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The Demand for Insurance With an Upper Limit on Coverage

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Abstract

The demand for insurance is examined when the indemnity schedule is subject to an upper limit. The optimal contract is shown to display full insurance above a deductible up to the cap. Some results derived in the standard model with no upper limit on coverage turn out to be invalid; the optimal deductible of an actuarially fair policy is positive and insurance may be a normal good under decreasing absolute risk aversion. An increase in the upper limit would induce the policyholder with constant absolute risk aversion to reduce his or her optimal deductible and therefore this would increase the demand for insurance against small losses.

Introduction

A long-established result in the insurance literature due to Arrow (1971) is that, for a premium depending only on the policy's actuarial value, a risk-averse agent would prefer full insurance above a deductible. The optimal deductible is zero if insurance is sold at an actuarially fair price, i.e., the insurance premium equals the expected value of the loss, and it is positive otherwise. Such a result implicitly assumes that insurance contracts display no limit on coverage; indemnity is paid in full whatever may be the amount of the realized loss (above the deductible).

However, real-world insurance and reinsurance markets typically include limitations on coverage. Many insurance contracts are structured such that the insurer's payoff is determined as a nonlinear function of the policyholder's loss. For example, an insurer typically agrees to pay $I(x) = \max(x - D, 0) - \max(x - (D + \bar{I}), 0)$, where x is the policyholder's loss, $I(x)$ is the insurer's payment to the policyholder, D is the deductible, and $(D + \bar{I})$ is the policy coverage limit. Such contracts are common even in lines of insurance subject to a high degree of diversification such as personal automobile

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liability insurance. Nonlinear indemnity schedules with upper limits also characterize nonproportional reinsurance treaties, such as those protecting primary insurers against liability claims and catastrophic property losses. Securitized risk transfer instruments, such as CAT bonds, have a similar payoff structure (Cummins, Lalonde, and Phillips, 2002).

Limited coverage on insurance and reinsurance contracts might be justified, among other things, by the limited financial capacity of the insurance and reinsurance industry in comparison with the huge losses that could be generated by events such as liability lawsuits and natural catastrophes. For example, a $75 billion catastrophe represents about one-fifth of the capital of the insurance and reinsurance industry (see Froot, 1999, for additional explanations for the paucity of catastrophe risk sharing). The rationale for upper limits on coverage might also stem from the existence of regulatory constraints (Raviv, 1979), the policyholder's option of bankruptcy (Huberman, Mayers, and Smith, 1983), or the manipulation of audit cost with an infinitely risk-averse auditor (Picard, 2000).

The purpose of this article is to investigate the demand for insurance when the coverage is constrained by an upper bound. Limitations on coverage create market incompleteness; very high losses cannot be insured because of the absence of corresponding markets. The article in the prior literature most similar to ours is Doherty and Schlesinger (D&S) (1990). D&S (1990) analyzed the insurance-purchasing decision in a three-state model where contracts are subject to an exogenous risk of total default. The presence of a risk of nonperformance or of upper limits on coverage creates wealth states that are worse when the agent purchases insurance than when he or she is uninsured. However, it is noteworthy that an upper limit on coverage is not a special case of nonperformance, as defined by D&S (1990), because partial payments are in accordance with this contract.

The assumptions and the model are specified in the next section. In the third section, the design of an optimal insurance contract with an upper limit on coverage is derived. It is shown to display full marginal coverage above the deductible up to the cap. The fourth section deals with the optimal level of deductibility. It is found to be positive when the contract is sold at an actuarially fair price. The intuition behind this novel result is that raising the deductible reduces the premium and thus increases the amount of wealth available in the states of the world when the policy limit constraint is binding. Thus, by increasing the deductible, buyers are shown to shift funds from relatively low loss states to high loss states. Changes in risk aversion and in initial wealth are shown to have ambiguous effects on the optimal deductible. Insurance may thus be a normal good under decreasing absolute risk aversion (DARA). If the policyholder's preferences exhibit constant absolute risk aversion, an increase in the upper limit would increase the demand for insurance against small losses through a decrease in the deductible. The final section provides a summary and concluding remarks.

