

A PRICING MODEL FOR QUANTITY CONTRACTS

Knut K. Aase

ABSTRACT

An economic model is proposed for a combined price futures and yield futures market. The innovation of the article is a technique of transforming from quantity and price to a model of two genuine pricing processes. This is required in order to apply modern financial theory. It is demonstrated that the resulting model can be estimated solely from data for a yield futures market and a price futures market. We develop a set of pricing formulas, some of which are partially tested, using price data for area yield options from the Chicago Board of Trade. Compared to a simple application of the standard Black and Scholes model, our approach seems promising.

INTRODUCTION

In the farming industry as well as for many other primary commodity producers it is possible to effectively manage price risk by the use of futures price contracts and options on futures. However, in many of these industries there is still considerable uncertainty left when it comes to revenue, since quantities produced can be volatile, depending on many factors, such as e.g., weather conditions in the growing season. Until recently, similar market-based instruments for managing yield risk have not been available. Instead, federal agricultural support programs and subsidized Crop Yield Insurance (CYI) programs have served as alternatives. In an important development in 1995, the Chicago Board of Trade (CBOT) has launched its CYI Futures and Options contracts. The first CYI contract that began trading on June 2, 1995 was Iowa Corn Yield Insurance Futures and Options. On January 19, 1996, the CBOT added a U.S. contract plus four additional state corn yield contracts for Illinois, Indiana, Ohio, and Nebraska. So far the trading volumes have been fairly modest. Regardless of the status of this particular market for the moment, we want to discuss such contracts from a principle point of view, and develop a pricing theory for this kind of markets.¹

Knut K. Aase works at the Norwegian School of Economics and Business Administration, Bergen, Norway. The author would like to thank Jørgen Aase Nielsen and Bent Jesper Christensen for valuable comments during a seminar presentation at the University of Aarhus. Sten Harr helped to organize the data we obtained from Dr. Eugene Kunda at the CBOT.

¹ At the time of publishing of this article, CYI futures and options are, to my knowledge, not traded any longer on the CBOT due to the low trading volumes. One reason of this failure may be yield basis risk, i.e., individual yields were not sufficiently correlated with yield indices

The CYI contracts are designed to provide a hedge for crop yield risk. For example, CYI futures users can lock in a certain crop yield several months into the future as a temporary substitute for a later yield-based commitment, or they can alternatively lock in the revenue of a given acreage by combining yield contracts with futures price contracts.²

The emergence of markets like these can be thought of as a result of *dynamic efficiency*; if agents think that such instruments will improve economic efficiency, they will somehow be created.

The focus of this article is to construct a pricing model for yield futures and futures option contracts. The innovation is in the modeling stage. In order to apply modern financial theory, one has to start with genuine pricing models. The starting point here is, on the other hand, a model for yield and a model for the spot price of corn. A transformation is proposed in order to overcome this difficulty. It is demonstrated that the resulting technique is consistent with financial pricing theory, and also possible to implement in practice.

There is a large literature on non-market-based risk management and insurance of crop yield, which we will not address here. Yield contracts have been dealt with from the perspective of hedging, using a mean variance approach by Vukina, Li, and Holthausen (1996), while minimizing the variance of revenue was the objective in Li and Vukina (1998). In both these papers the yield contracts traded at CBOT are explained, so we need not elaborate on the market structure here.

There was another securitized insurance market at the CBOT centered around certain catastrophe indexes, these indexes playing a similar role to the yield index of the present article (e.g., Aase, 1999, 2001). The analysis of such markets must typically differ from the model chosen in the present article, since catastrophes cannot be modeled well by a continuous stochastic process.

The article is organized as follows. In the first section we present the economic model, which we develop in the subsequent section to a pricing model for any combination of yield and price futures and futures option contracts, like a futures contract on revenue (if it were to exist). In the third section, we specialize to pure yield contracts, where in Proposition 2 we present pricing formulas for yield futures and yield option contracts. These we calibrate and estimate from price data at the CBOT. Two proofs are relegated to Appendix A. The mechanism of using yield futures can best be illustrated by an example, which can be found in Appendix B. Some contract specifications are given in Appendix C. We round off the article with some remarks on risk management, related in particular to the example of Appendix B. The last section concludes.

to be of interest to U.S. farmers (e.g., in the trade off between basis risk and transaction costs).

² In an earlier paper (Aase, 2002), it is shown what strategy can be used to lock in a certain revenue, when combining these two markets. Here it is abstracted from production costs, and assumed zero local price basis (i.e., local cash price equals futures price) and zero yield basis (i.e., individual farm yield equals index yield).

AREA YIELD FUTURES AND OPTIONS

Introduction

Imagine a country, or another area, sectioned into regions that are uniform in terms of growing conditions for a certain crop, say corn. In each area there is a quantity index y_t , for time t running from 0 to T , where T is the time of sale and 0 is the time of sowing. As an example, for agricultural yield contracts in the United States traded at the CBOT the values of y are provided by the United States Department of Agriculture (USDA). One may think of y_t as a forecast at each time t of quantity, measured in bushels per acre, up for sale in this specific region at the final time T . On this index we assume it is possible to trade futures, and futures options contracts. In order to bring in the quantum uncertainty, we assume that this index can be modeled as a stochastic process. A farmer in this region may have production uncertainty that is well represented by this index, where the relevant number of contracts can be determined from each farmer's production area.

The idea is that if the producer can buy options on this quantity index or on its corresponding futures index, the farmer can lock in a prespecified quantity by buying an appropriate number of such contingent claims. This strategy is of course 100 percent efficient only if the farmer's yield uncertainty is perfectly represented by the index, an unlikely event, but a careful selection of homogeneous regions may make such markets useful for practical risk-management purposes. Presumably one can use a yield market in combination with an ordinary futures market for the price of the crop to secure a prespecified revenue, abstracting from production costs. Exactly how this can be done is the subject of another paper (Aase, 2002).

The Economic Model

In this section, we present a simple, testable model of a futures market for both price and yield. Quantity $y(t)$ at time t , measured in bushels per acre, is not a price process. In order to be able to use the framework of no-arbitrage pricing theory of financial economics, we start with two pricing processes: (i) the spot price process $q(t)$ of the crop, and (ii) another spot price process denoted $p(t)$ such that the fraction

$$y(t) := \frac{p(t)}{q(t)}. \quad (1)$$

One may wonder what $p(t)$ will be the spot price of. First, note the units of measurement of p must be in dollars per acre. Since q is measured in dollars per bushel, y is measured in bushels per acre as it certainly should. Second, consider the price of a leasing contract of agrarian land for the crop in the particular region of consideration, which expires at time T . Then, under certain presumptions, one may think of $p(t)$ as the spot price of such a leasing contract.

The idea is now that in using a no-arbitrage argument in the pricing of a futures contract on the quantity index, it is an implicit assumption that the index can be traded. Since such an index is not a traded commodity, we can nevertheless set up the

required no-arbitrage argument in the leasing market for agricultural land. This will solve our theoretical pricing problem of quantity futures.³

Other crops than corn can, of course, be produced on the agrarian land, so the above interpretation cannot be strictly valid if this is possible. But, if we assume that the particular crop is the dominating agricultural product in the area under consideration, this interpretation of p is fruitful, at least as a thought experiment. The introduction of the pricing process p primarily plays a consistency role in the model. It turns out that the parameters of the pricing process $p(t)$ will indirectly be available from observations in the following three markets: the yield futures market, the yield futures options market, and the ordinary price futures market of corn. In other words, we will never need to study separately the leasing market for corn land.

Associated with the crop there is a *convenience yield*, which we model by a constant fraction of the relevant price process. For the crop there is assumed to be a world futures market, which can be used to determine the convenience yield rate δ_q for the crop. Associated with the leasing market for agrarian crop land, the “convenience yield” can better be interpreted as a leasing rate, say δ_p .

Convenience yields on the crop is related to the return, and is assumed stochastic in our model, but is a bounded variation process. It is represented by a fixed percentage δ_q of the price $q(t)$. The accumulated convenience yields in the time interval $(0, t]$ is $\int_0^t \delta_q q_s ds$, and is thus a stochastic process.

The notion of convenience yield was introduced by the economists Kaldor and Working who, among other things, studied the theory of storage. In the present context, it may reflect the relative advantage a holder of the crop has compared to someone who only has a claim to a future delivery of the crop. Convenience yield can, of course, vary through the season, and can in fact be negative and equal to physical storage cost, as with corn from December to March. Since we typically consider time durations of say one year or less, it may not be unreasonable to consider the percentage δ_q to be a constant in this time interval.

