valuable fund in regard to the future surrender value is $g > r - q$, precisely the same cutoff line as for the GMDB, only with the opposite fund being more dominant. It will turn out that the effect on the surrender value in the constant force of mortality case always dominates the effect on the GMDB and people will want to transfer to fixed only if $g > r - q$ even if the effect on the GMDB is included. First, we will consider the variable annuity surrender value in isolation from the GMDB. We still assume that the account value is received on death, but no additional guarantee is included. In this case, the differential equation that must be satisfied is

$$-[r + \mu_x + \lambda]\, f_a + (r - q)S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_a}{\partial S^2} = -\mu_x S - \lambda S, \qquad (17)$$

with solution

$$f_a(S,t) = \left(\frac{\mu_x + \lambda}{q + \mu_x + \lambda}\right) S. \qquad (18)$$

Interestingly, this only equals the fund value if there are no asset fees being taken out. This odd result occurs in a world where the individual could withdraw his funds at a value of S at any time because the world of variable annuities is not the world of capital market theory with no taxes or transaction costs. If the individual withdraws his money from the annuity and places it in a mutual fund, there will be tax consequences as well as mutual fund fees. Milevsky and Posner (2001) suggest that much of the value

TABLE 2
GMDB Values for Various Combinations of Parameters With Optimal Transfers Evaluated Analytically

$r(\%)$	5	10	5	5	5	5	5
q(bp)	100	100	200	100	100	100	100
$\sigma(\%)$	20	20	20	30	20	20	20
μ_x	2	2	2	2	12	2	2
$\lambda(\%)$	0	0	0	0	0	10	0
$g(\%)$	1	1	1	1	1	1	2
ξ_1	0.300000	0.550000	0.250000	0.283333	0.300000	0.300000	0.300000
ξ_2	0.100000	0.350000	0.050000	−0.016667	0.100000	0.100000	0.100000
m_1	1.436492	1.260399	1.637459	1.304011	2.458040	2.458040	1.436492
m_2	−2.436492	−4.760399	−2.137459	−1.192900	−3.458040	−3.458040	−2.436492
m_f	7.000000	12.000000	7.000000	7.000000	17.000000	17.000000	3.500000
a	0.030837	0.006057	0.039502	0.068044	0.065787	0.010965	0.030837
b_1	0.411790	0.506057	0.253788	0.448997	0.282982	0.047164	0.411790
b_2	0.003198	0.003410	0.001632	0.001978	0.004183	0.000697	0.000731
b_f	0.318840	0.058005	0.705561	7.907713	6.515250	1.085875	0.259941
C	0.557023	0.784900	0.461344	0.290595	0.627673	0.627673	0.436133
S/X	Values	Values	Values	Values	Values	Values	Values
2.0	0.006287	0.000349	0.009349	0.030629	0.006367	0.001061	0.005832
1.9	0.007124	0.000446	0.010432	0.032562	0.007603	0.001267	0.006608
1.8	0.008127	0.000577	0.011710	0.034731	0.009166	0.001528	0.007538
1.7	0.009342	0.000757	0.013232	0.037182	0.011169	0.001862	0.008665
1.6	0.010829	0.001010	0.015062	0.039970	0.013774	0.002296	0.010044
1.5	0.012673	0.001374	0.017290	0.043169	0.017218	0.002870	0.011754
1.4	0.014993	0.001908	0.020038	0.046872	0.021857	0.003643	0.013906
1.3	0.017960	0.002715	0.023477	0.051205	0.028242	0.004707	0.016658
1.2	0.021827	0.003975	0.027858	0.056335	0.037248	0.006208	0.020245
1.1	0.026982	0.006014	0.033552	0.062496	0.050324	0.008387	0.025026
1.0	0.034035	0.009467	0.041133	0.070022	0.069971	0.011662	0.031568
0.9	0.043800	0.015423	0.051330	0.079315	0.099552	0.016592	0.040612
0.8	0.056747	0.025191	0.064453	0.090600	0.139983	0.023330	0.052498
0.7	0.073368	0.040197	0.080734	0.104074	0.191851	0.031975	0.067485
0.6	0.094509	0.057702	0.100528	0.120000	0.256985	0.042831	0.085945
0.5	0.121539	0.075772	0.124466	0.138745	0.330932	0.055155	0.108479
0.4	0.152903	0.093940	0.153537	0.160880	0.405883	0.067647	0.136236
0.3	0.185784	0.112121	0.185869	0.187441	0.480882	0.080147	0.169558
0.2	0.219052	0.130303	0.219057	0.219149	0.555882	0.092647	0.206644
0.1	0.252381	0.148485	0.252381	0.252382	0.630882	0.105147	0.245796
0.0	0.285714	0.166667	0.285714	0.285714	0.705882	0.117647	0.285714

of q is payment for the fact that variable annuity investment income is tax deferred. Also, some part of q must be payment for the value of the GMDB and would not be present in this case. Without some benefits of this type, a rational person surely would surrender any contract that charges fees that would not be charged by other substitutable investments. Even as a practical matter, substitutable investments with no GMDB also have fees. Mutual funds charge fees and individual stocks require brokers commissions and bid-ask spreads to sell, so they are still in some sense not worth S to an individual holding them. In addition, the variable annuity cannot be sold to a third party for its account value.

Next, we look for the solution to the differential equation that includes both the effect of the account itself and the effect of the GMDB. The differential equation satisfied is

$$-[r+\mu_x+\lambda]\, f_a + (r-q)S\frac{\partial f_a}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_a}{\partial S^2} = -\mu_x \,\mathrm{Max}(X,S) - \lambda S, \qquad (19)$$

which still has the general solution given in Equation (7) but has the following particular solutions:

$$f_a(S,t) = \left(\frac{\mu_x+\lambda}{q+\mu_x+\lambda}\right) S, \qquad S > X; \qquad (20a)$$

$$f_a(S,t) = \frac{\mu_x}{r+\mu_x+\lambda} X + \frac{\lambda}{q+\mu_x+\lambda} S, \qquad S < X. \qquad (20b)$$

If we take into account the boundary conditions that $f_a(0,t)$ is finite, $f_a(S,t)$ approaches the value without the GMDB as $S \to \infty$, and the function and first derivative are continuous across the boundary $S = X$, we get the following solution:

$$\begin{aligned} f_a(S,t) &= \left(\frac{\mu_x+\lambda}{q+\mu_x+\lambda}\right) S \\ &\quad + \left\{ \frac{\mu_x X}{2(r+\mu_x+\lambda)}\left[1 - \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r+\mu_x+\lambda)}}\right] - \frac{\mu_x X}{2(q+\mu_x+\lambda)} \right. \\ &\quad \left. \times \left[1 - \frac{\xi_1}{\sqrt{\xi_1^2 + 2(q+\mu_x+\lambda)}}\right]\right\} \left(\frac{S}{X}\right)^{m_2}, \quad S > X; \end{aligned} \qquad (21a)$$

and

$$\begin{aligned} f_a(S,t) &= \frac{\mu_x}{r+\mu_x+\lambda} X - \frac{\mu_x}{q+\mu_x+\lambda} X\left(\frac{S}{X}\right) + \frac{\mu_x+\lambda}{q+\mu_x+\lambda} X\left(\frac{S}{X}\right) \\ &\quad + \left\{ \frac{\mu_x X}{2(q+\mu_x+\lambda)}\left[1 + \frac{\xi_1}{\sqrt{\xi_1^2 + 2(q+\mu_x+\lambda)}}\right] \right. \\ &\quad \left. - \frac{\mu_x X}{2(r+\mu_x+\lambda)}\left[1 + \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r+\mu_x+\lambda)}}\right]\right\} \left(\frac{S}{X}\right)^{m_1}, \quad S < X; \end{aligned} \qquad (21b)$$

which is seen to be just the sum of the value of the GMDB and underlying policy separately. The final step is to compute the entire solution for the integrated policy assuming transfers between fixed and variable funds are allowed and will be executed optimally.

It is not immediately clear how to divide the regions and set boundaries in this case. We will begin by examining the area very near to $S = 0$. If the money were in the fixed fund, the solution would be

$$f_a(S,t) = \frac{\mu_x}{r + \mu_x + \lambda} X + \frac{\lambda}{r - g + \mu_x + \lambda} X \left(\frac{S}{X}\right) + b_f S^{m_f}, \tag{22}$$

and if the money were in the variable fund, the solution would be

$$f_a(S,t) = \frac{\mu_x}{r + \mu_x + \lambda} X + \frac{\lambda}{q + \mu_x + \lambda} X \left(\frac{S}{X}\right) + b_1 S^{m_1}. \tag{23}$$

The money will be held in whichever account is larger. Near $S = 0$, the first terms in Equations (22) and (23) cancel out, and the final term is negligible compared to the second term, which fortunately does not depend on the specifics of the solution. When the slope of Equation (23) is larger than Equation (22), the money will start in the variable fund. When the slope of Equation (23) is smaller than Equation (22), the money will start in the fixed fund. In other words, if $r - q > g$, the funds will start in the variable account, and if $r - q < g$, the funds will start in the fixed account. This is the opposite situation as in the case where the GMDB was considered in isolation.