THE SETTING

Consider a risk-averse individual with preferences represented by the twice continuously differentiable von Neumann–Morgenstern utility function u, with $u' > 0$ and

$u'' < 0$. The individual is endowed with a nonrandom initial wealth w_0 and a random loss $\tilde{x}$, where the distribution of $\tilde{x}$ is continuous with support $[0, \bar{x}]$, $\bar{x} \leq w_0$.[1,2]

The insurance policy is characterized by the premium P paid by the insured and by a coverage function $I(x)$ specifying the reimbursement of the insurance company for each possible loss x. The coverage function is assumed to be nonnegative as follows:

$$I(x) \geq 0 \quad \text{for all } x. \tag{1}$$

Also it is often required that the indemnity is less than the loss in order to prevent moral hazard problems. It will be shown that this condition is always satisfied in our setting. The original contribution of the current model is that the indemnity function is constrained to be less than an upper limit $\bar{I} > 0$ as follows:

$$I(x) \leq \bar{I} \quad \text{for all } x. \tag{2}$$

The cap $\bar{I}$ is assumed to be exogenous and lower than the maximum loss $\bar{x}$, consistent with what is typically observed in real-world insurance markets. Otherwise, the constraint (2) would never be binding and standard results implicitly derived with no limitations on coverage apply. This upper limit on coverage provides the insurance company with limited liability with respect to this indemnity schedule. It can be viewed as a special case of partial payment; valid claims are paid in full up to the cap $\bar{I}$ and then the indemnity payment is constant and independent of the realized loss.

Finally, the insurance premium is assumed to be proportional to the policy's actuarial value as follows:

$$P = mEI(\tilde{x}), \tag{3}$$

where the loading factor m, with $m \geq 1$, is assumed to be sufficiently small so that the individual purchases insurance. This tariff is sustained by a competitive insurance market with risk-neutral insurers and transaction costs that are proportional to claims.

OPTIMAL INSURANCE DESIGN

An optimal insurance contract maximizes the insured's expected utility of final wealth subject to the above-mentioned constraints:

$$\begin{cases} \underset{I(.),P}{\text{Max}}\ Eu(w_0 - \tilde{x} + I(\tilde{x}) - P) \\ \text{subject to conditions (1), (2), and (3).} \end{cases} \tag{4}$$

The following proposition characterizes the optimal indemnity schedule, solution to problem (4), when the premium is fixed.

[1] We could assume that the random variable is distributed according to a mixture of discrete and continuous probability distributions. However, this would complicate the exposition without adding any content.

[2] Tildes are used to denote random variables, and the same variables without the tilde denote realizations of random variables.

Proposition 1: *The optimal insurance contract, solution to problem* (4), *provides full marginal coverage above a deductible* $D \in [0, \bar{I}]$ *up to the limit* $\bar{I}$:

$$I^*(x) = \min[\max(x - D, 0), \bar{I}] = \max(x - D, 0) - \max(x - (\bar{I} + D), 0).$$

The proof of this proposition is shown in the Appendix.

The optimal insurance schedule based on the solution of the maximization problem in (4) contains three segments in the (x, I) graph: a first horizontal line, $I^*(x) = 0$, where the nonnegativity constraint is binding, a line whose slope is equal to the unity, $I^*(x) = x - D$, where both constraints are not binding, and a second horizontal line, $I^*(x) = \bar{I}$, where the constraint on the upper limit on coverage is binding. Therefore, the optimality of full insurance above a deductible shown by Arrow (1971) in a model with no limit on coverage is robust to the introduction of limited coverage (when the realized loss is lower than $D + \bar{I}$).

It can be shown that this result holds under the assumptions of risk aversion, defined as an aversion to mean-preserving spreads, and of preferences for higher levels of wealth, including the expected utility framework as a special case. The proof, in the line of Gollier and Schlesinger (1996) and Gollier (2000), consists in showing that the final wealth under the optimal insurance policy $[I^*(\cdot), P]$ is a mean-preserving contraction in risk of the final wealth generated by any other insurance contracts. Once an indemnity, positive and lower than the upper limit, is paid, any marginal increase in the loss is fully indemnified.