For the price process p , interpreted loosely as the price of leasing agricultural land, there is also a “convenience yield” here, but now interpreted simply as a rent of this land. We make similar assumptions for the rental rate δ_p as above.

We now give the formal description of the model. Given is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where Ω is the set of states with generic element ω ; P is a probability measure, the “objective probability”; $\mathcal{F}$ is the set of events in Ω given by a σ -algebra; $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is a filtration satisfying the usual conditions, where $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$, $\mathcal{F}_t$ signifying the possible events that could happen by time t , or “the information available by time t .” We assume $\mathcal{F}_0$ to be trivial, containing only events of probability 0 or 1, meaning roughly that there is no information available at time zero, and $\mathcal{F}_T = \mathcal{F}$, i.e., at time T all the uncertainty is resolved.

We assume there is a risk-free asset having rate of return r and price at time t , β_t , given by

³ In practice such a hedging procedure may not be entirely accurate, but then one should perhaps have in mind that in the real world the only “perfect hedge” is found in a Japanese garden.

$$d\beta_t = r\beta_t dt, \quad \beta_0 = 1. \quad (2)$$

On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ are given two stochastic price processes; one price process p_t related to leasing agrarian land measured in dollars per acre and a price process q_t of the crop measured in dollars per bushel satisfying the following stochastic differential equations:

$$dp(t) = \mu_p p(t) dt + p(t)(\sigma_{p,1} dB_1(t) + \sigma_{p,2} dB_2(t)), \quad (3)$$

$$dq(t) = \mu_q q(t) dt + q(t)(\sigma_{q,1} dB_1(t) + \sigma_{q,2} dB_2(t)). \quad (4)$$

Here B_1 and B_2 are two independent, standard Brownian motions generating the filtration $\mathbb{F}$. In the present setting the Brownian motions seem reasonable as the driving sources of uncertainty, since we primarily want to carry out a principal analysis, and then we have the advantages of a complete model, as will be apparent from what follows. If one is very concerned about realism, a jump Itô–Lévy system of processes would seem more natural, which could model the occurrence of a natural disaster on the gross revenue. In this case, we would not, however, in general enjoy the benefits from working with a complete model, and a full equilibrium type model seems to be the most reasonable approach in this case.

The quantity μ_q is the conditional expected rate of change of the capital gain of the crop, sometimes termed as the instantaneous expected capital gain, with a similar interpretation for μ_p related to agrarian land, and $\sigma_{q,1}, \sigma_{q,2}, \sigma_{p,1}, \sigma_{p,2}$ are volatility parameters. To explain the latter more precisely, let

$$\sigma_{p,q} := \sigma_{p,1}\sigma_{q,1} + \sigma_{p,2}\sigma_{q,2}, \quad (5)$$

$$\sigma_p^2 := \sigma_{p,1}^2 + \sigma_{p,2}^2, \quad (6)$$

$$\sigma_q^2 := \sigma_{q,1}^2 + \sigma_{q,2}^2, \quad (7)$$

$$\rho := \frac{\sigma_{p,q}}{\sigma_p \sigma_q}. \quad (8)$$

Then σ_q^2 is the rate of change of the conditional variance of the return on the crop, with a similar interpretation for σ_p^2 related to the return on leasing agrarian land. Since

$$\begin{aligned} \sigma_{p,q} &= \frac{1}{(t-s)} \text{cov}_s(\sigma_{p,1}(B_1(t) - B_1(s)) \\ &\quad + \sigma_{p,2}(B_2(t) - B_2(s)), \sigma_{q,1}(B_1(t) - B_1(s)) + \sigma_{q,2}(B_2(t) - B_2(s))), \end{aligned} \quad (9)$$

the parameter $\sigma_{p,q}$ is the rate of change of the conditional covariance between the return on the crop and the return on leasing land, and ρ is the corresponding instantaneous correlation coefficient.⁴

⁴ Here, $\text{cov}_s(\cdot)$ denotes conditional covariance, given the information available at time $s \leq t$.

Now let us consider the quantity variable y measured in bushels per acre. Using Itô's lemma on $y(t) = p(t)/q(t)$ in (1), we find that also y satisfies a stochastic differential equation of the form given for p and q above, i.e.,

$$dy(t) = \mu_y y(t) dt + y(t)(\sigma_{y,1} dB_1(t) + \sigma_{y,2} dB_2(t)), \quad (10)$$

where

$$\mu_y = \mu_p - \mu_q - \sigma_{p,q} + \sigma_q^2, \quad \sigma_{y,1} = \sigma_{p,1} - \sigma_{q,1}, \quad \sigma_{y,2} = \sigma_{p,2} - \sigma_{q,2}. \quad (11)$$

This follows by the Itô differentiation rule, since

$$\begin{aligned} dy(t) &= \frac{1}{q_t} dp_t - \frac{p_t}{q_t^2} dq_t - \frac{dp_t dq_t}{q_t^2} + \frac{1}{2} \left(\frac{2p_t}{q_t^3} \right) (dq_t)^2 \\ &= y_t \frac{dp_t}{p_t} - y_t \frac{dq_t}{q_t} - y_t \left(\frac{dp_t}{p_t} \frac{dq_t}{q_t} \right) + y_t \left(\frac{dq_t}{q_t} \right)^2 \\ &= y_t (\mu_p - \mu_q - \sigma_{p,q} + \sigma_q^2) dt + y_t ((\sigma_{p,1} - \sigma_{q,1}) dB_1(t) + (\sigma_{p,2} - \sigma_{q,2}) dB_2(t)). \end{aligned}$$

From this we also notice that

$$\sigma_{y,q} = \sigma_{p,q} - \sigma_q^2, \quad (12)$$

where $\sigma_{y,q} := \sigma_{y,1}\sigma_{q,1} + \sigma_{y,2}\sigma_{q,2}$ by the convention in Equation (5). Since there are no restrictions on the values of the parameter $\sigma_{p,q}$, any values of the covariance rate $\sigma_{q,y}$ is allowed. Normally, we would expect that $\sigma_{y,q} < 0$, at least if y_T represented the total quantum per acre up for sale at time T . Similarly, if y refers to an important region in terms of produced quantity brought to the market, we also anticipate a negative covariance, but typically the correlation may be close to zero in smaller regions with less impact. Notice that if $\sigma_{p,q} = 0$ for some region, not an unreasonable assumption, then $\sigma_{y,q} < 0$ for this area, which follows from the relation (12) because $\sigma_q^2 > 0$.

Discussion of the Model

The reader will have noticed that we have chosen correlated geometric Brownian motions as models for the quantities p and q . These processes are strictly positive for all $t > 0$ almost surely, an important property here, since both prices in question are positive, and quantum y becomes well defined and positive as well. In all, as a first approach, we believe this model can serve as a reasonable choice.

The convenience yield rates δ_q and δ_p are assumed to be constants, although there are few problems to allow for these to be stochastic processes as well, e.g., Gibson and Schwartz (1990), who used the Ornstein–Uhlenbeck process in this regard. In our case this would serve to unnecessarily complicate matters, in particular since these quantities are not marketed assets, so we choose parsimony.

Regarding our choice for the interest rate r , it could of course also have been modeled by a stochastic process, say a mean reverting one, but we choose simplicity here as well.

In a relatively recent paper, Miltersen and Schwartz (1998) develop a fairly general model to value options on commodity futures in the presence of stochastic interest rates as well as stochastic convenience yields. However, they do not consider quantity contracts and contracts on revenue as we do.

Suppose there is a futures market for the crop under consideration. Brennan and Schwartz (1985) in their pioneering research incorporated the convenience yield in the valuation of commodity derivatives, and established in particular the relationship between the spot price q_t and the futures price F_t^q at time t for delivery of the commodity at the future time T , given by

$$F_t^q = q_t e^{(r - \delta_q)(T-t)} \quad \text{for } t \leq T. \quad (13)$$

Since all the quantities in this formula, including the left-hand side, are directly observable except for the convenience yield rate δ_q , this parameter can be estimated from this relationship, using observations in the spot and the futures market for the crop.