Now consider the situation where $S \to +\infty$. The solution if the money is in the fixed account is

$$f_a(S,t) = \left(\frac{\mu_x + \lambda}{r - g + \mu_x + \lambda}\right) S, \tag{24}$$

and the value if the money is in the variable account is

$$f_a(S,t) = \left(\frac{\mu_x + \lambda}{q + \mu_x + \lambda}\right) S + a_2 S^{m_2}. \tag{25}$$

The last term in Equation (25) is positive, as it reflects the extra value of the GMDB. Again, if Equation (25) is larger than Equation (24), the money will be held in the variable fund, otherwise the money will be held in the fixed fund. In other words, if $r - q > g$, the funds will start in the variable account, and if $r - q < g$, the funds will start in the fixed account. This is the identical situation as the one at $S = 0$. Therefore, it seems likely that no boundary exists, and the money will be entirely in the fixed account or entirely in the variable account depending on which account earns the larger rate in a risk-neutral world.

TRINOMIAL LATTICE APPROACH

We now describe a procedure for determining option values and boundary locations for more realistic mortality conditions. We use a trinomial lattice approach similar to that in Hull (1997), with mortality based on the 2,000 male GAM table. We use time increments of 1/365th of a year, and assume the individual dies at age 115. The option value at this time is therefore the maximum of $X - S$ and 0. We then work our way backward through the lattice until age 0 is reached. At each age, we compute the value of the option assuming the fund is fully invested in the market, and the value of the option assuming the fund is fully invested in the fixed account. The option value at the earlier time is the maximum of these two values.

If the fund is fully invested in the stock market, the trinomial lattice model of Hull (1997, p. 360) is used. Using $r = 5\%$, $q = 100\,bp$, and $\sigma = 20\%$, we find a lattice spacing of $u = e^{\sigma\sqrt{3\Delta t}} = 1.018297318$, where the ratio between adjacent lattice points is held constant. This leads to probabilities of up, middle, and down moves of

$$\begin{aligned} p_u &= \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6} = 0.168177661, \\ p_m &= \frac{2}{3} = 0.666666667, \\ p_d &= -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6} = 0.165155672, \end{aligned} \tag{26}$$

respectively. The value of the option when fully invested in the market in the previous period is therefore

$$\begin{aligned} V(i,j) = e^{-r\Delta t}[&(1 - \mu_x \Delta t - \lambda \Delta t) \\ &\times (p_d V(i+1, j-1) + p_m V(i+1, j) + p_u V(i+1, j+1)) \\ &+ \mu_x \Delta t (p_d \operatorname{Max}(X - S/u, 0) + p_m \operatorname{Max}(X - S, 0) + p_u \operatorname{Max}(X - Su, 0))]. \end{aligned} \tag{27}$$

The value of the option when fully invested in the fixed fund must be found by interpolation of the future values, leading to an equation similar to Equation (27) but with the probabilities replaced by

$$\begin{aligned} p_u &= \frac{e^{g\Delta t} - 1}{u - 1} = 0.001497358, \\ p_m &= \frac{u - e^{g\Delta t}}{u - 1} = 0.998502642, \\ p_d &= 0, \end{aligned} \tag{28}$$

assuming $g = 1\%$.

Trinomial Lattice Results

In this section, we compare the results from the trinomial lattice to the analytic results of Section 2. We will reproduce the results of Table 2 by assuming constant force of mortality until age 115 and looking at the value of the option at age 0. This is a sufficiently long time to remove the effect of forced death at age of 115. It turns out that there is a boundary below $S = X$ as predicted by the analytic results. Below this boundary, the money is entirely invested in the fixed account and above this boundary, the money is entirely invested in the variable account. Table 3 gives the values from the trinomial lattice. Values that do not correspond to an exact lattice point are obtained by linear interpolation. Because the lattice is discrete, it is not possible to determine an exact location for the boundary. Instead, the upper and lower limits that differ by a factor of u are given. The results in Table 3 compare very favorably with the results in Table 2.

Table 4 gives the results of the trinomial lattice calculation for the specific assumptions: $r = 5\%$, $q = 100\,bp$, $\sigma = 20\%$, $\lambda = 10\%$, and $g = 1\%$. Mortality follows the 2,000 Male GAM table. Results are given at integral ages from 0 to 115. Again, it is not possible to determine the exact boundary location, so instead the geometric mean of the upper and lower boundary is given. The 2 percent constant mortality assumption is very similar to the age 64 results. For comparison, the age 64 table mortality is 1.00 percent and rises to 2.00 percent between ages 70 and 71.

TABLE 3

GMDB Values for Various Combinations of Parameters With Optimal Transfers Evaluated by Trinomial Lattice

r(%)	5	10	5	5	5	5	5
q(bp)	100	100	200	100	100	100	100
s(%)	20	20	20	30	20	20	20
m_x(%)	2	2	2	2	12	12	2
l(%)	0	0	0	0	0	0	0
g(%)	1	1	1	1	1	1	2
C_1	0.570016	0.804461	0.466946	0.300821	0.635528	0.635528	0.450316
C_2	0.559774	0.790006	0.458556	0.295415	0.624109	0.624109	0.442225
S/X	Values	Values	Values	Values	Values	Values	Values
2.0	0.006309	0.000352	0.009381	0.030722	0.006372	0.001062	0.005846
1.9	0.007148	0.000449	0.010466	0.032657	0.007610	0.001268	0.006623
1.8	0.008154	0.000581	0.011746	0.034828	0.009174	0.001529	0.007555
1.7	0.009370	0.000762	0.013270	0.037281	0.011178	0.001863	0.008683
1.6	0.010859	0.001017	0.015101	0.040071	0.013782	0.002297	0.010062
1.5	0.012709	0.001383	0.017334	0.043270	0.017234	0.002872	0.011776
1.4	0.015034	0.001922	0.020085	0.046979	0.021879	0.003646	0.013931
1.3	0.018007	0.002735	0.023528	0.051314	0.028270	0.004712	0.016686
1.2	0.021877	0.004001	0.027910	0.056447	0.037271	0.006212	0.020273
1.1	0.027044	0.006058	0.033612	0.062614	0.050371	0.008395	0.025062
1.0	0.034104	0.009530	0.041195	0.070135	0.070010	0.011668	0.031604
0.9	0.043889	0.015536	0.051402	0.079436	0.099627	0.016604	0.040659
0.8	0.056859	0.025394	0.064536	0.090726	0.140095	0.023349	0.052556
0.7	0.073516	0.040355	0.080831	0.104205	0.192031	0.032005	0.067560
0.6	0.094715	0.057745	0.100648	0.120139	0.257219	0.042870	0.086047
0.5	0.121770	0.075770	0.124625	0.138896	0.330951	0.055159	0.108632
0.4	0.153026	0.093927	0.153701	0.161050	0.405863	0.067644	0.136465
0.3	0.185852	0.112106	0.185965	0.187644	0.480862	0.080144	0.169744
0.2	0.219134	0.130287	0.219148	0.219296	0.555866	0.092644	0.206745
0.1	0.252529	0.148468	0.252529	0.252528	0.630869	0.105145	0.245866
0.0	0.285928	0.166648	0.285928	0.285928	0.705868	0.117645	0.285928

COMPARISON TO THE VACATION PUT

In this section, we compare the GMDB optimal strategy to that of the somewhat similar vacation put. Shreve and Vecer (2000) have found a closed-form solution for the value of the vacation put as well as the optimal strategy $q_t(t, S_t, \pi_t)$. q_t is the investment in

TABLE 4
GMDB Values for 2,000 Male GAM Mortality With Optimal Transfers Evaluated by Trinomial Lattice

	Age									
Stock	0	1	2	3	4	5	6	7	8	9
2.0	0.000034	0.000035	0.000036	0.000038	0.000039	0.000041	0.000043	0.000045	0.000047	0.000049
1.9	0.000039	0.000040	0.000042	0.000044	0.000046	0.000048	0.000050	0.000052	0.000054	0.000056
1.8	0.000046	0.000047	0.000049	0.000051	0.000053	0.000056	0.000058	0.000061	0.000063	0.000066
1.7	0.000055	0.000056	0.000058	0.000060	0.000063	0.000066	0.000069	0.000071	0.000074	0.000078
1.6	0.000066	0.000067	0.000069	0.000072	0.000075	0.000078	0.000082	0.000085	0.000089	0.000092
1.5	0.000081	0.000081	0.000083	0.000086	0.000090	0.000094	0.000098	0.000102	0.000107	0.000111
1.4	0.000102	0.000100	0.000102	0.000106	0.000110	0.000115	0.000120	0.000125	0.000130	0.000136
1.3	0.000132	0.000126	0.000128	0.000131	0.000136	0.000142	0.000148	0.000155	0.000161	0.000168
1.2	0.000178	0.000163	0.000162	0.000166	0.000171	0.000178	0.000186	0.000195	0.000203	0.000212
1.1	0.000255	0.000219	0.000212	0.000214	0.000220	0.000229	0.000239	0.000250	0.000262	0.000273
1.0	0.000390	0.000306	0.000286	0.000286	0.000291	0.000301	0.000313	0.000328	0.000344	0.000359
0.9	0.000628	0.000445	0.000398	0.000391	0.000395	0.000405	0.000420	0.000440	0.000463	0.000484
0.8	0.000960	0.000636	0.000550	0.000534	0.000534	0.000545	0.000563	0.000590	0.000623	0.000653
0.7	0.001372	0.000876	0.000744	0.000715	0.000712	0.000724	0.000748	0.000783	0.000829	0.000870
0.6	0.001837	0.001167	0.000985	0.000944	0.000937	0.000951	0.000982	0.001028	0.001090	0.001144
0.5	0.002336	0.001496	0.001267	0.001217	0.001210	0.001229	0.001269	0.001329	0.001408	0.001478
0.4	0.002849	0.001839	0.001566	0.001506	0.001499	0.001524	0.001575	0.001649	0.001746	0.001832
0.3	0.003365	0.002186	0.001868	0.001800	0.001794	0.001825	0.001886	0.001975	0.002090	0.002193
0.2	0.003883	0.002534	0.002171	0.002095	0.002090	0.002127	0.002199	0.002303	0.002436	0.002556
0.1	0.004400	0.002882	0.002475	0.002391	0.002385	0.002429	0.002512	0.002630	0.002783	0.002919
0.0	0.004917	0.003230	0.002778	0.002686	0.002681	0.002731	0.002825	0.002958	0.003129	0.003283
Boundary	0.741428	0.702175	0.653051	0.641316	0.618476	0.607363	0.596450	0.596450	0.596450	0.596450