OPTIMAL LEVEL OF DEDUCTIBILITY

In this section, the choice of the insurance deductible is examined. Following the purchase of the insurance contract expressed in Proposition 1, the insured's expected utility of final wealth is given by

$$\begin{aligned} U(D) &= Eu(w_0 - \tilde{x} + \min[\max(\tilde{x} - D, 0), \bar{I}] - P) \\ &= \int_0^D u(w_1)\, dF(x) + [F(D + \bar{I}) - F(D)]u(w_2) + \int_{D+\bar{I}}^{\bar{x}} u(w_3)\, dF(x), \end{aligned} \tag{5}$$

where $w_1 = w_0 - x - P$, $w_2 = w_0 - D - P$, $w_3 = w_0 - x - P + \bar{I}$, $F(x)$ is the cumulative distribution function (CDF) of the random loss $\tilde{x}$ and the premium is $P = m\{\int_D^{D+\bar{I}} (x - D)\, dF(x) + \bar{I}[1 - F(D + \bar{I})]\}$. The first-order condition for a choice of D^* that maximizes Equation (5) is

$$\begin{aligned} U'_D(D) = \frac{dEu}{dD} &= -P'_D \int_0^D u'(w_1)\, dF(x) - \left(1 + P'_D\right)[F(D + \bar{I}) - F(D)]u'(w_2) \\ &- P'_D \int_{D+\bar{I}}^{\bar{x}} u'(w_3)\, dF(x) = 0, \end{aligned} \tag{6}$$

where $P'_D = \partial P / \partial D = -m[F(D + \bar{I}) - F(D)] < 0$. Using the method provided by Meyer and Ormiston (1999), we show in the Appendix that the second-order condition of this maximization problem holds.

Evaluating $U'_D(D)$ at $D = 0$ yields

$$U'_D(D)|_{D=0} = -(1 - mF(\bar{I}))F(\bar{I})u'(w_0 - P) + mF(\bar{I})\int_{\bar{I}}^{\bar{x}} u'(w_3)\,dF(x) > (m-1)F(\bar{I})u'(w_0 - P). \quad (7)$$

The strict inequality in Equation (7) comes from the fact that $w_3 < w_2 \equiv w_0 - P$ for all $x > \bar{I}$ and therefore, $\int_{\bar{I}}^{\bar{x}} u'(w_3)\,dF(x) > [1 - F(\bar{I})]u'(w_0 - P)$ under risk aversion. This implies that $U'_D(D)|_{D=0} > 0$ for all $m \geq 1$. Consequently, the optimal level of deductibility is positive under fair insurance, $m = 1$. Equation (7) shows that a marginal increase of the deductible from zero will increase the "low-wealth" state w_3 and decrease the "intermediate-wealth" state w_2 from a marginal decrease in the fair premium. This thus generates a mean-preserving contraction in risk, in the Rothschild–Stiglitz (1970) sense, that is preferred by every risk-averse agent.

Evaluating $U'_D(D)$ at $D = \bar{x} - \bar{I}$ under fair insurance, $m = 1$, yields

$$U'_D(D)|_{D=\bar{x}-\bar{I}} = (1 - F(\bar{x} - \bar{I}))\left[\int_0^{\bar{x}-\bar{I}} u'(w_1)\,dF(x) - F(\bar{x} - \bar{I})u'(w_0 - \bar{x} + \bar{I} - P)\right]. \quad (8)$$

Because $w_1 > w_0 - \bar{x} + \bar{I} - P$ for all $x < \bar{x} - \bar{I}$, $\bar{I} < \bar{x}$, and $u'' < 0$, this yields $U'_D(D)|_{D=\bar{x}-\bar{I}} < 0$. The optimal deductible is thus lower than $(\bar{x} - \bar{I})$ under fair insurance. This means that the upper limit constraint is always binding. This discussion is summarized in the following proposition.

Proposition 2: *If the insurance contract subject to an upper limit on coverage, with $\bar{I} < \bar{x}$, is sold at a fair price, $m = 1$, then the optimal deductible satisfies $0 < D^* < \bar{x} - \bar{I}$.*

The optimality of full insurance under actuarially fair prices is a well-known result when there is no upper limit on coverage. However, full coverage is not possible when the indemnity schedule is constrained to be lower than a cap; the policyholder fully bears any marginal increase in the loss above this upper limit. Selecting a positive deductible allows him or her to shift wealth from the intermediate-wealth state w_2 to the low-wealth state w_3 through a decrease in the premium and therefore to increase his or her wealth in states of nature where the marginal utility is the highest.