Now consider the leasing of agrarian land. It is not common to have an associated futures markets on $p(t)$ directly, so the question then comes up how to estimate the rental rate δ_p . From our results in the next section (Proposition 2, Equation (34)) it follows that the futures price at time t on the *quantity* variable y depends on the parameters δ_q , δ_p , and $\sigma_{y,q}$ in addition to the value y_t . The only remaining unknown parameter here is δ_p , which can then be estimated from this relationship, using the observations for the index $y(t)$ noted in the quantity futures market, and the observed futures prices F_t^y for yield y in this market. Thus the existence of a futures market for quantity will effectively resolve this estimation problem. This we illustrate later.

We notice in particular that we do not need to estimate the parameters associated to the process p from observations of the leasing market for agricultural land, in order to employ the present model to the futures yield market. Thus the inclusion of a market for leasing of land was necessary primarily to establish a consistent pricing model, in addition to the possibility of using this market for hedging purposes, the latter to support the no-arbitrage pricing of quantity futures.

The Financial Pricing Model

We are now in position to use the pricing theory of financial economics (e.g., Duffie, 1996, Chapter 6). To this end, consider the following linear system of equations:

$$\begin{pmatrix} p_t \sigma_{p,1} & p_t \sigma_{p,2} \\ q_t \sigma_{q,1} & q_t \sigma_{q,2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} (\mu_p + \delta_p - r) p_t \\ (\mu_q + \delta_q - r) q_t \end{pmatrix}. \quad (14)$$

By assumption both p_t and q_t are positive for all t with probability one, so in this system of equations both these quantities cancel.

The right-hand side in (14) follows because p and q are price processes where the drift terms must be adjusted for the relevant convenience yields and the risk-free interest rate r . In general, when there is a risk-free asset, it is the drift and diffusion terms of the discounted, adjusted price processes that appear in this equation, and since the convenience yields can be treated as dividend rates, the drift terms of the discounted gains processes are given by $(\mu_p + \delta_p - r)p_t$ for the price process of leasing agrarian land, and $(\mu_q + \delta_q - r)q_t$ for the price process of the crop, while the associated diffusion terms of the discounted, adjusted price processes are both unaltered from that of the price processes p and q , since both the risk-free asset and the convenience yields are of bounded variation.

The solution of the system of equations given in (14) is as follows:

$$\eta_1 = \frac{\sigma_{p,2}(\mu_q + \delta_q - r) - \sigma_{q,2}(\mu_p + \delta_p - r)}{\sigma_{p,2}\sigma_{q,1} - \sigma_{p,1}\sigma_{q,2}} \quad (15)$$

$$\eta_2 = \frac{\sigma_{q,1}(\mu_p + \delta_p - r) - \sigma_{p,1}(\mu_q + \delta_q - r)}{\sigma_{p,2}\sigma_{q,1} - \sigma_{p,1}\sigma_{q,2}}, \quad (16)$$

assuming the determinant in the denominator different from zero. Thus the *market-price-of-risk* parameters η_1 and η_2 are determined in terms of the parameters of the model, including the convenience yield rate δ_q and the rental rate δ_p .

The model as outlined above is complete, which means that if X_T represents the payoff of any asset or contingent claim at time T , having no intermediate dividends, then the market price X_t at time $t \leq T$ is given by

$$X_t = \frac{1}{\xi_t} E_t \{ e^{-r(T-t)} \xi_T X_T \}, \quad (17)$$

where the density process ξ_t in our model is given by the expression

$$\xi_t = \exp \left\{ -\eta_1 B_1(t) - \eta_2 B_2(t) - \frac{1}{2}(\eta_1^2 + \eta_2^2)t \right\} \quad \text{for } t \leq T, \quad (18)$$

or, in differential form, by Itô's lemma

$$d\xi_t = -\xi_t(\eta_1 dB_1(t) + \eta_2 dB_2(t)), \quad \xi_0 = 1. \quad (19)$$

Here $\pi_t := \xi_t e^{-rt}$ is the state price deflator (the pricing kernel, or the shadow price).

An equivalent martingale measure Q is given by $\frac{dQ}{dP} = \xi_T$, where P is the given probability measure under which the joint probability distribution of (q, y) can be found from Equations (4) and (10). By completeness of the model, the measure Q is uniquely determined, and the pricing formulas above can alternatively be expressed in terms of discounted expectations under Q . For example, can the market value in Equation (17) be written as

$$X_t = E_t^Q \{ e^{-r(T-t)} X_T \}. \quad (20)$$

This formula usually gives the most direct way to carry out the computations of prices once the probability distributions of p and q are known under the measure Q . Here, we notice that the drift rate of the process p is $(r - \delta_p)$ under Q , the drift rate of q is similarly $(r - \delta_q)$, whereas the variance and covariance rate parameters are the same as under P . As a consequence it follows from relations (11) and (12) that the drift rate μ_y^Q of the yield variable y is given as

$$\mu_y^Q = \delta_q - \delta_p - \sigma_{y,q} \quad \text{under } Q. \quad (21)$$

Note that the futures prices are lognormally distributed, and futures contracts are traded assets. Thus the futures indexes serve as underlying *traded* assets, supporting the no-arbitrage arguments behind the pricing results, known to hold under these assumptions.

In the extant literature there are several papers dealing with pricing of products of processes. A typical area of application is exchange rate models, as in e.g., Babel and Eisenberg (1993). This article also treats a variety of other issues. Other applications are to the valuation of the option to exchange one asset for another (Margarbe, 1978), options on the minimum and the maximum of two risky assets (Stulz, 1982; Johnsen, 1987), or the valuation of a random number of put options (Marcus and Modest, 1986). There are also papers treating the situation with an uncertain exercise price, e.g., Fisher (1978). A related literature on real options is of course of interest, as e.g., Bjerksund and Ekern (1990), Paddock, Siegel, and Smith (1988), Majd and Pindyck (1987), among others.

In the next section, we illustrate how to apply the above model in the valuation of general financial contracts. For example, we show how to price futures on revenue directly. Of course, there is no market for such contracts, therefore these examples will mainly serve as theoretical benchmarks for the subsequent analysis of pure yield and price contracts.

THE FUTURES OPTION PRICE OF REVENUE

Introduction

In this section, we use the valuation theory outlined above to compute market values $V_t(X_T)$ at any time $t \leq T$ of a claim on the future delivery of X_T at time T for various contingent claims X , and also associated futures prices and futures options prices. We then use the insights obtained from this to find the market values of yield contracts.

It seems reasonable to start with the processes y and q , and the revenue $R := yq$, since it is the uncertainty in the revenue the farmers presumably are concerned with. This is natural, since the sources of information will be the spot and futures markets for the crop, as well as the quantity index y and its associated futures market.

To this end, let us consider the problem of finding the current value, at time t , of a claim on the future delivery of $R_T = q_T y_T$ at time T . This value we denote by $V_t(R_T)$. We claim it is given by the expression

$$V_t(R_T) = q(t) y(t) e^{-\delta_p(T-t)}. \quad (22)$$

This expression follows from the valuation Equation (20):

$$V_t(R_T) = e^{-r(T-t)} E_t^Q\{q_T y_T\}.$$

Since $y_t = p_t/q_t$, this equals

$$e^{-r(T-t)} E_t^Q\{p_T\} = e^{-r(T-t)} p_t e^{(r-\delta_p)(T-t)}, \quad (23)$$

where the last equality follows since the drift rate of the price process p of agrarian land under the risk-adjusted pricing measure Q is $(r - \delta_p)$. Thus the conclusion follows from the definition of p .

Observe that the parameters of interest are reduced to

$$r, \delta_p, \delta_q, \sigma_{y,1}, \sigma_{y,2}, \sigma_{q,1}, \sigma_{q,2}, \quad \text{and} \quad \sigma_{y,q}.$$

Futures Options on Revenue

If there existed a futures market for the revenue process R itself, the futures price F_t^R at any time $t \leq T$ is determined by the relation⁵

$$E_t^Q(R_T - F_t^R) = 0,$$

which implies that

$$F_t^R = E_t^Q(q_T y_T) = q_t y_t e^{(r-\delta_p)(T-t)}, \quad (24)$$

since F_t^R is contained in the information set $\mathcal{F}_t$ at time t .

We can also consider contingent claims, or options, on the futures price index F^R . Suppose a contingent claim has payoff only at some time $T_1 \leq T$. If the futures price is $F_{T_1}^R$ at this time, the payoff of the contingent claim is $\varphi(F_{T_1}^R)$, where φ is some nonlinear, real function determined by the specific contract.