	Age									
Stock	10	11	12	13	14	15	16	17	18	19
2.0	0.000051	0.000053	0.000056	0.000058	0.000061	0.000064	0.000068	0.000071	0.000075	0.000079
1.9	0.000059	0.000061	0.000064	0.000067	0.000071	0.000074	0.000078	0.000082	0.000086	0.000091
1.8	0.000069	0.000072	0.000075	0.000078	0.000082	0.000086	0.000090	0.000095	0.000100	0.000105
1.7	0.000081	0.000084	0.000088	0.000092	0.000096	0.000101	0.000106	0.000111	0.000117	0.000123
1.6	0.000096	0.000100	0.000104	0.000109	0.000114	0.000119	0.000125	0.000131	0.000138	0.000145
1.5	0.000116	0.000120	0.000125	0.000131	0.000137	0.000143	0.000150	0.000157	0.000165	0.000173
1.4	0.000141	0.000147	0.000153	0.000159	0.000166	0.000174	0.000182	0.000190	0.000200	0.000209
1.3	0.000175	0.000182	0.000189	0.000197	0.000205	0.000214	0.000224	0.000234	0.000245	0.000257
1.2	0.000220	0.000229	0.000238	0.000247	0.000257	0.000268	0.000280	0.000293	0.000307	0.000321
1.1	0.000283	0.000294	0.000306	0.000318	0.000330	0.000344	0.000359	0.000375	0.000392	0.000410
1.0	0.000374	0.000388	0.000403	0.000418	0.000434	0.000452	0.000471	0.000491	0.000513	0.000536
0.9	0.000505	0.000524	0.000544	0.000564	0.000585	0.000608	0.000632	0.000659	0.000687	0.000717
0.8	0.000681	0.000707	0.000733	0.000760	0.000788	0.000817	0.000849	0.000884	0.000921	0.000961
0.7	0.000907	0.000943	0.000977	0.001012	0.001048	0.001086	0.001128	0.001173	0.001222	0.001274
0.6	0.001194	0.001240	0.001285	0.001330	0.001377	0.001427	0.001481	0.001539	0.001603	0.001671
0.5	0.001541	0.001601	0.001658	0.001716	0.001776	0.001841	0.001910	0.001985	0.002066	0.002153
0.4	0.001910	0.001984	0.002056	0.002128	0.002203	0.002284	0.002370	0.002464	0.002566	0.002675
0.3	0.002287	0.002376	0.002463	0.002551	0.002642	0.002740	0.002845	0.002959	0.003082	0.003215
0.2	0.002666	0.002771	0.002873	0.002976	0.003083	0.003198	0.003322	0.003457	0.003602	0.003758
0.1	0.003045	0.003165	0.003282	0.003401	0.003525	0.003657	0.003799	0.003954	0.004121	0.004301
0.0	0.003425	0.003560	0.003692	0.003826	0.003966	0.004115	0.004276	0.004452	0.004641	0.004844
Boundary	0.607363	0.607363	0.607363	0.607363	0.607363	0.607363	0.607363	0.607363	0.596450	0.596450

(*continued*)

the stock at time t and is restricted to the range $\alpha \leq q_t \leq \beta$. This places a limit, not on the money amount invested, but on the number of shares invested. S_t and π_t are the value of the stock index and total wealth at time t. The payoff on the option is $\max(\pi_T, 0)$ at maturity. It turns out that it is only optimal to hold the extremes α and β. For the specific case of the vacation put with no dividends on the stock and all proceeds

TABLE 4
(Continued)

	Age									
Stock	20	21	22	23	24	25	26	27	28	29
2.0	0.000084	0.000089	0.000095	0.000101	0.000108	0.000116	0.000124	0.000134	0.000144	0.000156
1.9	0.000096	0.000102	0.000108	0.000115	0.000123	0.000131	0.000141	0.000152	0.000164	0.000177
1.8	0.000111	0.000118	0.000125	0.000133	0.000141	0.000151	0.000162	0.000174	0.000187	0.000202
1.7	0.000130	0.000137	0.000145	0.000154	0.000164	0.000174	0.000187	0.000200	0.000215	0.000232
1.6	0.000153	0.000161	0.000170	0.000180	0.000191	0.000204	0.000217	0.000233	0.000250	0.000270
1.5	0.000182	0.000192	0.000202	0.000214	0.000227	0.000241	0.000256	0.000274	0.000294	0.000316
1.4	0.000220	0.000231	0.000244	0.000257	0.000272	0.000288	0.000306	0.000326	0.000349	0.000375
1.3	0.000270	0.000283	0.000298	0.000314	0.000331	0.000350	0.000371	0.000395	0.000421	0.000451
1.2	0.000337	0.000353	0.000371	0.000390	0.000411	0.000433	0.000458	0.000485	0.000516	0.000551
1.1	0.000429	0.000450	0.000472	0.000495	0.000520	0.000547	0.000577	0.000609	0.000646	0.000687
1.0	0.000561	0.000587	0.000615	0.000644	0.000675	0.000709	0.000745	0.000785	0.000829	0.000878
0.9	0.000750	0.000784	0.000820	0.000858	0.000898	0.000941	0.000986	0.001035	0.001089	0.001149
0.8	0.001004	0.001049	0.001096	0.001146	0.001198	0.001253	0.001310	0.001371	0.001438	0.001511
0.7	0.001330	0.001389	0.001451	0.001515	0.001582	0.001652	0.001725	0.001801	0.001884	0.001974
0.6	0.001743	0.001820	0.001900	0.001983	0.002069	0.002157	0.002248	0.002344	0.002447	0.002559
0.5	0.002245	0.002343	0.002445	0.002550	0.002660	0.002772	0.002888	0.003010	0.003141	0.003285
0.4	0.002791	0.002914	0.003042	0.003176	0.003314	0.003458	0.003607	0.003764	0.003933	0.004118
0.3	0.003356	0.003506	0.003662	0.003826	0.003996	0.004172	0.004356	0.004550	0.004759	0.004988
0.2	0.003925	0.004101	0.004286	0.004479	0.004680	0.004889	0.005108	0.005339	0.005588	0.005861
0.1	0.004493	0.004696	0.004909	0.005132	0.005365	0.005607	0.005860	0.006128	0.006417	0.006734
0.0	0.005061	0.005291	0.005533	0.005785	0.006049	0.006324	0.006612	0.006917	0.007246	0.007607
Boundary	0.596450	0.607363	0.607363	0.607363	0.607363	0.607363	0.607363	0.607363	0.607363	0.607363
	Age									
Stock	30	31	32	33	34	35	36	37	38	39
2.0	0.000170	0.000184	0.000200	0.000218	0.000237	0.000258	0.000280	0.000304	0.000329	0.000356
1.9	0.000192	0.000209	0.000227	0.000247	0.000269	0.000292	0.000318	0.000345	0.000374	0.000405
1.8	0.000219	0.000238	0.000259	0.000281	0.000306	0.000334	0.000363	0.000395	0.000428	0.000464
1.7	0.000252	0.000273	0.000297	0.000323	0.000352	0.000383	0.000417	0.000454	0.000493	0.000535
1.6	0.000292	0.000316	0.000343	0.000374	0.000407	0.000444	0.000484	0.000527	0.000573	0.000622
1.5	0.000341	0.000369	0.000401	0.000436	0.000475	0.000518	0.000565	0.000616	0.000671	0.000730
1.4	0.000404	0.000437	0.000474	0.000515	0.000561	0.000612	0.000668	0.000728	0.000794	0.000865
1.3	0.000485	0.000523	0.000566	0.000615	0.000669	0.000730	0.000797	0.000871	0.000950	0.001036
1.2	0.000591	0.000635	0.000687	0.000744	0.000810	0.000883	0.000964	0.001054	0.001152	0.001258
1.1	0.000734	0.000787	0.000848	0.000917	0.000996	0.001085	0.001185	0.001296	0.001417	0.001550
1.0	0.000934	0.000998	0.001071	0.001155	0.001250	0.001359	0.001483	0.001622	0.001776	0.001945
0.9	0.001217	0.001294	0.001382	0.001485	0.001602	0.001737	0.001892	0.002068	0.002265	0.002485
0.8	0.001594	0.001688	0.001796	0.001922	0.002068	0.002236	0.002431	0.002657	0.002912	0.003197
0.7	0.002075	0.002191	0.002325	0.002481	0.002663	0.002876	0.003124	0.003413	0.003743	0.004113
0.6	0.002684	0.002827	0.002994	0.003189	0.003419	0.003690	0.004007	0.004379	0.004804	0.005285
0.5	0.003445	0.003628	0.003842	0.004093	0.004389	0.004737	0.005146	0.005627	0.006177	0.006799
0.4	0.004325	0.004562	0.004837	0.005159	0.005536	0.005978	0.006495	0.007101	0.007792	0.008572
0.3	0.005244	0.005536	0.005874	0.006268	0.006729	0.007267	0.007895	0.008627	0.009461	0.010400
0.2	0.006166	0.006513	0.006913	0.007381	0.007925	0.008558	0.009296	0.010155	0.011132	0.012230
0.1	0.007088	0.007490	0.007953	0.008493	0.009120	0.009850	0.010698	0.011684	0.012804	0.014061
0.0	0.008009	0.008467	0.008993	0.009605	0.010316	0.011141	0.012100	0.013212	0.014475	0.015891
Boundary	0.607363	0.596450	0.596450	0.585732	0.575208	0.564872	0.554722	0.544755	0.534966	0.534966