Proposition 2 provides another exception to the claim that a risk-averse individual will purchase full insurance given actuarially fair premiums (Mossin, 1968). The optimality of partial coverage under fair insurance was previously mentioned by D&S (1990) in a three-state model in the presence of an exogenous risk of total default. Proposition 2 shows that partial insurance holds in an alternative framework where the indemnity payoff is subject to an upper limit. In addition, partial coverage is shown to be caused by full (marginal) coverage above a positive deductible.

Change in Initial Wealth and in Risk Aversion

In the standard insurance model with no limit on coverage, a higher degree of Arrow–Pratt absolute risk aversion leads to the purchase of more insurance coverage (sold at an unfair price), i.e., to a lower deductible (Schlesinger, 1981). A direct consequence is that, under DARA, an increase in initial nonrandom wealth will induce the policyholder to increase his or her deductible; insurance is an inferior good. These results are reexamined when the insurance policy is subject to an upper limit on coverage.

To determine the effect on the optimal deductible of a change in the initial wealth, it is sufficient to determine the sign of $U''_{Dw_0}(D^*) = d^2Eu/dDdw_0$, because the objective function has been shown to be concave in the level of deductibility at local optima. This yields

$$U''_{Dw_0}(D^*) = P'_D \int_0^{D^*} A(w_1)u'(w_1)\,dF(x) + \left(1 + P'_D\right)[F(D^* + \bar{I}) - F(D^*)]A(w_2)u'(w_2) + P'_D \int_{D^*+\bar{I}}^{\bar{x}} A(w_3)u'(w_3)\,dF(x), \tag{9}$$

where $A(w) = -u''(w)/u'(w)$ is the Arrow–Pratt index of absolute risk aversion. Observe that for all $x < D^*$ we have $w_1 > w_2$ and thus $A(w_1) < (=, >)A(w_2)$ under decreasing (constant, increasing) absolute risk aversion, and that for all $x > D^* + \bar{I}$ we have $w_3 < w_2$ and thus $A(w_3) > (=, <)A(w_2)$ under decreasing (constant, increasing) absolute risk aversion. From the first-order necessary condition (6), this implies that a marginal change in w_0 on D^*, i.e., the wealth effect, is null under constant absolute risk aversion (CARA) and is indeterminate under decreasing or increasing absolute risk aversion. As a consequence, insurance may be a normal good under DARA; an increase in initial wealth may induce the policyholder to increase his or her coverage against small losses by reducing his or her deductible.

A higher degree of absolute risk aversion has an indeterminate impact on the optimal deductible because an increase in nonrandom initial wealth under DARA is a special case of a reduction in risk aversion. This is the consequence of two counteracting effects. On one hand, the policyholder more highly values a lower deductible in order to have a better coverage against intermediate losses. On the other hand, he or she more highly values a higher deductible that generates a lower premium and thus creates "savings" to protect him or her against high losses. In particular, a more risk-averse policyholder may be induced to select a higher level of deductibility.

The indeterminate effect of risk aversion was shown by Schlesinger and Schulenburg (1987) about insurance purchasing under an exogenous risk of total default and by Briys and Schlesinger (1990) about self-protection activity. In the spirit of Briys and Schlesinger's analysis, the break in the link between risk aversion and the deductible can be intuitively explained as follows. Consider a reduction in the deductible D when the insurance contract is sold at a fair price. This corresponds to a mean-preserving change in risk. Lowering the deductible would decrease the "high-wealth" state w_1 through an increase in the premium, would increase the intermediate-wealth state w_2 through an increase in the indemnity net of the premium, but it would also decrease the low-wealth state w_3 through a higher premium. Rather than improving the low-wealth state w_3, such a decrease in the deductible makes it even worse. Therefore, a decrease in the deductible is neither a mean-preserving spread nor a mean-preserving contraction in the Rothschild–Stiglitz (1970) sense. Risk-averse policyholders may thus either reject or accept a reduction in the deductible.