As an example, suppose a farmer wants to secure at least a revenue p^0 (a constant) by time T_1 . Then he could consider buying a put option on the futures index for revenue (if it were to exist), in which case $\varphi(x) = (p - x)^+$. In the case where his own production is closely correlated with the quantity index, this futures option may give adequate protection against a bad season.

If we consider the contingent claim as an *option* on F^R , we have a *conventional futures option*, with market price at time t is given by

$$V_t(\varphi(F_{T_1}^R)) = e^{-r(T_1-t)} E_t^Q(\varphi(F_{T_1}^R)). \quad (25)$$

⁵ Notice that the forward price and the futures price are equal in this model, since the interest rate is deterministic.

In this case the premium in (25) is payable at time t .⁶ On the other hand, if we consider the contingent claim as a *futures contract*, we have a *pure futures option*, in which case we determine the futures price F_t^φ from

$$E_t^Q(\varphi(F_{T_1}^R) - F_t^\varphi) = 0$$

or

$$F_t^\varphi = E_t^Q(\varphi(F_{T_1}^R)). \quad (26)$$

Here nothing is paid at the initiation of the contract.

We will consider the latter interpretation when treating options on futures contracts. In the present model the price of the conventional contract is simply the discounted value of the pure futures option price.

As an illustration, let us evaluate a put option on the futures price of revenue in our model. The easiest way to accomplish this is to use the results of the previous section.

Proposition 1: *The value of a European put option, with exercise price p^0 and expiration time $T_1 \leq T$, on the futures price process of revenue, the latter with expiration time T , is given as follows:*

$$F_t^p = E_t^Q\{(p^0 - F_{T_1}^R)^+\} = p^0 \Phi(y_1) - R_t e^{(r - \delta_p)(T - t)} \Phi(y_2), \quad (27)$$

where $\Phi(\cdot)$ is the cumulative probability distribution function of the standard normal distribution, where

$$y_1 = \frac{\ln\left(\frac{p^0}{R_t}\right) - \left(r - \delta_p - \frac{1}{2}\hat{\sigma}^2\right)(T_1 - t) - (r - \delta_p)(T - T_1)}{\hat{\sigma}\sqrt{T_1 - t}},$$

$$y_2 = \frac{\ln\left(\frac{p^0}{R_t}\right) - \left(r - \delta_p + \frac{1}{2}\hat{\sigma}^2\right)(T_1 - t) - (r - \delta_p)(T - T_1)}{\hat{\sigma}\sqrt{T_1 - t}},$$

and where $\hat{\sigma}$ is

$$\hat{\sigma}^2 = (\sigma_{y,1} + \sigma_{q,1})^2 + (\sigma_{y,2} + \sigma_{q,2})^2. \quad (28)$$

The proof of Proposition 1 can be found in Appendix A.

Notice how the formula for this price simplifies somewhat if the expiration time of the option coincides with that of the underlying futures contract, i.e., when $T = T_1$.

⁶ The current value of a claim on the future delivery of the revenue $R(T_1)$ is precisely a conventional futures option, where $\varphi(x) = x$ for all real x .

We notice that

$$\frac{\partial F_t^p}{\partial \delta_p} > 0,$$

so the futures put is more valuable as the rent on agrarian land increases, *ceteris paribus*. When this rental rate increases, the probability increases that $R_{T_1} \exp\{(r - \delta_p)(T - T_1)\}$ falls below p^0 , so the futures put option increases in value.

The Futures Price of General "Product Contracts"

In the present model, we can find market prices of more involved financial contracts, and for later comparisons with "product markets" to be treated in the next section, an analysis of such contracts will be useful. Consider a general contingent claim with payoff at time T given by $X = g(y_T) \cdot h(q_T)$, where $g(\cdot)$ and $h(\cdot)$ are some functions determined by the contract. We use the pure futures option interpretation, and for simplicity of exposition we let the expiration time coincide with the expiration time T for the underlying futures contract on revenue. This "product contract" cannot be analyzed by only knowing the probability distribution of revenue, as we did for the futures put contract of the previous section. In general, the futures price $F_t^{g(y)h(q)}$ in question is given by

$$F_t^{g(y)h(q)} = E_t^Q(g(y_T)h(q_T)) \quad \text{for any } t \leq T. \quad (29)$$

As an illustration of such contracts, and for later comparisons, consider a farmer who is concerned with having at least a harvest of k bushels per acre by time T . In this case, the following contract is of interest:

$$g(x) = (k - x)^+, \quad h(x) = x,$$

that is, a put option on the quantity variable separately, having value $(k - y_T)^+ q_T$ at the expiration time T . Let us denote the corresponding option futures price by $F_t^{(k-y)^+q}$ at time $t \leq T$. We then have the following simple expression for this futures option price.

Theorem 1: *The separate futures option described above has price $F_t^{(k-y)^+q}$ at any time $t \leq T$ given by*

$$F_t^{(k-y)^+q} = kq_t e^{(r-\delta_q)(T-t)} \Phi(d_1) - R_t e^{(r-\delta_p)(T-t)} \Phi(d_2), \quad (30)$$

where

$$d_1 = \frac{\ln\left(\frac{k}{y_t}\right) - \left(\delta_q - \delta_p - \frac{1}{2}\sigma_y^2\right)(T-t)}{\sigma_y \sqrt{T-t}}, \quad (31)$$

and

$$d_2 = \frac{\ln\left(\frac{k}{y_t}\right) - \left(\delta_q - \delta_p + \frac{1}{2}\sigma_y^2\right)(T-t)}{\sigma_y\sqrt{T-t}}. \quad (32)$$

The proof of this theorem can be found in Appendix A.

Non-Separable Futures Options

Situations with more complicated futures options are also possible in this market. Although slightly outside the scope of this article, the story can quickly be told: Instead of the contract of Equation (29), consider a contract on $h(q_T, y_T)$, where h is some function of two variables. The futures price of this futures option is given by

$$F_t^{h(q,y)} = E_t^Q(h(q_T, y_T)) \quad \text{for any } t \leq T. \quad (33)$$

In cases where this expectation is difficult to compute, we may alternatively solve a partial differential equation. For reasonable functions h , there exists a function $f \in C^{2,2,1}(\mathbb{R}_+^2 \times [0, T])$, such that

$$f(q_t, y_t, t) = E_t^Q(h(q_T, y_T))$$

satisfying

$$\mathcal{D}f(q, y, t) = 0, \quad (q, y, t) \in \mathbb{R}_+^2 \times [0, T]$$

with boundary condition

$$f(q, y, T) = h(q, y), \quad (q, y) \in \mathbb{R}_+^2,$$

where

$$\begin{aligned} \mathcal{D}f(q, y, t) = & \frac{\partial}{\partial t} f(q, y, t) + \frac{\partial}{\partial q} f(q, y, t)(r - \delta)q + \frac{\partial}{\partial y} f(q, y, t)(\delta_q - \delta_R - \sigma_{y,q})y \\ & + \frac{1}{2} \left(\frac{\partial^2}{\partial q^2} f(q, y, t) q^2 \sigma_q^2 + 2 \frac{\partial^2}{\partial q \partial y} f(q, y, t) q y \sigma_{q,y} + \frac{\partial^2}{\partial y^2} f(q, y, t) y^2 \sigma_y^2 \right). \end{aligned}$$

The price process $f(q_t, y_t, t) = F_t^{h(q,y)}$ is a Q -martingale having stochastic differential equation given by

$$\begin{aligned} df(q_s, y_s, s) = & \frac{\partial}{\partial q} f(q_s, y_s, s) q_s (\sigma_{q,1} d\tilde{B}_1(s) + \sigma_{q,2} d\tilde{B}_2(s)) \\ & + \frac{\partial}{\partial y} f(q_s, y_s, s) y_s (\sigma_{y,1} d\tilde{B}_1(s) + \sigma_{y,2} d\tilde{B}_2(s)) \quad \text{for } t \leq s \leq T. \end{aligned}$$

We shall not be concerned with contracts of the type where $h(q, y)$ is not a separable function of q and y .

In the next section, we turn to the analysis of yield contracts.

YIELD FUTURES AND FUTURES OPTIONS

We now utilize the results of the last section to construct a pricing model for yield futures and futures option contracts. These we calibrate to price data at the CBOT, and estimate the model. In particular we derive an estimate of the convenience yield δ_p of the price process p , the “rental rate” of agricultural corn land in Iowa.