(*continued*)

able to be invested at the risk-free rate r, $\alpha = -1$ and $\beta = 0$, and $q_t^{\text{opt}} = -I_{\{\pi_t^{\text{opt}} \geq -\frac{1}{2} S_t\}}$, where I is the indicator function.

The two state variables S_t and π_t are, in fact, necessary. The fund value alone is insufficient because the holder of the "vacation put" is allowed to transfer any amount

TABLE 4
(Continued)

	Age									
Stock	40	41	42	43	44	45	46	47	48	49
2.0	0.000385	0.000415	0.000447	0.000480	0.000516	0.000554	0.000594	0.000636	0.000681	0.000730
1.9	0.000438	0.000473	0.000510	0.000548	0.000589	0.000633	0.000679	0.000728	0.000780	0.000836
1.8	0.000502	0.000543	0.000585	0.000630	0.000678	0.000729	0.000782	0.000839	0.000900	0.000964
1.7	0.000580	0.000627	0.000677	0.000730	0.000786	0.000845	0.000908	0.000975	0.001046	0.001121
1.6	0.000675	0.000731	0.000790	0.000852	0.000918	0.000988	0.001063	0.001142	0.001225	0.001315
1.5	0.000793	0.000859	0.000930	0.001005	0.001084	0.001168	0.001256	0.001351	0.001451	0.001557
1.4	0.000941	0.001021	0.001107	0.001197	0.001293	0.001394	0.001501	0.001615	0.001736	0.001865
1.3	0.001129	0.001227	0.001332	0.001443	0.001561	0.001685	0.001816	0.001956	0.002104	0.002262
1.2	0.001372	0.001495	0.001625	0.001763	0.001910	0.002065	0.002229	0.002403	0.002588	0.002784
1.1	0.001694	0.001849	0.002014	0.002190	0.002377	0.002574	0.002783	0.003004	0.003238	0.003487
1.0	0.002129	0.002329	0.002543	0.002771	0.003014	0.003270	0.003542	0.003829	0.004134	0.004457
0.9	0.002725	0.002987	0.003269	0.003572	0.003895	0.004236	0.004598	0.004980	0.005385	0.005813
0.8	0.003512	0.003857	0.004231	0.004633	0.005063	0.005519	0.006001	0.006510	0.007048	0.007618
0.7	0.004525	0.004977	0.005469	0.006000	0.006567	0.007169	0.007806	0.008478	0.009188	0.009939
0.6	0.005820	0.006409	0.007052	0.007745	0.008487	0.009274	0.010107	0.010985	0.011912	0.012892
0.5	0.007492	0.008255	0.009087	0.009985	0.010944	0.011962	0.013038	0.014173	0.015371	0.016638
0.4	0.009439	0.010392	0.011430	0.012550	0.013746	0.015015	0.016357	0.017774	0.019269	0.020850
0.3	0.011442	0.012588	0.013834	0.015177	0.016611	0.018133	0.019741	0.021439	0.023231	0.025126
0.2	0.013448	0.014786	0.016240	0.017805	0.019478	0.021252	0.023126	0.025105	0.027194	0.029402
0.1	0.015454	0.016984	0.018645	0.020434	0.022344	0.024371	0.026512	0.028771	0.031157	0.033679
0.0	0.017460	0.019181	0.021051	0.023063	0.025211	0.027489	0.029897	0.032438	0.035120	0.037955
Boundary	0.534966	0.544755	0.544755	0.554722	0.554722	0.554722	0.564872	0.564872	0.564872	0.564872

	Age									
Stock	50	51	52	53	54	55	56	57	58	59
2.0	0.000781	0.000835	0.000893	0.000954	0.001019	0.001088	0.001161	0.001238	0.001317	0.001400
1.9	0.000895	0.000958	0.001025	0.001096	0.001171	0.001252	0.001337	0.001426	0.001520	0.001618
1.8	0.001033	0.001106	0.001184	0.001267	0.001355	0.001449	0.001549	0.001655	0.001766	0.001882
1.7	0.001201	0.001287	0.001379	0.001476	0.001580	0.001691	0.001809	0.001935	0.002067	0.002206
1.6	0.001410	0.001511	0.001619	0.001735	0.001858	0.001990	0.002131	0.002281	0.002440	0.002608
1.5	0.001671	0.001792	0.001921	0.002059	0.002207	0.002366	0.002535	0.002716	0.002909	0.003113
1.4	0.002002	0.002148	0.002305	0.002472	0.002651	0.002843	0.003049	0.003270	0.003506	0.003757
1.3	0.002430	0.002609	0.002801	0.003006	0.003225	0.003461	0.003715	0.003987	0.004279	0.004591
1.2	0.002993	0.003216	0.003454	0.003708	0.003982	0.004275	0.004591	0.004931	0.005297	0.005691
1.1	0.003752	0.004034	0.004336	0.004658	0.005004	0.005376	0.005776	0.006208	0.006674	0.007177
1.0	0.004800	0.005166	0.005556	0.005973	0.006420	0.006900	0.007417	0.007976	0.008580	0.009235
0.9	0.006268	0.006752	0.007268	0.007819	0.008409	0.009042	0.009723	0.010458	0.011256	0.012122
0.8	0.008222	0.008865	0.009549	0.010279	0.011059	0.011895	0.012795	0.013766	0.014821	0.015970
0.7	0.010735	0.011580	0.012480	0.013439	0.014464	0.015562	0.016742	0.018018	0.019404	0.020919
0.6	0.013931	0.015034	0.016207	0.017457	0.018792	0.020222	0.021759	0.023423	0.025233	0.027216
0.5	0.017981	0.019407	0.020924	0.022542	0.024271	0.026124	0.028119	0.030278	0.032630	0.035208
0.4	0.022526	0.024307	0.026200	0.028219	0.030376	0.032685	0.035169	0.037855	0.040777	0.043972
0.3	0.027134	0.029267	0.031535	0.033952	0.036533	0.039295	0.042265	0.045474	0.048961	0.052771
0.2	0.031743	0.034228	0.036870	0.039685	0.042690	0.045906	0.049361	0.053093	0.057146	0.061570
0.1	0.036352	0.039188	0.042205	0.045418	0.048847	0.052516	0.056458	0.060712	0.065330	0.070369
0.0	0.040960	0.044149	0.047539	0.051151	0.055004	0.059126	0.063553	0.068331	0.073514	0.079168
Boundary	0.575208	0.575208	0.575208	0.575208	0.575208	0.575208	0.575208	0.575208	0.575208	0.575208

(*continued*)

of money he wishes between 0 and $-S_t$. Equivalently, he can invest money in between 0 and -1 shares of stock. If the investor has, at some point, not been invested in the stock then the value of $-S_t$ has become decoupled from the value of π_t and must be known independently.