TABLE 1
Optimal Level of Deductibility and Risk Aversion

CRRA Coefficient	Deductible With Fair Insurance $m = 1$		Deductible With Unfair Insurance $m = 1.1$	
β	$\bar{I} = 1.0$	$\bar{I} = 0.5$	$\bar{I} = 1.0$	$\bar{I} = 0.5$
0.5	0	0.269	0.433	0.442
1.0	0	0.281	0.307	0.369
1.5	0	0.293	0.251	0.351
2.0	0	0.304	0.221	0.348
2.5	0	0.315	0.200	0.350
3.0	0	0.325	0.186	0.354
3.5	0	0.335	0.174	0.361
4.0	0	0.344	0.164	0.366
4.5	0	0.352	0.157	0.372
5.0	0	0.359	0.150	0.377

Note: The table assumes that a risk-averse insurance buyer is choosing a deductible for a policy characterized by a cap $\bar{I}$ and a loading factor m. The buyer's utility function is $u(w) = w^{1-\beta}/(1-\beta)$ for $\beta \neq 1$ and $u(w = \ln(w)$ for $\beta = 1$. The table shows the optimal deductible for various values of β, $\bar{I}$, and m.

The existence of a positive relationship between risk aversion and the optimal deductible is illustrated using a numerical example. Consider an individual with initial wealth $w_0 = 1$ facing a random loss that follows a uniform distribution $U_{[0,1]}$. His or her preferences exhibit constant relative risk aversion, i.e., $u(w) = w^{1-\beta}/(1-\beta)$ with $\beta \neq 1$ and $u(w) = \ln(w)$ when $\beta = 1$. In our example, β varies from 0.5 to 5.0, with 0.5 increments.

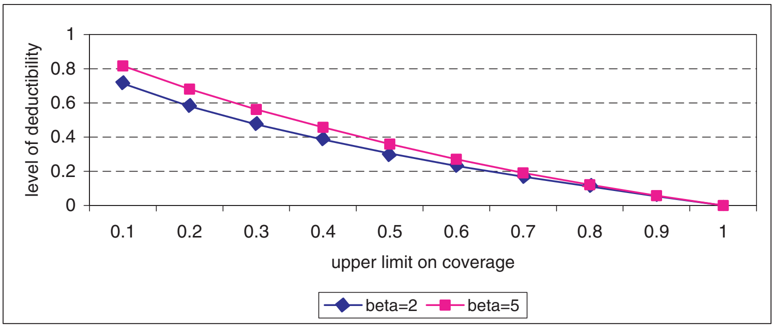
Table 1 presents the optimal level of deductibility when the loading factor is equal to either 1.0 (fair insurance) or 1.1 (costly insurance) and when the insurance policy has either no upper limit on coverage, $\bar{I} = 1.0$, or limited coverage with $\bar{I} = 0.5$. Under fair insurance and no upper limit on coverage, full insurance ($D = 0$) is obviously optimal. Under limited coverage, the optimal deductible of the fair insurance policy increases with risk aversion. Observe that introducing an upper limit on coverage has a significant effect on the optimal deductible whatever the degree of risk aversion. Under unfair insurance and no upper limit on coverage ($\bar{I} = 1.0$), the more risk averse the policyholder, the lower the deductible, as predicted by the standard model. When coverage is limited ($\bar{I} = 0.5$), the optimal deductible first decreases to a minimum of 0.348 at $\beta = 2.0$ and increases afterward as the index of relative risk aversion increases.