Pure Yield Contracts

We are now in position to discuss a pure futures market for quantity. What we mean by this is the following: Consider a contingent claim having final payoff X of the form $X = g(y_T) \cdot 1$, where $g(\cdot)$ is some real function specifying the terms of the contract. Here we have simply chosen the function $h(x) = \$1$ per bushel for all $x \geq 0$. According to the general futures formula (29), the futures price $F_t^{g(y) \cdot 1}$ at any time $t \leq T$ is given by $F_t^{g(y) \cdot 1} = E_t^Q(g(y_T) \cdot 1)$.

The dimension of this quantity is value, i.e., in units of the risk-free asset, while the dimension of the quantity $g(y_T)$ is, say bushels per acre, so the number one in these formulas is meant to signify one unit of the numeraire per bushel. In other words, settlement in this market is in cash, not in bushels of wheat, say.⁷

The important example of the futures price for quantity is found as follows: Using the fact that $F_t^y = E_t^Q(y_T \cdot 1)$, we have that

$$F_t^y = E_t^Q(y_T \cdot 1) = y_t \cdot 1e^{(\delta_q - \delta_p - \sigma_{y,q})(T-t)} \quad \text{for any } t \leq T, \quad (34)$$

which follows from the relation (21), where we showed that the drift rate of y_t under Q is given by

$$\mu_y^Q = \delta_q - \delta_p - \sigma_{y,q}.$$

Notice from the formula (34) that when $\sigma_{y,q} > 0$, F_t^y is smaller than in the case where $\sigma_{y,q} < 0$, *ceteris paribus*. If this covariance rate is positive, this roughly means that the agricultural area under consideration is such that, on the average, it harvests larger quantities of the crop when the rest of the world production tends to be low. In this situation, a farmer who sells a quantity futures contract at time t , receives $(F_t^y - y_T)$ at time T , if he holds the position until expiration, which is a lower payout than in a region where $\sigma_{y,q} < 0$, *ceteris paribus*. This seems reasonable, since a quantity insurance would be more needed in an area having $\sigma_{y,q} < 0$ with a low world production, compared to an area where $\sigma_{y,q} > 0$.

⁷ Since there is a risk-free asset, the futures price at time t of 1 at time T is 1. Also, recall the multiplication factor of \$100 per bushel for the Iowa corn contracts in Example 1 of Appendix B.

As an example of a futures option, consider a quantity put option with strike price k . The futures price of this contract is

$$F_t^{(k-y_T)^+,1} = E_t^Q((k - y_T)^+ \cdot 1) = \int_{-\infty}^{\infty} \left(k - y_t e^{\left(\mu_y^Q - \frac{1}{2}\sigma_y^2\right)(T-t) + \sigma_y z}\right)^+ f(z) dz,$$

where $f(\cdot)$ is the probability density of a normal variate with mean zero and variance $(T - t)$. Thus, we summarize our findings as the following proposition.

Proposition 2: *The futures price at time t of a pure futures yield put option with strike price k and expiration time T is given by*

$$F_t^{(k-y_T)^+,1} = k\Phi(x_1) - y_t \cdot 1e^{(\delta_q - \delta_p - \sigma_{y,q})(T-t)}\Phi(x_2), \quad (35)$$

where

$$x_1 = \frac{\ln\left(\frac{k}{y_t}\right) - \left(\delta_q - \delta_p - \sigma_{y,q} - \frac{1}{2}\sigma_y^2\right)(T-t)}{\sigma_y\sqrt{T-t}}, \quad (36)$$

$$x_2 = \frac{\ln\left(\frac{k}{y_t}\right) - \left(\delta_q - \delta_p - \sigma_{y,q} + \frac{1}{2}\sigma_y^2\right)(T-t)}{\sigma_y\sqrt{T-t}}. \quad (37)$$

Furthermore, the futures price at time t of a futures contract of yield y , expiring at time T , is given by

$$F_t^y = y_t \cdot 1e^{(\delta_q - \delta_p - \sigma_{y,q})(T-t)} \quad \text{for any } t \leq T. \quad (38)$$

Proof: Direct integration in the case of the futures yield put option.

Q.E.D.

One should not expect that contracts of the nonseparable type given in (33) can be obtained equivalently in the two separate markets for quantity and price that we discuss. Such contracts are not our concern, however, and for all contracts of the *separable* type given in (29), it is shown in Aase (2002) that one may, in principle, restrict attention to these two separate markets rather than the idealized, nonexistent market of revenue, outlined above. This statement is, strictly speaking, true only if the correlation rate $\sigma_{y,q} = 0$.

Consider the contracts $g(y_T) = (k - y_T)^+$ and $h(q_T) = q_T$. In the situation where $\sigma_{y,q} = 0$,

$$F_t^{(k-y)^+} F_t^q = (kq_t\Phi(x_1) - y_te^{(\delta_q - \delta_R)(T-t)}q_t\Phi(x_2))e^{(r - \delta_q)(T-t)}$$

by Equations (13) and (35), which coincides with the expression for $F_t^{(k-y)^+q}$ given in Theorem 1 when $\sigma_{y,q} = 0$. Note that this is also consistent with the well-known property that a zero correlation between to bivariate normal random variables implies that they

are statistically independent. In this case, one can hedge a prespecified revenue by a combined use of the yield and price markets (see Aase, 2002).

When $\sigma_{q,y} \neq 0$ the futures price of $(k - y_T)^+ q_T$ is given by Theorem 1, and can now only approximately be secured by a combined use of the yield and price markets separately (for details, see Aase, 2002).

Notice that central conclusions (except the given formulas) of the article are not confined to the special probability distributions chosen. Of particular interest is to allow for models where the volatilities display seasonal effects (observed for, e.g., Iowa corn yield options). Seasonal effects will be treated in a separate paper.

We now discuss the use of the formulas in Proposition 2 in the light of some yield put option data obtained from the CBOT.

Calibrating the Parameters

In order to see how the theory presented above can be used, we now calibrate the model to trading data at the CBOT.

One possible test of the put option formula above could be to estimate implied volatilities and compare to historic estimates. In order to single out the parameter σ_y , we need separate estimates for the parameters δ_p , δ_q , and $\sigma_{y,q}$. In principle it is clear what we must do; further details may be found in Harr (1999).

First consider the world futures market for corn. Contract specifications of this market are given in Appendix C. Typically the basis, or, the difference between spot price and futures price, is negative. Based on historic estimates, we use the value $\hat{\delta}_q = -0.10$.⁸ This we may interpret to mean that the storage costs are dominating compared to the advantage to utilize a sudden increase in the demand for corn.

Next we consider the market for corn yield futures for the region of Iowa. Contract specifications for this market are presented in Appendix C. Iowa is a major producer of corn, and an estimate of the covariance rate $\sigma_{y,q}$ turns out to be negative. Based on historic values we use $\hat{\sigma}_{y,q} = -0.20$.

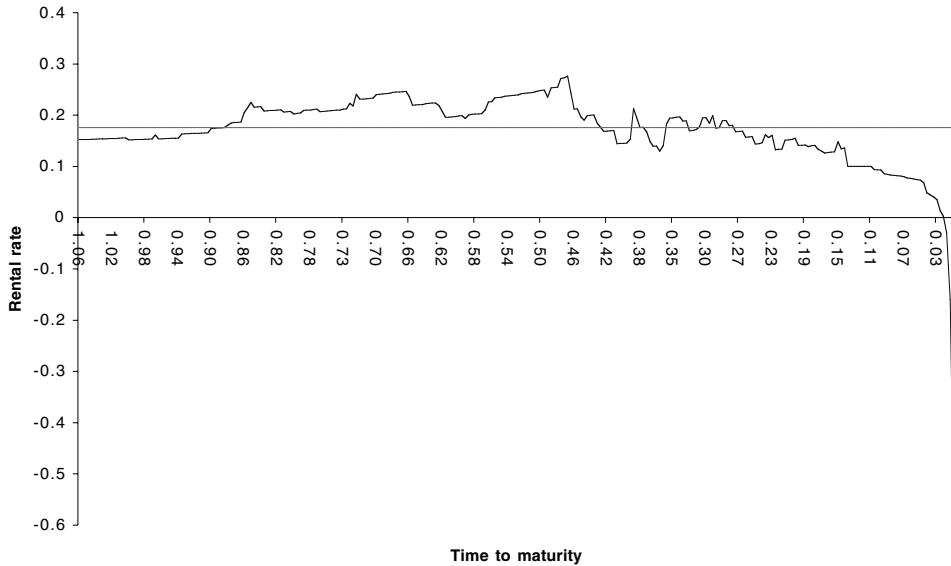
For the yield markets one has observed that the spot price is sometimes above, and sometimes below, the futures price. Considering the expression for the yield futures price in Proposition 2 given by $F_t^y = y_t \exp\{(\delta_q - \delta_p - \sigma_{y,q})(T - t)\}$, this will depend on the parameters of this expression. Analyzing data of corn yield futures for the years 1995, 1997, and 1998, we obtained an estimate of the rental rate for agricultural corn land δ_p in Iowa to be around 17 percent. A graph of how this estimate varies as a function of time to maturity shows that it does not fluctuate much around the mean value during the year, but gets a sharp drop towards the end of the year to levels around -40 to 50 percent (see Figure 1 for the year 1997). The other years show similar patterns.