TABLE 4
(Continued)

	Age									
Stock	60	61	62	63	64	65	66	67	68	69
2.0	0.001486	0.001573	0.001661	0.001750	0.001839	0.001927	0.002013	0.002097	0.002178	0.002256
1.9	0.001719	0.001823	0.001929	0.002036	0.002143	0.002250	0.002355	0.002459	0.002560	0.002657
1.8	0.002003	0.002127	0.002255	0.002385	0.002515	0.002646	0.002776	0.002904	0.003030	0.003153
1.7	0.002351	0.002502	0.002657	0.002815	0.002975	0.003137	0.003298	0.003459	0.003617	0.003773
1.6	0.002784	0.002967	0.003157	0.003351	0.003550	0.003752	0.003954	0.004157	0.004358	0.004557
1.5	0.003329	0.003554	0.003789	0.004031	0.004280	0.004534	0.004791	0.005049	0.005308	0.005565
1.4	0.004023	0.004303	0.004597	0.004902	0.005217	0.005540	0.005869	0.006203	0.006539	0.006876
1.3	0.004924	0.005276	0.005647	0.006035	0.006439	0.006856	0.007284	0.007721	0.008164	0.008611
1.2	0.006112	0.006560	0.007036	0.007537	0.008062	0.008608	0.009173	0.009753	0.010346	0.010949
1.1	0.007719	0.008299	0.008919	0.009577	0.010271	0.010998	0.011757	0.012542	0.013351	0.014180
1.0	0.009943	0.010708	0.011530	0.012408	0.013343	0.014332	0.015371	0.016457	0.017585	0.018750
0.9	0.013064	0.014088	0.015196	0.016390	0.017671	0.019038	0.020487	0.022014	0.023614	0.025281
0.8	0.017225	0.018595	0.020088	0.021707	0.023456	0.025334	0.027339	0.029467	0.031712	0.034063
0.7	0.022579	0.024399	0.026389	0.028559	0.030913	0.033453	0.036178	0.039082	0.042157	0.045392
0.6	0.029394	0.031790	0.034417	0.037290	0.040416	0.043799	0.047437	0.051326	0.055453	0.059804
0.5	0.038041	0.041157	0.044575	0.048311	0.052374	0.056768	0.061490	0.066531	0.071876	0.077502
0.4	0.047479	0.051327	0.055540	0.060137	0.065127	0.070515	0.076296	0.082458	0.088983	0.095843
0.3	0.056947	0.061524	0.066529	0.071984	0.077898	0.084276	0.091114	0.098396	0.106099	0.114193
0.2	0.066415	0.071721	0.077519	0.083830	0.090669	0.098038	0.105932	0.114333	0.123215	0.132542
0.1	0.075884	0.081918	0.088508	0.095677	0.103440	0.111800	0.120750	0.130271	0.140332	0.150891
0.0	0.085352	0.092115	0.099496	0.107523	0.116210	0.125561	0.135567	0.146208	0.157447	0.169240
Boundary	0.564872	0.564872	0.564872	0.564872	0.564872	0.575208	0.575208	0.575208	0.575208	0.585732

	Age									
Stock	70	71	72	73	74	75	76	77	78	79
2.0	0.002330	0.002400	0.002465	0.002523	0.002575	0.002619	0.002655	0.002682	0.002699	0.002707
1.9	0.002751	0.002840	0.002923	0.003001	0.003070	0.003131	0.003183	0.003225	0.003255	0.003275
1.8	0.003272	0.003386	0.003495	0.003596	0.003690	0.003774	0.003848	0.003910	0.003960	0.003997
1.7	0.003925	0.004072	0.004214	0.004349	0.004475	0.004591	0.004695	0.004787	0.004864	0.004926
1.6	0.004753	0.004945	0.005132	0.005311	0.005482	0.005642	0.005789	0.005921	0.006037	0.006137
1.5	0.005821	0.006073	0.006321	0.006562	0.006795	0.007016	0.007224	0.007416	0.007590	0.007744
1.4	0.007213	0.007549	0.007881	0.008208	0.008528	0.008836	0.009131	0.009410	0.009669	0.009906
1.3	0.009062	0.009513	0.009965	0.010414	0.010858	0.011293	0.011716	0.012122	0.012508	0.012871
1.2	0.011560	0.012179	0.012802	0.013428	0.014053	0.014673	0.015285	0.015883	0.016462	0.017019
1.1	0.015027	0.015890	0.016766	0.017654	0.018551	0.019452	0.020353	0.021247	0.022129	0.022992
1.0	0.019949	0.021179	0.022439	0.023727	0.025041	0.026376	0.027728	0.029091	0.030456	0.031817
0.9	0.027009	0.028795	0.030636	0.032533	0.034485	0.036491	0.038545	0.040643	0.042774	0.044931
0.8	0.036515	0.039062	0.041702	0.044436	0.047268	0.050199	0.053225	0.056342	0.059540	0.062807
0.7	0.048777	0.052304	0.055973	0.059786	0.063751	0.067873	0.072152	0.076580	0.081150	0.085841
0.6	0.064365	0.069128	0.074094	0.079267	0.084660	0.090283	0.096135	0.102209	0.108490	0.114954
0.5	0.083392	0.089535	0.095928	0.102578	0.109497	0.116694	0.124167	0.131905	0.139886	0.148081
0.4	0.103018	0.110494	0.118265	0.126340	0.134734	0.143455	0.152501	0.161858	0.171499	0.181389
0.3	0.122651	0.131457	0.140606	0.150106	0.159974	0.170219	0.180838	0.191814	0.203115	0.214700
0.2	0.142284	0.152421	0.162948	0.173872	0.185214	0.196984	0.209175	0.221770	0.234731	0.248011
0.1	0.161916	0.173385	0.185289	0.197639	0.210454	0.223748	0.237512	0.251726	0.266347	0.281322
0.0	0.181548	0.194348	0.207630	0.221404	0.235693	0.250511	0.265848	0.281680	0.297962	0.314631
Boundary	0.585732	0.585732	0.596450	0.596450	0.596450	0.607363	0.607363	0.607363	0.618476	0.618476

(*continued*)

If the starting wealth π_0 is positive, the investor will invest in -1 units of the stock, and receive S_0 to invest at the risk-free rate. At any later time before a switch in investment strategy, $\pi_t = e^{rt}(\pi_0 + S_0) - S_t$. This leads to the continuing state $q_t^{\text{opt}} = -1$ as long as the condition $e^{rt}(\pi_0 + S_0) \geq \frac{1}{2} S_t$ remains true. The left-hand side only gets

TABLE 4
(Continued)

	Age									
Stock	80	81	82	83	84	85	86	87	88	89
2.0	0.002705	0.002694	0.002673	0.002643	0.002604	0.002557	0.002503	0.002442	0.002376	0.002306
1.9	0.003283	0.003280	0.003265	0.003240	0.003204	0.003158	0.003102	0.003038	0.002967	0.002890
1.8	0.004020	0.004030	0.004027	0.004010	0.003980	0.003937	0.003883	0.003818	0.003744	0.003661
1.7	0.004972	0.005003	0.005017	0.005015	0.004997	0.004964	0.004916	0.004853	0.004779	0.004693
1.6	0.006218	0.006280	0.006323	0.006347	0.006350	0.006335	0.006300	0.006248	0.006179	0.006096
1.5	0.007878	0.007989	0.008078	0.008143	0.008184	0.008201	0.008195	0.008165	0.008115	0.008044
1.4	0.010119	0.010308	0.010469	0.010602	0.010706	0.010780	0.010825	0.010841	0.010829	0.010791
1.3	0.013207	0.013515	0.013793	0.014037	0.014247	0.014421	0.014558	0.014659	0.014723	0.014754
1.2	0.017548	0.018047	0.018512	0.018940	0.019328	0.019674	0.019975	0.020232	0.020442	0.020609
1.1	0.023832	0.024642	0.025418	0.026155	0.026850	0.027497	0.028094	0.028637	0.029126	0.029559
1.0	0.033165	0.034493	0.035795	0.037064	0.038294	0.039478	0.040611	0.041688	0.042704	0.043657
0.9	0.047102	0.049279	0.051451	0.053611	0.055748	0.057855	0.059924	0.061945	0.063910	0.065813
0.8	0.066129	0.069494	0.072889	0.076302	0.079722	0.083135	0.086531	0.089895	0.093213	0.096473
0.7	0.090639	0.095527	0.100486	0.105500	0.110556	0.115633	0.120714	0.125784	0.130815	0.135785
0.6	0.121576	0.128332	0.135197	0.142144	0.149151	0.156191	0.163237	0.170260	0.177227	0.184105
0.5	0.156459	0.164987	0.173634	0.182371	0.191166	0.199988	0.208804	0.217579	0.226274	0.234847
0.4	0.191490	0.201763	0.212170	0.222675	0.233242	0.243833	0.254409	0.264928	0.275343	0.285605
0.3	0.226523	0.238541	0.250708	0.262982	0.275321	0.287681	0.300017	0.312280	0.324415	0.336366
0.2	0.261557	0.275319	0.289246	0.303289	0.317400	0.331530	0.345626	0.359632	0.373487	0.387129
0.1	0.296591	0.312098	0.327784	0.343597	0.359480	0.375379	0.391235	0.406984	0.422560	0.437891
0.0	0.331623	0.348874	0.366321	0.383902	0.401557	0.419225	0.436841	0.454334	0.471630	0.488650
Boundary	0.618476	0.629793	0.629793	0.629793	0.641316	0.641316	0.641316	0.653051	0.653051	0.653051