Increase in the Upper Limit on Coverage

In this section, the impact of a change in the cap on the optimal deductible is analyzed. From the theorem of implicit functions, one can show (see the Appendix) that

$$\frac{dD^*}{d\bar{I}} = \frac{-U''_{D\bar{I}}}{U''_{DD}} = \frac{1}{U''_{DD}} P'_D \int_{D^*+\bar{I}}^{\bar{x}} u''(w_0 - x + \bar{I} - P)\, dF(x) - P'_{\bar{I}} \frac{dD^*}{dw_0}, \tag{10}$$

FIGURE 1
Optimal Level of Deductibility and Upper Limits on Coverage



Note: The figure assumes that a risk-averse insurance buyer is choosing a deductible for an actuarially fair policy characterized by a cap $\bar{I}$. The buyer's utility function is $u(w) = w^{1-\beta}/(1-\beta)$ for $\beta \neq 1$ and $u(w) = \ln(w)$ for $\beta = 1$. The figure shows the optimal deductible for various values of $\bar{I}$ when the buyer is moderately risk averse ($\beta = 2$) and highly risk averse ($\beta = 5$).

where $P'_{\bar{I}} = \partial P/\partial \bar{I} > 0$. An increase in the upper limit on coverage $\bar{I}$ has two effects on the optimal level of deductibility. The first right-hand side (RHS) term in Equation (10) is the direct effect. It is negative; an increase in $\bar{I}$ increases the low-wealth state w_3 and thus it induces the policyholder to choose a lower deductible. The second RHS term is the wealth effect. It is indeterminate under DARA. Therefore, increasing $\bar{I}$ may induce the policyholder to choose a higher deductible if this wealth effect is negative, i.e., insurance is a normal good, and stronger than the direct effect. Under CARA, this wealth effect vanishes and the policyholder would respond to an increase in $\bar{I}$ by reducing his or her deductible, i.e., by increasing his or her coverage against small losses.

Using the previous numerical illustration, Figure 1 shows the optimal fair insurance deductible for various upper limits on coverage when the policyholder is moderately risk averse, $\beta = 2$, or highly risk averse, $\beta = 5$. The optimal level of deductibility turns out to be decreasing in the cap. In our illustrative example, this means that either the wealth effect is positive or the direct effect dominates the negative wealth effect.

If the existence of an upper limit on coverage is caused by the limited financial capacity of the insurance and reinsurance industry, an increase in the financial capacity would allow insurers to increase the upper limit and thus this would increase the demand for insurance against small losses. Therefore, providing a better coverage for high losses would induce the policyholder with exponential utility function to select a better coverage for small losses.

THE UPPER LIMIT AS A DECISION VARIABLE

The upper limit on coverage $\bar{I}$ has been assumed to be unique and exogenous. In real-world insurance and reinsurance markets, it is usually selected from a menu of caps. Because the optimal insurance contract with no limit on coverage, when the premium is only based on the expected indemnity, displays full marginal coverage above a deductible (Arrow, 1971; Raviv, 1979), it follows that the risk-averse policyholder will choose the highest cap from this menu, unless of course the maximum possible loss is lower than this highest cap.

When the premium does not depend only on the actuarial value of the insurance policy, the optimal cap within the insurance policy providing full marginal coverage above the deductible may be lower than the highest loss. As an example, consider that the insurance premium is proportional to the expected indemnity and that the loading factor increases with the upper limit on coverage; $P = m(\bar{I})EI(\tilde{x})$, with $m \geq 1$ and $m' \geq 0$. It can be easily shown that the optimal indemnity schedule is $I^*(x) = \min[\max(x - D, 0), \bar{I}]$ under this specific form of insurance premium. Differentiating the policyholder's objective function expressed in Equation (5) with respect to $\bar{I}$ and evaluating it at $\bar{I} = \bar{x} - D$ yields

$$U'_{\bar{I}}|_{\bar{I}=\bar{x}-D} = -P'_{\bar{I}}\left[\int_0^D u'(w_1)\,dF(x) + (1 - F(D + \bar{I}))u'(w_2)\right], \tag{11}$$

where $P'_{\bar{I}}|_{\bar{I}=\bar{x}-D} = (>)0$ if $m' = (>)0$. Because the term in the brackets in Equation (11) is positive, $U'_{\bar{I}}|_{\bar{I}=\bar{x}-D}$ is equal to zero (positive) if the marginal loading factor is zero (positive). Assuming the second-order conditions are satisfied, the optimal cap is thus lower than $(\bar{x} - D)$, i.e., highest losses are not insured, as long as $m' > 0$.