This sharp drop towards the end of the life of the contract probably reflects that the crop has been harvested, so the agricultural land has no immediate use, which can give the owner any positive expected return the rest of the year.

⁸ These estimates we have obtained at the courtesy of Dr. Eugene Kunda at the CBOT.

FIGURE 1

Estimated Rental Rate $\hat{\delta}_p$ of Agricultural Corn Land in Iowa as a Function of Time to Maturity $\tau = T - t$ (Year 1997)



Finally, we turn to the market for corn yield futures options for the region of Iowa, with contract specifications given in Appendix C. We considered put options expiring in January of the years 1996, 1997, and 1999. These years were chosen because trade then took place in the put options with several different strikes. The implicit volatility σ_y in the futures option formula (35) was then inverted from the pricing formula using the Newton–Raphson algorithm. Common to all the put options analyzed is that time to expiration runs from about 3 months to around 1 year.⁹

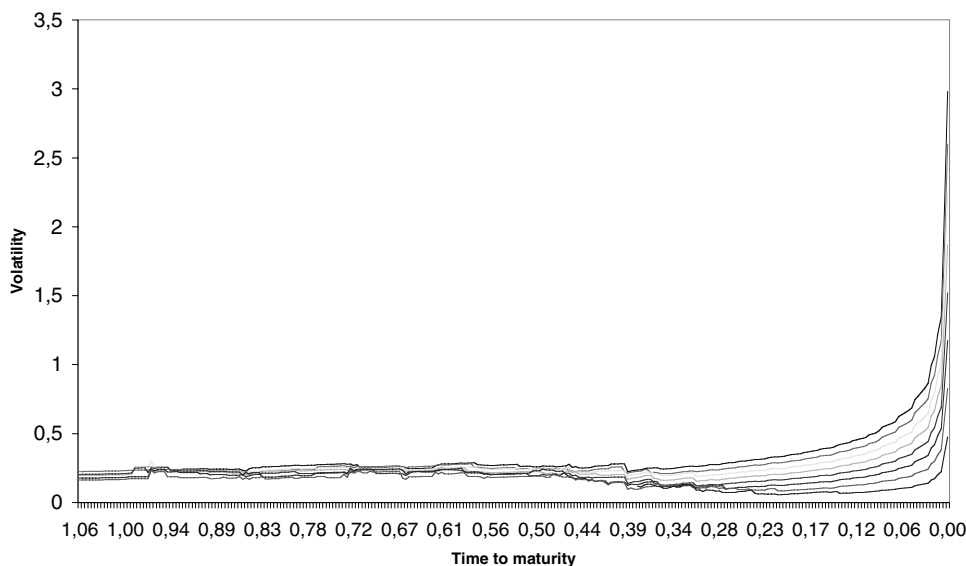
For the year 1995, we estimated an implied volatility σ_y of 17 percent. We have also investigated the volatility structure during 1995 for put options with five different strikes, 1,050, 1,100, 1,150, 1,200, and 1,250, as a function of the time to expiration. The typical picture is that the implied volatility decreases slowly during the year as time to maturity decreases, but then there is a sharp increase towards the end of the year, stronger for the put with the lower strike. This year the yield index of Iowa ended at 123 bushels per acre, which was higher than the markets expectations a few months earlier, based on trading at that time.

For the year 1996, an the average estimate $\hat{\sigma}_y = 0.23$. The eight different strikes were 1,000, 1,050, 1,100, 1,150, 1,200, 1,250, 1,300, and 1,350. Here the picture shows very little variation during the year, where it has been steady at around 20 percent, but with a sharper increase at the end, than for the year 1995. The yield index of Iowa ended at 138 bushels per acre, lower than expected a few months earlier.

⁹ All the data are again at the courtesy of Dr. Kunda.

FIGURE 2

Estimated Volatility as a Function of Time τ to Maturity for Five Different Strikes: 1,100, 1,150, 1,200, 1,250, 1,300, and 1,350. The Lowest Strike Has the Sharpest Increase, Then the Next Lowest, etc. (Year 1998, With Contracts Expiring in January 1999)



For the year 1998 the average turned out to be 20.66 percent. The six different strikes were 1,100, 1,150, 1,200, 1,250, 1,300, and 1,350. The estimated volatilities for the different strikes stayed approximately constant up until just a short time before expiration (see Figure 2 for this year). The other two years roughly show similar patterns. The yield index of Iowa ended at 145 bushels per acre, lower than expected based on trade a few months earlier.

The estimates of the implied volatilities for all the three years display a similar structure: Relatively constant through the year, but with a sharp increase just before expiration. This increase we partly attribute to the release at that time of the harvest report by USDA. In this report USDA updates its forecasts of the different types of corn in the different states. The yield index of Iowa is an average of all the harvests of this state, and the estimates of USDA published just a short time before expiration will constitute an important piece of information in forming the market's expectations of the final level of the index. One reason for the sharp increase in the volatility may be that the market participants held expectations different from the USDA forecast.

For all the put options, we observed the estimates of the implied volatilities to be quite similar when time to expiration was more than one-half year. For shorter durations, the picture changes so that the put option with the lowest strike gets a higher estimated implied volatility. These are the options furthest from the market's expectations of the yield index level, given the prices the yield futures were traded at shortly before expiration. Knowing that the put prices increase with the volatility, this seems reasonable. We will not observe the usual U-form of the "volatility smile" for yield

futures options, but a smile skewed to the right, since there is no trade in these options having a strike sufficiently high for the put option to be “in the money.”

A paper published by the CBOT (E. Kunda) refers to a similar investigation for options on yield futures with expiration in September 1995, but where a standard Black–Scholes model was used. This model gave an average estimated implied volatility of 28 percent. The historic volatility for the same period was estimated to 13 percent. Also here was observed a significant increase in the estimated implied volatility when USDA presented its yield forecasts.

The historic yield for corn in the period of 1972–1994 for the state of Iowa shows an average value of 112.2 bushels per acre, with an estimated volatility of 18.7 percent (source: USDA).

Finally, we remark that we did not update the yield index y_t daily, but instead based the analysis on USDA monthly forecasts.

Compared to a simple application of the standard Black and Scholes model, our approach seems promising because the difference between the implied volatilities and the historic ones are smaller for our model. Also for this case, however, there seems to be clear indications that, e.g., the parameter σ_y ought to be modeled by, say, a time a varying deterministic function, or perhaps, even a stochastic process.

IMPLICATIONS FOR RISK MANAGEMENT

In the following, we make some remarks regarding risk management. We abstract from production costs, and assume zero local price basis (i.e., local cash price equals futures price) and zero yield basis (i.e., individual farm yield equals index yield). This is to say, we only address market risk, not idiosyncratic risk. We also ignore asymmetric information. Intuitively, one would think that a combined use of yield contracts and futures price contracts ought to provide a “reasonable” strategy for insuring revenue. In Aase (2002) this intuition is made precise. It is shown there that revenue can be approximately hedged by a combined, dynamic use of these two markets. This procedure is *exact* if the correlation between yield and price is zero. Moreover, the relevant strategy is also characterized. This strategy depends only on observable price information in these two separate markets.

More precisely, consider a strategy that holds F_s^y futures contracts on price and F_s^q futures contracts on quantity at any time s , where $0 \leq t \leq s \leq T$, t signifying the present. The resettlement gain from this strategy is given by

$$\int_t^T F_s^y dF_s^q + \int_t^T F_s^q dF_s^y. \quad (39)$$

It can then be shown that this strategy is equivalent to a futures contract on revenue R_T having futures price at each time t given by $F_t^R := E_t^Q\{y(T)q(T)\}$, in the case where the correlation rate between yield and price is zero. In the general case, there is a correction term, which is identified in Aase (2002).