	Age									
Stock	90	91	92	93	94	95	96	97	98	99
2.0	0.002232	0.002156	0.002079	0.002000	0.001920	0.001839	0.001756	0.001670	0.001580	0.001485
1.9	0.002809	0.002724	0.002636	0.002546	0.002454	0.002361	0.002265	0.002165	0.002060	0.001948
1.8	0.003572	0.003478	0.003380	0.003279	0.003175	0.003068	0.002957	0.002842	0.002718	0.002586
1.7	0.004599	0.004497	0.004389	0.004277	0.004160	0.004039	0.003913	0.003780	0.003637	0.003482
1.6	0.006001	0.005895	0.005780	0.005658	0.005530	0.005396	0.005255	0.005105	0.004941	0.004761
1.5	0.007957	0.007855	0.007740	0.007615	0.007480	0.007337	0.007185	0.007021	0.006839	0.006636
1.4	0.010730	0.010648	0.010548	0.010433	0.010304	0.010163	0.010010	0.009841	0.009649	0.009430
1.3	0.014753	0.014724	0.014670	0.014593	0.014497	0.014383	0.014252	0.014102	0.013923	0.013708
1.2	0.020735	0.020821	0.020873	0.020892	0.020884	0.020851	0.020796	0.020716	0.020602	0.020444
1.1	0.029939	0.030268	0.030550	0.030788	0.030988	0.031153	0.031288	0.031395	0.031463	0.031482
1.0	0.044546	0.045372	0.046139	0.046849	0.047507	0.048121	0.048699	0.049245	0.049756	0.050226
0.9	0.067649	0.069413	0.071109	0.072738	0.074305	0.075819	0.077292	0.078737	0.080168	0.081593
0.8	0.099663	0.102774	0.105804	0.108756	0.111632	0.114442	0.117207	0.119954	0.122719	0.125531
0.7	0.140683	0.145486	0.150187	0.154794	0.159304	0.163729	0.168099	0.172469	0.176895	0.181429
0.6	0.190866	0.197490	0.203964	0.210288	0.216467	0.222522	0.228487	0.234423	0.240426	0.246558
0.5	0.243265	0.251504	0.259551	0.267403	0.275070	0.282578	0.289970	0.297320	0.304746	0.312327
0.4	0.295676	0.305527	0.315143	0.324522	0.333676	0.342635	0.351452	0.360216	0.369066	0.378094
0.3	0.348090	0.359553	0.370739	0.381645	0.392285	0.402696	0.412939	0.423117	0.433391	0.443866
0.2	0.400505	0.413581	0.426336	0.438768	0.450895	0.462758	0.474427	0.486019	0.497716	0.509639
0.1	0.452920	0.467608	0.481932	0.495892	0.509506	0.522820	0.535915	0.548920	0.562041	0.575411
0.0	0.505332	0.521632	0.537526	0.553012	0.568112	0.582879	0.597399	0.611818	0.626362	0.641180
Boundary	0.665000	0.665000	0.665000	0.677167	0.677167	0.677167	0.677167	0.689558	0.689558	0.689558

(*continued*)

TABLE 4
(Continued)

	Age									
Stock	100	101	102	103	104	105	106	107	108	109
2.0	0.001385	0.001281	0.001173	0.001063	0.000953	0.000845	0.000740	0.000640	0.000546	0.000458
1.9	0.001828	0.001702	0.001570	0.001435	0.001298	0.001161	0.001028	0.000899	0.000776	0.000661
1.8	0.002443	0.002291	0.002131	0.001964	0.001792	0.001620	0.001449	0.001282	0.001121	0.000968
1.7	0.003313	0.003130	0.002935	0.002729	0.002515	0.002296	0.002076	0.001859	0.001648	0.001444
1.6	0.004563	0.004345	0.004109	0.003856	0.003589	0.003313	0.003031	0.002748	0.002469	0.002196
1.5	0.006407	0.006152	0.005870	0.005563	0.005233	0.004886	0.004526	0.004159	0.003790	0.003425
1.4	0.009176	0.008887	0.008560	0.008195	0.007797	0.007367	0.006914	0.006442	0.005960	0.005474
1.3	0.013450	0.013144	0.012786	0.012374	0.011911	0.011400	0.010846	0.010257	0.009642	0.009009
1.2	0.020232	0.019958	0.019616	0.019202	0.018713	0.018153	0.017525	0.016836	0.016095	0.015314
1.1	0.031441	0.031327	0.031132	0.030846	0.030464	0.029984	0.029406	0.028734	0.027977	0.027146
1.0	0.050645	0.050999	0.051277	0.051466	0.051556	0.051538	0.051408	0.051163	0.050811	0.050359
0.9	0.083013	0.084419	0.085800	0.087148	0.088445	0.089679	0.090841	0.091920	0.092917	0.093840
0.8	0.128407	0.131349	0.134344	0.137384	0.140455	0.143525	0.146582	0.149604	0.152573	0.155509
0.7	0.186102	0.190943	0.195922	0.201021	0.206221	0.211467	0.216715	0.221922	0.227053	0.232093
0.6	0.252855	0.259324	0.265951	0.272704	0.279539	0.286402	0.293239	0.299999	0.306642	0.313155
0.5	0.320102	0.328082	0.336249	0.344563	0.352970	0.361404	0.369799	0.378092	0.386235	0.394214
0.4	0.387348	0.396840	0.406546	0.416422	0.426400	0.436405	0.446357	0.456183	0.465827	0.475272
0.3	0.454599	0.465601	0.476848	0.488285	0.499835	0.511411	0.522921	0.534280	0.545424	0.556335
0.2	0.521850	0.534364	0.547151	0.560149	0.573271	0.586418	0.599485	0.612377	0.625022	0.637399
0.1	0.589102	0.603127	0.617454	0.632013	0.646707	0.661425	0.676050	0.690475	0.704620	0.718463
0.0	0.656349	0.671885	0.687752	0.703872	0.720139	0.736427	0.752610	0.768568	0.784214	0.799523
Boundary	0.702175	0.702175	0.702175	0.702175	0.715023	0.715023	0.728106	0.728106	0.728106	0.741428

	Age				
Stock	110	111	112	113	114
2.0	0.000374	0.000288	0.000189	0.000076	0.000003
1.9	0.000549	0.000435	0.000302	0.000137	0.000009
1.8	0.000820	0.000667	0.000488	0.000251	0.000026
1.7	0.001245	0.001040	0.000799	0.000461	0.000069
1.6	0.001928	0.001652	0.001328	0.000853	0.000185
1.5	0.003062	0.002690	0.002255	0.001599	0.000490
1.4	0.004987	0.004485	0.003907	0.003026	0.001268
1.3	0.008364	0.007699	0.006946	0.005808	0.003202
1.2	0.014502	0.013658	0.012720	0.011348	0.007862
1.1	0.026257	0.025319	0.024297	0.022874	0.018982
1.0	0.049827	0.049245	0.048631	0.047825	0.045071
0.9	0.094709	0.095589	0.096590	0.097908	0.099304
0.8	0.158415	0.161409	0.164751	0.169204	0.177221
0.7	0.237074	0.242135	0.247703	0.255040	0.268076
0.6	0.319582	0.326106	0.333276	0.342707	0.359394
0.5	0.402084	0.410069	0.418840	0.430364	0.450702
0.4	0.484584	0.494030	0.504402	0.518019	0.542008
0.3	0.567090	0.577996	0.589970	0.605680	0.633320
0.2	0.649596	0.661963	0.675538	0.693343	0.724633
0.1	0.732103	0.745931	0.761107	0.781005	0.815946
0.0	0.814605	0.829893	0.846671	0.868662	0.907253
Boundary	0.741428	0.754994	0.754994	0.768809	0.782876

larger with time, so if S_t drops, creating more wealth, the investor remains in the state $q_t^{\mathrm{opt}} = -1$.

The only way to reach the state $q_t^{\mathrm{opt}} = 0$ is if S_t rises far enough to create the condition $S_t > 2e^{rt}(\pi_0 + S_0)$. If this occurs, the investor closes out his short stock position with negative wealth $\pi_t = -e^{rt}(\pi_0 + S_0)$. It seems counterintuitive to make this transfer, as negative wealth will only get more negative with time, and the value of the option is now "locked in" at 0. How can this be optimal?

It turns out that if the investor is in this new state, $q_t^{\text{opt}} = 0$, for the first time, the wealth is now $\pi_t = -e^{rt_1}(\pi_0 + S_0)e^{r(t-t_1)} = -e^{rt}(\pi_0 + S_0)$, where t_1 is the time of switch. He remains in this state as long as $-e^{rt}(\pi_0 + S_0) < -\frac{1}{2}S_t$ or $e^{rt}(\pi_0 + S_0) > \frac{1}{2}S_t$. Again, if S_t drops, the investor remains in state $q_t^{\text{opt}} = 0$. He only switches if S_t rises far enough to create the condition $S_t \geq 2e^{rt}(\pi_0 + S_0)$, in which case he switches back to state $q_t^{\text{opt}} = -1$. In the new state, $\pi_t = -e^{rt_2}(\pi_0 + S_0)e^{r(t-t_2)} + 2e^{rt_2}(\pi_0 + S_0)e^{r(t-t_2)} - S_t = e^{rt}(\pi_0 + S_0) - S_t$, the same wealth as if the initial switch had never been made.

This answers the question of why the investor would want to wait even if his fund value is below zero. If he maintained the short position, there is a small chance that his option will be in the money. On the other hand, if he closed his position, waited for the stock price to become larger and then shorted the stock again, there would be a greater chance that his option would eventually be in the money due to the larger volatility that has not been accompanied by a corresponding drop in wealth. This greater chance must be balanced against the probability that a switch back will never occur. We might call it a "wait and see" strategy.