CONCLUSION

This article has examined the demand for insurance in a specific case of incomplete markets; the indemnity payment is subject to an upper limit. This particular case of partial coverage is shown not to alter the insurance contract design; full insurance above a deductible (up to the cap) is optimal. Nevertheless, three standard results derived in the model with no limit on coverage fail. First, the optimal deductible is positive even though the insurance policy is sold at a fair price. Second, a more risk-averse policyholder may choose a higher deductible. This is illustrated with numerical simulations. Third, insurance may be a normal good under DARA because the wealth effect is indeterminate. Under constant absolute risk aversion, an increase in the upper limit would increase the demand for insurance against small losses through a lower deductible.

Some of these results, e.g., the optimality of partial insurance under fair pricing, were pointed out by D&S (1990) in a three-state model with an exogenous risk of total default. This is due to the fact that the existence of a default risk in their model and the presence of an upper limit on coverage in our model create wealth states in which the policyholder cannot be fully insured and in which an increase in the insurance premium makes them worse. We have also shown that optimal partial coverage stems from the optimality of full (marginal) coverage above a deductible.

This insurance model with limited coverage seems to be appropriate to investigate the reinsurance strategy with existing layers of reinsurance and options spreads. This analysis can be generalized to models with option spread-like payoff where basis risk should be introduced to take into account the imperfect correlation between individual and industry-wide losses. Further research would also investigate the demand for insurance when the layer depends on the size of the deductible and the optimal level of the cap when the premium does not depend only on the actuarial value of the insurance policy.

APPENDIX

Proof of Proposition 1: The Euler equation simplifies to a succession of pointwise first-order conditions for $I(\cdot)$ because $I'(x)$ appears neither in the objective nor in the constraints. Therefore, Equation (4) can be solved using Kuhn-Tucker conditions for $I(x)$ for all x. The Lagrangian is

$$L = \{u(w_0 - x + I(x) - P) + \lambda(x)I(x) + \Phi(x)[\bar{I} - I(x)]\}f(x) + \mu[P - mEI(\tilde{x})], \quad \text{(A1)}$$

where $f(\cdot)$ is the density function of the random loss $\tilde{x}$, and μ, $\lambda(x)$, and $\Phi(x)$ are the multipliers associated respectively to constraint (3), (1), and (2) with

$$\lambda(x) \begin{cases} =0 & \text{if } I(x) > 0 \\ \geq 0 & \text{otherwise,} \end{cases} \quad \text{(A2)}$$

$$\Phi(x) \begin{cases} =0 & \text{if } I(x) < \bar{I} \\ \geq 0 & \text{otherwise.} \end{cases} \quad \text{(A3)}$$

The first-order necessary condition with respect to the indemnity function $I(x)$ is

$$u'(w_0 - x + I(x) - P) + \lambda(x) - \Phi(x) - \mu m = 0 \quad \text{for all } x. \quad \text{(A4)}$$

For all $x : 0 < I(x) < \bar{I}$, Equation (A4) becomes

$$u'(w_0 - x + I(x) - P) = \mu m. \quad \text{(A5)}$$

In every state of the world where indemnity payments are made, the marginal utility of the policyholder must be constant. This implies that $I'^{*}(x) = 1$ in those states. From the concavity of the insured's utility function and constraints (1) and (3), there exists a deductible $D \geq 0$ such that the optimal insurance policy is

$$I^*(x) = \begin{cases} 0 & \text{if } x \leq D \\ x - D & \text{if } D \leq x \leq \bar{I} + D \\ \bar{I} & \text{if } x \geq \bar{I} + D. \end{cases} \quad \text{(A6)}$$

Q.E.D.

Second-Order Condition of the Maximization Program

We use the alternate method of finding the optimal level of deductibility proposed by Meyer and Ormiston (1999). Denote $Q = EI^*(\tilde{x})$, the actuarially fair premium, and define $V(Q) = U(D)$. Meyer and Ormiston (1999) point out that, at local optima, the second order conditions $U''_{DD} < 0$ and $V''_{QQ} < 0$ are identical. We reconsider the problem when the deductible insurance policy is subject to an upper limit on coverage.