There is an interesting connection to *optimal insurance*. Referring to Example 1 in Appendix B, consider the payoff of the following “sell and hold” strategy that sells F_t^q

quantity contracts, priced at F_t^y at time $t \leq T$, and holds this position until maturity, and sells F_t^y price contracts, priced at F_t^q at time $t \leq T$, and holds this position till maturity as well.

The payoff at expiration for the hypothetical contract on revenue would be $(F_t^R - y_T q_T)$, for an agent selling one such contract. On the other hand, the combined contracts described above would yield the following payoff:

$$(F_t^q - q_T)F_t^y + (F_t^y - y_T)F_t^q, \quad (40)$$

where the first term is the payoff of F_t^y short futures contracts on price q , and the second term is the corresponding payoff of F_t^q short contracts on quantity y .

This latter sum can be seen to be equal to

$$(F_t^R - y_T q_T) + (F_t^y - y_T)(F_t^q - q_T) \quad (41)$$

in the situation where $F_t^R = F_t^y F_t^q$, e.g., when the cross-correlation rate is zero.

Assuming no transaction costs, basis risk and correlation between yield and price, Mahul and Wright (2003) characterize the Pareto optimal indemnity payoff net of the premium for any risk-averse agent, and risk-neutral insurer. It is shown to be $(F_t^R - y_T q_T)$, which is really a consequence of Borch's theorem. This argument requires risk-neutral pricing, which in our model amounts to equating the risk-adjusted probability measure Q and the given one P . As a consequence of this, market prices are determined as $F_t^R = E_t(q_T)E_t(y_T)$, and the optimal revenue insurance has payoff $(E_t(q_T)E_t(y_T) - y_T q_T)$. From the relation in (41) it is seen that this payoff results, but in addition there is a remainder term, the last term in Equation (41). The latter is caused by the *sell and hold* strategy. If the *dynamic* resettlement strategy (39) is used instead, this "correction term" vanishes, and the optimal payoff is exactly achieved.

This demonstrates an interesting connection to optimal insurance coverage, showing that the Pareto optimal net indemnity payoff can be dynamically replicated by using separate yield and price futures contracts. In particular, we here have the rare situation of giving an explicit measure of the gain from being able to trade dynamically (in continuous time), as compared to only being able to sell and hold.

CONCLUSIONS

An economic model is proposed for a joint price and yield futures market. We develop a set of pricing formulas, some of which are partially tested. Compared to a simple application of the standard Black and Scholes model, our approach seems promising because the difference between the implied volatilities and the historic ones are smaller for our model.

The innovation of the article lies in the modeling stage. In order to apply modern financial theory, one has to start with genuine pricing models. The starting point here was, on the other hand, a model for yield and a model for the spot price of corn. A transformation was proposed in order to overcome this difficulty. It was demonstrated,

both theoretically and empirically, that the resulting technique is consistent with financial pricing theory, and also possible to implement in practice.

We rounded off the article with some remarks on risk management.

APPENDIX A

In this section, we present the proofs of Proposition 1 and Theorem 1.

We start with Proposition 1: It follows from the definition of the revenue process R that it has the following dynamic equation under the equivalent martingale measure Q :

$$dR_t = R_t(r - \delta_R)dt + R_t[(\sigma_{y,1} + \sigma_{q,1})d\tilde{B}_1(t) + (\sigma_{y,2} + \sigma_{q,2})d\tilde{B}_2(t)]. \quad (A1)$$

Here $\tilde{B}_1$ and $\tilde{B}_2$ are two independent, standard Brownian motions under the probability measure Q . Thus, we know the distribution of R under Q ; it is lognormal and the random variable $[(\sigma_{y,1} + \sigma_{q,1})\tilde{B}_1(t) + (\sigma_{y,2} + \sigma_{q,2})\tilde{B}_2(t)]$ is normally distributed with mean zero and variance $t\hat{\sigma}^2$, where

$$\hat{\sigma}^2 = (\sigma_{y,1} + \sigma_{q,1})^2 + (\sigma_{y,2} + \sigma_{q,2})^2, \quad (A2)$$

and any computation of market values that depend only on R is in principle straightforward.

Proof of Proposition 1: We have to compute

$$E_t^Q[(p^0 - R_{T_1}e^{(r-\delta_p)(T-T_1)})^+].$$

By substitution in the associated integral to the standard normal distribution, we get

$$\int_{-\infty}^{y_1} \left(p^0 - R_t e^{(r-\delta_p-\frac{1}{2}\hat{\sigma}^2)(T_1-t)+(r-\delta_p)(T-T_1)+\hat{\sigma}x\sqrt{T_1-t}} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx,$$

where y_1 is as given above.

The first integral above simply equals $p\Phi(y_1)$, whereas for the second integral the only difficulty is to compute the term

$$\int_{-\infty}^{y_1} e^{\hat{\sigma}x\sqrt{T_1-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = e^{\frac{1}{2}\hat{\sigma}^2(T_1-t)} \int_{-\infty}^{y_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = e^{\frac{1}{2}\hat{\sigma}^2(T_1-t)} \Phi(y_2),$$

where $y_2 = y_1 - \hat{\sigma}\sqrt{T_1-t}$. Here, we have made a full square in the exponent in order to transform to the standard normal distribution. Putting this together with the above expression, gives the result of the proposition. Q.E.D.

Proof of Theorem 1: We have to compute

$$F_t^{(k-y)^+q} = E_t^Q[(k - y_T)^+ q_T] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(k - y_t e^{(\mu_y^Q - \frac{1}{2}\sigma_y^2)(T-t) + \sigma_y u \sqrt{T-t}} \right)^+ \cdot q_t e^{(r - \delta_q - \frac{1}{2}\sigma_q^2)(T-t) + \sigma_q v \sqrt{T-t}} f(u, v) du dv,$$

where $f(u, v)$ is the joint probability density of two random variables U and V which are binormally distributed with zero means, unit variances and correlation coefficient equal to ρ given in Equation (8). First, we get rid of the $+$ in the function $(\cdot)^+$, and the above futures price equals

$$F_t^{(k-y)^+q} = \int_{-\infty}^{\infty} q_t e^{(r - \delta_q - \frac{1}{2}\sigma_q^2)(T-t) + \sigma_q v \sqrt{T-t}} \times \left(\int_{-\infty}^c \left(k - y_t e^{(\mu_y^Q - \frac{1}{2}\sigma_y^2)(T-t) + \sigma_y u \sqrt{T-t}} \right) f(u, v) du \right) dv, \quad (A3)$$

where

$$c = \frac{\ln\left(\frac{k}{y_t}\right) - \left(\delta_q - \delta_p - \sigma_{y,q} - \frac{1}{2}\sigma_y^2\right)(T-t)}{\sigma_y \sqrt{T-t}}. \quad (A4)$$

The above expression for the futures price is a sum of two integrals, and we start with the first one, which can be written as

$$k q_t e^{(r - \delta_q - \frac{1}{2}\sigma_q^2)(T-t)} \int_{-\infty}^{\infty} e^{\sigma_q v \sqrt{T-t}} \int_{-\infty}^c f(u, v) du dv. \quad (A5)$$

Let us concentrate on the integral, abstracting from the multiplying constant. By Fubini's theorem we get,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\sigma_q v \sqrt{T-t}} \int_{-\infty}^c f(u, v) du dv &= \int_{-\infty}^c du \int_{-\infty}^{\infty} e^{\sigma_q v \sqrt{T-t}} f(u, v) dv \\ &= \int_{-\infty}^c E\left(e^{\sigma_q V \sqrt{T-t}} \mid U = u\right) f_U(u) du, \end{aligned}$$

where $f_U(u)$ is the probability density of U , a standard normal variate having mean zero and variance one. The conditional distribution of V given $U = u$ is again normal with mean ρu and variance $(1 - \rho^2)$, and using again the well-known expression for the moment generating function of a normal variate, we get that the above integral equals

$$\begin{aligned}
& e^{\frac{1}{2}\sigma_q^2(1-\rho^2)(T-t)} \int_{-\infty}^c \frac{1}{\sqrt{2\pi}} e^{\rho\sigma_q u \sqrt{T-t} - \frac{1}{2}u^2} du \\
&= e^{\frac{1}{2}\sigma_q^2(T-t)} \int_{-\infty}^c \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-\sigma_q\rho\sqrt{T-t})^2} du \\
&= e^{\frac{1}{2}\sigma_q^2(T-t)} \int_{-\infty}^{c-\sigma_q\rho\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = e^{\frac{1}{2}\sigma_q^2(T-t)} \Phi(c - \sigma_q\rho\sqrt{T-t}).
\end{aligned}$$

Here, we have made a full square in the exponent inside the integral, and used the substitution $x = u - \sigma_q\rho\sqrt{T-t}$. Picking up the constant multiplier of the integral from the expression in (A5), we have that the first integral equals

$$kq_t e^{(r-\delta_q - \frac{1}{2}\sigma_q^2)(T-t) + \frac{1}{2}\sigma_q^2(T-t)} \Phi(d_1) = kq_t e^{(r-\delta_q)(T-t)} \Phi(d_1), \quad (\text{A6})$$

where the expression for d_1 , given in (31), follows from (A4) and the above substitution.