The GMDB benefits described in this article differ from vacation puts in several important ways. First, the vacation put option has a fixed maturity date while the GMDB can be viewed as an integral over maturity date probabilities. More importantly, the vacation put investor is invested in exactly 0 or -1 shares of the stock. The funds underlying the GMDB are invested based on the number of shares that can be bought with the current wealth at the current price.

In the variable annuity case, only one state variable is necessary. This is because the investment is not limited by number of shares, but by money. The owner of the annuity cannot transfer more money to the stock than he already has, that is, he cannot allow his fixed fund value to drop below \$0. For the "vacation put," a doubling of the stock price when the investor is not invested in it has the effect of doubling the amount of money that can be invested, and therefore doubling the maximum volatility achievable. For a variable annuity, however, a doubling of the stock price when the investor is not invested in it does not change the amount of money that can be invested, but cuts the number of shares that can be purchased in half. This has no effect on the maximum volatility achievable, and the stock index does not need to be known independently of the fund value.

According to the optimal strategy for vacation puts found in Shreve and Vecer, the option holder should begin by investing in -1 shares of the stock as long as $\pi_t \geq -\frac{1}{2}S_t$. If the stock price falls, then $\pi_t \geq 0$ and the investor remains invested in the stock. This optimal strategy, however, is derived only in the "symmetric" case with no dividends and the stock growth rate equal to the interest rate. This is equivalent to the case $g = r - q$ discussed in this article in which no boundary was found as the stock price decreased. These results are therefore in agreement with previous work on vacation put options. The boundary only appears as the fixed fund begins to earn less than the stock index.

The "wait and see" strategy does not work in the case of a variable annuity because the potential volatility of switching back does not change regardless of any change in stock price. Any stock price increase is exactly balanced by a decrease in the number of shares that could be purchased. Therefore, while an "out-of-the-money" boundary exists for vacation put options, a similar boundary does not exist for transfers in variable annuity accounts.

CONCLUSIONS

In this article, we have used a differential equation that must be satisfied for a return of premium GMDB attached to a variable annuity contract. Solving this equation in the situation where transfers are allowed between the variable account and a fixed fund gives some insight into the value of the real option to transfer. An important general result is that when the GMDB is considered in isolation the policyholder will only transfer funds to the fixed account if the rate that account earns is sufficiently low. This should alleviate somewhat a fear that individuals will transfer funds to a fixed account to "lock in" an in-the-money GMDB. This will only be done in situations where the profits earned by the company on the fixed funds are larger than those earned on the variable funds. The value of the fund and GMDB in total has been shown to be highest when the GMDB is ignored entirely and the policyholder invests in the funds that have the largest return in a risk-neutral world. This means that any transfers to the fixed funds to lock in the death benefit will only lower the value of the total contract to the customer. Therefore, this option does not present a serious danger to the economic value of an insurance company's liabilities.

APPENDIX A

General Derivation That $g < r - q$ Is a Necessary Condition for Transfers to the Fixed Fund

Assume for at least one value of S and t,

$$f_f(S,t) > f_v(S,t), \tag{A1}$$

where $f_f(S, t)$ is the value of the option if the funds are instantaneously invested in the fixed account, and $f_v(S, t)$ is the value of the option if the funds are instantaneously invested in the variable account. The value of the option at time $t + dt$ is $f_a(S, t + dt)$. The inequality (A1) becomes

$$f_a(S,t+dt) - \frac{\partial f_f}{\partial t} > f_a(S,t+dt) - \frac{\partial f_v}{\partial t}, \tag{A2}$$

or

$$\frac{\partial f_f}{\partial t} < \frac{\partial f_v}{\partial t}. \tag{A3}$$

The differential Equations (3) and (11) can be substituted in for the time derivatives in Equation (A3) to give

$$
\begin{aligned}
&-gS\frac{\partial f_f}{\partial S} + [r + \mu_x(t) + \lambda(S,t)]\left[f_a(S,t+dt) - \frac{\partial f_f}{\partial t}dt\right] - [\mu_x(t)]\,\mathrm{Max}(Xe^{pt} - S, 0) \\
&< -(r-q)S\frac{\partial f_v}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f_v}{\partial S^2} + [r + \mu_x(t) + \lambda(S,t)]\left[f_a(S,t+dt) - \frac{\partial f_v}{\partial t}dt\right] \\
&\quad - [\mu_x(t)]\,\mathrm{Max}(Xe^{pt} - S, 0), \qquad \text{(A4)}
\end{aligned}
$$

where terms in dt disappear as $dt \to 0$ and many terms can be canceled to give

$$
g\frac{\partial f_f}{\partial S} > (r-q)\frac{\partial f_v}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 f_v}{\partial S^2}. \qquad \text{(A5)}
$$

Also, as $dt \to 0$, the derivatives with respect to S approach the derivatives of $f_a(S, t + dt)$ with respect to S. Since the first derivative $\frac{\partial f_a}{\partial S}$ must be negative, the inequality is now

$$
g < (r-q) + \frac{1}{2}\sigma^2 S\frac{\left[\frac{\partial^2 f_a}{\partial S^2}\right]}{\left[\frac{\partial f_a}{\partial S}\right]}. \qquad \text{(A6)}
$$

Since the first partial is negative, and the second partial is positive, the last term is negative and $g < (r - q)$ for the fixed fund to be preferable under any circumstances.

APPENDIX B

Derivation of the Boundary Conditions at the Fixed-to-Variable Transfer Boundary

The boundary conditions at the fixed-to-variable transfer boundary are similar to those found in Merton (1973). First, the continuity of the function follows from an arbitrage argument. If the function is discontinuous, there would be a risk-free profit with probability one at this point. See Wilmott, Howison, and DeWynne (1995, pp. 109-110).

Second, the continuity of the first derivative follows from the fact that at the boundary C, the GMDB owner should be indifferent between being in the fixed fund or being in the variable fund. Assume that the individual chooses to be in the fixed fund. We can determine the value of his option by discounting the value exactly Δt ahead, where Δt is infinitesimally small. Ignoring all terms of order Δt^2 or higher, we get

$$
\begin{aligned}
f_a(C,t) = {} & f_a(C + gC\Delta t, t + \Delta t)(1 - r\Delta t)(1 - \mu_x(t)\Delta t - \lambda\Delta t) \\
& + [\mu_x(t)\Delta t](1 - r\Delta t)(Xe^{pt} - C), \qquad \text{(B1)}
\end{aligned}
$$

now

$$
f_a(C + gC\Delta t, t + \Delta t) = f_a(C,t) + \frac{\partial f_a}{\partial t}\Delta t + gC\frac{\partial f_a}{\partial S}\Delta t, \qquad \text{(B2)}
$$

where $\frac{\partial f_a}{\partial S}$ is in the region just *above* C.

Substituting Equation (B1) into Equation (B2), and eliminating terms of order Δt^2 we get

$$f_a(C,t) = f_a(C,t) + \frac{\partial f_a}{\partial t}\Delta t + gC\frac{\partial f_a}{\partial S}\Delta t - r\Delta t f_a(C,t) - \mu_x(t)\Delta t f_a(C,t) - \lambda\Delta t f_a(C,t) + \mu_x(t)\Delta t(Xe^{pt} - C). \tag{B3}$$

Canceling terms and dividing by Δt gives

$$\frac{\partial f_a}{\partial t} + gC\frac{\partial f_a}{\partial S} = (r + \mu_x(t) + \lambda)f_a(C,t) - \mu_x(t)\Delta t(Xe^{pt} - C). \tag{B4}$$

Equation (B4) is obeyed by $\frac{\partial f_a}{\partial S}$ in the region just above the boundary. It is identical to Equation (11) obeyed by $\frac{\partial f_a}{\partial S}$ in the region just below the boundary. Therefore, $\frac{\partial f_a}{\partial S}$ is continuous at the boundary.

Finally, the continuity of the second derivative can be seen in the following argument. Assume S is just slightly higher than C. The money is entirely invested in the variable account, and f_a obeys Equation (3). In addition, we can compute the value of $f_a(C, t)$ as the discounted expected value of the fund a small time Δt later. We will ignore any terms of order greater than Δt. This means we can assume the fund at the later time can be represented in the range $S > C$ as

$$f_a(S,t+\Delta t) = f_a(C + (r-q)C\Delta t, t+\Delta t) + \frac{\partial f_a}{\partial S}(S - [C + (r-q)C\Delta t]) + \frac{1}{2}\left.\frac{\partial^2 f_a}{\partial S^2}\right|_u (S - [C + (r-q)C\Delta t])^2, \tag{B5}$$

where the subscript u denotes the value of the second derivative in the upper region. We do not need a subscript on the first derivative because we know it is continuous in the upper and lower regions.