We denote $D \equiv \gamma(Q)$ and $P \equiv \psi(\gamma(Q))$. Observe that $\gamma'(Q) = [F(\gamma) - F(\gamma + K)]^{-1}$ and $\gamma''(Q) = [\gamma'(Q)]^2[f(\gamma + K) - f(\gamma)]$, where $f(x) = F'(x)$. Differentiating $V(Q)$ twice with respect to Q and rearranging the terms yield

$$
\begin{aligned}
V''_{QQ} = &\int_0^{\gamma(Q)} [u''(w_1)(\psi')^2 - u'(w_1)\psi''] \, dF(x) + (\gamma' + \psi')^2 [F(\gamma(Q) + \bar{I}) - F(\gamma(Q))] u''(w_2) \\
&- \psi''[F(\gamma(Q) + \bar{I}) - F(\gamma(Q))] u'(w_2) \\
&+ \int_{\gamma(Q)+\bar{I}}^{\bar{x}} [u''(w_3)(\psi')^2 - u'(w_3)\psi''] \, dF(x),
\end{aligned} \tag{A7}
$$

where $w_1 = w_0 - x - \psi(Q)$, $w_2 = w_0 - \gamma(Q) - \psi(Q)$, and $w_3 = w_0 - x - \psi(Q) + \bar{I}$. Equation (A7) is a slight extension of Meyer and Ormiston (1999, Equation (10)) to the case where the deductible insurance can be subject to an upper limit on coverage. It contains a new term characterized by the last RHS integral in (A7). Observe that this term vanishes when there is no upper limit on coverage. It follows that $V(Q)$ is strictly concave if $u' \geq 0$, $u'' < 0$, $\psi(0) = 0$, $\psi' \geq 1$, and $\psi'' \geq 0$. In particular, the objective function is strictly concave when $\psi(Q) = mQ$ with $m \geq 1$, as we assume in Equation (3). It follows that, at local optima, $V''_{QQ} < 0$ if and only if $U''_{DD} < 0$.

Effect of a Change in the Upper Limit on the Optimal Deductible

Differentiating $U'_D(D)$ with respect to $\bar{I}$ yields

$$
\begin{aligned}
U''_{D\bar{I}}(D) = \frac{d^2 Eu}{dD d\bar{I}} = &-P'_{\bar{I}} \left\{ -P'_D \int_0^D u''(w_1) \, dF(x) \right. \\
&\left. - (1 + P'_D)[F(D + \bar{I}) - F(D)] u''(w_2) - P'_D \int_{D+\bar{I}}^{\bar{x}} u''(w_3) \, dF(x) \right\} \\
&+ P''_{D\bar{I}} \left\{ - \int_0^D u'(w_1) \, dF(x) - [F(D + \bar{I}) - F(D)] u'(w_2) - \int_{D+\bar{I}}^{\bar{x}} u'(w_3) \, dF(x) \right\} \\
&- P'_D \int_{D+\bar{I}}^{\bar{x}} u''(w_3) \, dF(x) - f(D + \bar{I}) u'(w_2),
\end{aligned} \tag{A8}
$$

where $P'_D = -m[F(D + \bar{I}) - F(D)] < 0$, $P'_{\bar{I}} = \partial P / \partial \bar{I} = m[1 - F(D + \bar{I})] > 0$ and $P''_{D\bar{I}} = \partial^2 P / \partial D \partial \bar{I} = -mf(D + \bar{I}) < 0$. From the first-order necessary condition expressed in Equation (6), the above expression can be rewritten as

$$
\begin{aligned}
U''_{D\bar{I}}(D) = &-P'_D \int_{D+\bar{I}}^{\bar{x}} u''(w_3) \, dF(x) + P'_{\bar{I}} \left\{ -P'_D \int_0^D u'(w_1) A_u(w_1) \, dF(x) \right. \\
&- (1 + P'_D)[F(D + \bar{I}) - F(D)] u'(w_2) A_u(w_2) \\
&\left. - P'_D \int_{D+\bar{I}}^{\bar{x}} u'(w_3) A_u(w_3) \, dF(x) \right\}
\end{aligned} \tag{A9}
$$

The second RHS term in curly brackets in Equation (A9) is the wealth effect. Dividing Equation (A9) by $-U''_D$ and using Equation (7) gives Equation (10).

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