We now turn to the last term in the expression (A3). It is the negative of the following:

$$R_t e^{(r + \mu_y^Q - \delta_q - \frac{1}{2}(\sigma_q^2 + \sigma_y^2))(T-t)} \int_{-\infty}^{\infty} \int_{-\infty}^c e^{\sigma_q v \sqrt{T-t} + \sigma_y u \sqrt{T-t}} f(u, v) du dv. \quad (\text{A7})$$

Considering again just the integral, by Fubini's theorem we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^c e^{\sigma_q v \sqrt{T-t}} e^{\sigma_y u \sqrt{T-t}} f(u, v) du dv \\
&= \int_{-\infty}^c e^{\sigma_y u \sqrt{T-t}} \left(\int_{-\infty}^{\infty} e^{\sigma_q v \sqrt{T-t}} f(v | u) dv \right) f_U(u) du \\
&= e^{\frac{1}{2}\sigma_q^2(1-\rho^2)(T-t)} \int_{-\infty}^c e^{(\rho\sigma_q + \sigma_y)u \sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \\
&= e^{\frac{1}{2}(\sigma_q^2 + 2\rho\sigma_q\sigma_y + \sigma_y^2)(T-t)} \int_{-\infty}^{c-(\rho\sigma_q + \sigma_y)\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\
&= e^{\frac{1}{2}(\sigma_q^2 + 2\rho\sigma_q\sigma_y + \sigma_y^2)(T-t)} \Phi(d_2),
\end{aligned}$$

where $d_2 := (c - (\rho\sigma_q + \sigma_y)\sqrt{T-t})$ is given in Equation (32), and where $f(v | u)$ is the probability density of the conditional distribution of V given $U = u$. Returning to Equation (A7), we see that the last term in (A3) can be written

$$-R_t e^{(r + \mu_y^Q - \delta_q - \frac{1}{2}(\sigma_q^2 + \sigma_y^2))(T-t) + \frac{1}{2}(\sigma_q^2 + 2\rho\sigma_q\sigma_y + \sigma_y^2)(T-t)} \Phi(d_2) = -R_t e^{(r-\delta_R)(T-t)} \Phi(d_2).$$

Adding this term to the first integral given in Equation (A6), we obtain the conclusion of the theorem. Q.E.D.

APPENDIX B: CYI FUTURES CONTRACTS

This article presents a model of two combined futures markets, a quantity market and a price market. The mechanics of using yield futures can best be illustrated by an example.

Example 1: Consider a farm of 1,000 acres in Iowa, in an area with expected crop $F_t^y = 130$ bushels per acre at time t . The futures price of corn is $F_t^q = \$2.50$ per bushel, also at time t , in both cases for contracts expiring at a future time T .

Consider a strategy that sells 130,000 corn futures and similarly sells 2,500 area yield futures, both at time t and these positions are held until maturity. The payoff at expiration for this strategy would be

$$(F_t^q - q_T^{\text{obs}})F_t^y \cdot 1,000 + (F_t^y - y_T^{\text{obs}})F_t^q \cdot 1,000.$$

Consider four scenarios:

- (i) The observed price of corn at time T turns out to be $q_T^{\text{obs}} = \$2$ per bushel, the observed yield index y_T^{obs} ended up on 100 bushels per acre. This is the case of situation the farmer would like to insure against. The payoff from this strategy would be \$140,000. Without futures contracts, the farmers would end up \$125,000 below the expectation, assuming a perfect correlation between the farm output and the yield index, and after the gain from the futures contracts are taken into consideration, the “net gain” would be \$15,000.
- (ii) $q_T^{\text{obs}} = \$3$, $y_T^{\text{obs}} = 160$ bushels per acre. The payoff from the above strategy would be $-\$140,000$. Under the same simplifying assumptions as above, the farm would now end up with a result of \$155,000 higher than projected, in which case the “net gain” would also be \$15,000.
- (iii) $q_T^{\text{obs}} = \$2$, $y_T^{\text{obs}} = 160$ bushels per acre. The payoff from the above strategy would be $-\$10,000$. Under the same simplifying assumptions as above, the farm would now end up with a result of \$5,000 below the expectation, in which case the “net loss” would be \$15,000.
- (iv) $q_T^{\text{obs}} = \$3$, $y_T^{\text{obs}} = 100$ bushels per acre. The payoff from the above strategy would be \$10,000. Under the same simplifying assumptions as above, the farm would now end up with a result of \$25,000 below the expectation, in which case the “net loss” would also be \$15,000.

If these four cases were equally likely, the expected “net gain” would equal zero, so on average the insurance would then work.

When considering pure area yield contracts, one should notice that for yield futures contracts is used a multiplication factor of \$100 per bushel to convert production to income (here: in US\$), and also, the trading unit for corn futures is 5,000 bushels.

Here, we remark that it may be shown that the expression

$$\begin{aligned} & (F_t^q - q_T^{\text{obs}})F_t^y \cdot 1,000 + (F_t^y - y_T^{\text{obs}})F_t^q \cdot 1,000 \\ &= (F_t^R - y_T^{\text{obs}}q_T^{\text{obs}}) \cdot 1,000 + (F_t^y - y_T^{\text{obs}})(F_t^q - q_T^{\text{obs}}) \cdot 1,000 \end{aligned}$$

in the situation where the cross correlation $\sigma_{y,q} = 0$ (see, e.g., Aase, 2002). Thus the latter term in the above equation can be considered as a “second-order correction” to the first term in this particular situation, which is a better way to interpret the “net gains” and the “net losses” in the above example.

In Appendix C, we have relegated further specifications for three types of contracts considered in the article.

APPENDIX C: CONTRACT SPECIFICATIONS FOR CONTRACTS CONSIDERED IN THE ARTICLE

Corn Futures

Trading unit: 5,000 bu. Price quotations: cent and quarter per bushel. “Tick size”: 1/4 cent per bushels (\$12.50 per contract). Contract months: December, March, May, July, September. Last trading day: seven days before last day of trade in month of delivery, for contracts with date of delivery in March 2000 and later: trading day before 15th in contract month. Trading time: 9:30 to 13:15 Chicago time, Monday to Friday. Ticker symbol: C

Corn-Yield Futures

Underlying asset: Official forecast from USDA for each relevant state and for the entire USA throughout the corn season. Trading unit: Corn yield estimate multiplied by \$100. “Tick size”: 1/10 bushels per acre (\$10 per contract). Contract months: September, October, November, January. Last trading day: Last business day in the month before the release of the USDA forecasts of corn for the relevant states and for all of USA. Trading time: 10:30 to 12:45 Chicago time, Monday to Friday. Ticker symbol: CA (Iowa).

Corn-Yield Futures Options

Trading unit: One CBOT^R corn-yield insurance futures for a specific corn producing area (Iowa, Illinois, Indiana, Ohio, and all of USA) for a specific contract month. “Tick size”: 1/10 bushels per acre (\$10 per contract). Strike-yield: Intervals of 5 bushels per acre (for strike yield closest to the previous day’s settlement yield and the next 20 successive higher and lower strike yields). Strike yields will also be noted from 20 to 200 in 10 bushels per acre units over and under the 5 bushels strike interval. Contract months: September, October, November, January. Last trading day: Last business day in the month before the release of the USDA forecasts for the relevant states and for all of USA. Trading time: 10:30 to 12:45 Chicago time, Monday to Friday. Ticker symbol: CAC (Iowa Crop Yield Calls), CAP (Iowa Crop Yield Puts).

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