The continuity of the first derivative implies the function can be represented in the region $S < C$ by

$$f_a(S,t+\Delta t) = f_a(C + (r-q)C\Delta t, t+\Delta t) + \frac{\partial f_a}{\partial S}(S - [C + (r-q)C\Delta t]) + \frac{1}{2}\left.\frac{\partial^2 f_a}{\partial S^2}\right|_u (S - [C + (r-q)C\Delta t])^2 + \frac{1}{2}\left[\left.\frac{\partial^2 f_a}{\partial S^2}\right|_l - \left.\frac{\partial^2 f_a}{\partial S^2}\right|_u\right](S - C)^2. \tag{B6}$$

The distribution of returns at time Δt is normal with mean $(r - q - \frac{\sigma^2}{2})\Delta t$ and variance $\sigma^2\Delta t$. This implies, for small Δt, that the value of the index is approximately normal with mean $C + (r - q)C\Delta t$ and variance $\sigma^2 C^2 \Delta t$. This leads to the following equation for $f_a(C, t)$:

$$
\begin{aligned}
f_a(C,t) = {} & [1 - (r + \mu + \lambda)\Delta t] \\
& \times \left\{ \int_{-\infty}^{\infty} f_a(C + (r-q)C\Delta t, t + \Delta t) \frac{e^{-(S-[C-(r-q)C\Delta t])^2/2\sigma^2 C^2 \Delta t}}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} \, dS \right. \\
& + \int_{-\infty}^{\infty} \frac{\partial f_a}{\partial S} (S - [C + (r-q)C\Delta t]) \frac{e^{-(S-[C-(r-q)C\Delta t])^2/2\sigma^2 C^2 \Delta t}}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} \, dS \\
& + \int_{-\infty}^{\infty} \frac{1}{2} \left. \frac{\partial^2 f_a}{\partial S^2} \right|_u (S - [C + (r-q)C\Delta t])^2 \frac{e^{-(S-[C-(r-q)C\Delta t])^2/2\sigma^2 C^2 \Delta t}}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} \, dS \\
& \left. + \int_{-\infty}^{C} \frac{1}{2} \left[\left. \frac{\partial^2 f_a}{\partial S^2} \right|_l - \left. \frac{\partial^2 f_a}{\partial S^2} \right|_u \right] (S - C)^2 \frac{e^{-(S-[C-(r-q)C\Delta t])^2/2\sigma^2 C^2 \Delta t}}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} \, dS \right\} \\
& + \mu \Delta t \int_{-\infty}^{\infty} \left[X e^{pt} - S \right] \frac{e^{-(S-[C-(r-q)C\Delta t])^2/2\sigma^2 C^2 \Delta t}}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} \, dS. \qquad \text{(B7)}
\end{aligned}
$$

The first integral in Equation (B7) is just $f_a(C + (r - q)C\Delta t, t + \Delta t)$. The second integral is 0 by symmetry. The third integral can be done by parts to yield $\frac{\sigma^2}{2} C^2 \frac{\partial^2 f_a}{\partial S^2}|_u \Delta t$. The fourth integral is a bit tougher. By substituting $x = S - C - (r - q)C\Delta t$ we get

$$
\begin{aligned}
& \int_{-\infty}^{C} \frac{1}{2} \left[\left. \frac{\partial^2 f_a}{\partial S^2} \right|_l - \left. \frac{\partial^2 f_a}{\partial S^2} \right|_u \right] (S - C)^2 \frac{e^{-(S-[C-(r-q)C\Delta t])^2/2\sigma^2 C^2 \Delta t}}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} \, dS \\
& \quad = \frac{1}{2} \left[\left. \frac{\partial^2 f_a}{\partial S^2} \right|_l - \left. \frac{\partial^2 f_a}{\partial S^2} \right|_u \right] \int_{-\infty}^{-(r-q)C\Delta t} \frac{(x + (r-q)C\Delta t)^2}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} e^{-x^2/2\sigma^2 C^2 \Delta t} dx. \qquad \text{(B8)}
\end{aligned}
$$

Expanding the quadratic term yields three integrals

$$
\begin{aligned}
= \frac{1}{2} \left[\left. \frac{\partial^2 f_a}{\partial S^2} \right|_l - \left. \frac{\partial^2 f_a}{\partial S^2} \right|_u \right] & \left\{ \int_{-\infty}^{-(r-q)C\Delta t} \frac{(r-q)^2 C^2 \Delta t^2}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} e^{-x^2/2\sigma^2 C^2 \Delta t} dx \right. \\
& + \int_{-\infty}^{-(r-q)C\Delta t} \frac{2(r-q)C\Delta t}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} x e^{-x^2/2\sigma^2 C^2 \Delta t} \, dx \\
& \left. + \int_{-\infty}^{-(r-q)C\Delta t} \frac{1}{\sqrt{2\pi\sigma^2 C^2 \Delta t}} x^2 e^{-x^2/2\sigma^2 C^2 \Delta t} \, dx \right\}. \qquad \text{(B9)}
\end{aligned}
$$

The first integral in Equation (B9) is

$$
(r-q)^2 C^2 (\Delta t)^2 N\left(-\frac{(r-q)}{\sigma} \sqrt{\Delta t} \right) \approx O(\Delta t)^2, \qquad \text{(B10)}
$$

where $N(x)$ is the cumulative normal distribution function.

The second integral in Equation (B9) is

$$-\frac{2(r-q)C^2\sigma}{\sqrt{2\pi}}(\Delta t)^{3/2}\left[e^{-\frac{(r-q)^2}{\sigma^2}\Delta t}\right] \approx O(\Delta t)^{3/2}. \tag{B11}$$

The third integral must be done by parts to yield

$$\sigma^2 C^2 \Delta t\left[N\left(-\frac{(r-q)}{\sigma}\sqrt{\Delta t}\right)+\frac{(r-q)}{\sqrt{2\pi\sigma^2}}\sqrt{\Delta t}e^{-\frac{(r-q)^2}{\sigma^2}\Delta t}\right] \approx \frac{\sigma^2}{2}C^2\Delta t + O(\Delta t)^{3/2}. \tag{B12}$$

This indicated that the fourth integral in Equation (B7) is $\frac{\sigma^2}{4}C^2[\frac{\partial^2 f_a}{\partial S^2}|_l - \frac{\partial^2 f_a}{\partial S^2}|_u]\Delta t + O(\Delta t)^{3/2}$. The fifth integral in Equation (B7) is $\mu\Delta t\,[Xe^{pt} - C - (r-q)C\Delta t]$. We will now rewrite Equation (B7), keeping only terms of order Δt or less as

$$\begin{aligned} &f_a(C,t)\\ &=\left[f_a(C+(r-q)C\Delta t, t+\Delta t)+\frac{\sigma^2}{2}C^2\frac{\partial^2 f_a}{\partial S^2}\bigg|_u \Delta t+\frac{\sigma^2}{4}C^2\left(\frac{\partial^2 f_a}{\partial S^2}\bigg|_l-\frac{\partial^2 f_a}{\partial S^2}\bigg|_u\right)\Delta t\right]\\ &\quad\times[1-(r+\mu+\lambda)\Delta t]+\mu\Delta t(Xe^{pt}-C). \end{aligned} \tag{B13}$$

Now, since $f_a(C+(r-q)C\Delta t, t+\Delta t) \approx f_a(C,t) + \frac{\partial f_a}{\partial t}\Delta t + \frac{\partial f_a}{\partial S}(r-q)C\Delta t$, we get

$$\begin{aligned} f_a(C,t) = {} & f_a(C,t)+\frac{\partial f_a}{\partial t}\Delta t+\frac{\partial f_a}{\partial S}(r-q)C\Delta t-(r+\mu+\lambda)f_a(C,t)\Delta t\\ &+\frac{\sigma^2}{2}C^2\left(\frac{1}{2}\frac{\partial^2 f_a}{\partial S^2}\bigg|_l+\frac{1}{2}\frac{\partial^2 f_a}{\partial S^2}\bigg|_u\right)\Delta t+\mu(Xe^{pt}-C)\Delta t. \end{aligned} \tag{B14}$$

If we cancel the constant term, divide out Δt, and rearrange terms, we get

$$\begin{aligned} &\frac{\partial f_a}{\partial t}+(r-q)C\frac{\partial f_a}{\partial S}+\frac{1}{2}\sigma^2C^2\left(\frac{1}{2}\frac{\partial^2 f_a}{\partial S^2}\bigg|_l+\frac{1}{2}\frac{\partial^2 f_a}{\partial S^2}\bigg|_u\right)\\ &\quad=[r+\mu+\lambda]\,f_a-[\mu_x(t)](Xe^{pt}-C). \end{aligned} \tag{B15}$$

Since we know, from Equation (3) that

$$\frac{\partial f_a}{\partial t}+(r-q)C\frac{\partial f_a}{\partial S}+\frac{1}{2}\sigma^2C^2\frac{\partial^2 f_a}{\partial S^2}\bigg|_u=[r+\mu_x(t)+\lambda(S,t)]\,f_a-[\mu_x(t)](Xe^{pt}-C), \tag{B16}$$

it follows that

$$\frac{1}{2}\sigma^2C^2\left(\frac{1}{2}\frac{\partial^2 f_a}{\partial S^2}\bigg|_l+\frac{1}{2}\frac{\partial^2 f_a}{\partial S^2}\bigg|_u\right)=\frac{1}{2}\sigma^2C^2\frac{\partial^2 f_a}{\partial S^2}\bigg|_u, \tag{B17}$$

or

$$\left.\frac{\partial^2 f_a}{\partial S^2}\right|_l = \left.\frac{\partial^2 f_a}{\partial S^2}\right|_u . \tag{B18}$$

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