

APPLICATIONS OF FUZZY REGRESSION IN ACTUARIAL ANALYSIS

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ABSTRACT

In this article, we propose several applications of fuzzy regression techniques for actuarial problems. Our main analysis is motivated, on the one hand, by the fact that several articles in the financial and actuarial literature suggest using fuzzy numbers to model interest rate uncertainty but do not explain how to quantify these rates with fuzzy numbers. Likewise, actuarial literature has recently focused some of its attention in analyzing the Term Structure of Interest Rates (TSIR) because this is a key instrument for pricing insurance contracts. With these two ideas in mind, we show that fuzzy regression is suitable for adjusting the TSIR and discuss how to apply a fuzzy TSIR when pricing life insurance contracts and property-liability policies. Finally, we reflect on other actuarial applications of fuzzy regression and develop with this technique the *London Chain Ladder Method* for obtaining Incurred But Not Reported Reserves.

INTRODUCTION

To obtain the financial price of an insurance contract and in general the price of any other asset, we have to discount the cash flows that the asset produces throughout its life. We therefore need to know the discount rates that must be applied at each moment. The reference values of these discount rates are those interest rates that are free of default risk, i.e., those that correspond to public debt bonds. Of course, this is especially true in an actuarial pricing context because the profit to the insurer must be in accordance with the return from the insurer's investments of the premiums, and some of the premiums are invested in public debt securities. So, predicting the evolution of the default-free interest rate is a crucial question in actuarial pricing. This explains why yield curve analysis has become an important topic in actuarial science. Babbell and Merrill (1996) and Ang and Sherris (1997) provided a wide survey of Term Structure of Interest Rate (TSIR) models derived from the contingent claims theory, whereas Yao (1999) discussed the asymptotic properties of the rates fitted with some of these models, bearing in mind actuarial pricing. Delbaen and Lorimier (1992)

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used a nonparametric method based on quadratic programming to fit the short-term yield curve, whereas Carriere (1999) proposed combining bootstrapping and spline functions to estimate long-term yield rates embedded in the TSIR.

However, many actuarial analyses are concerned with the medium and long term and, in our opinion, modeling the behavior of interest rates in the long term by means of a stochastic model is not very realistic. As Gerber (1995) pointed out, there is no commonly accepted stochastic model for predicting long-term discount rates.

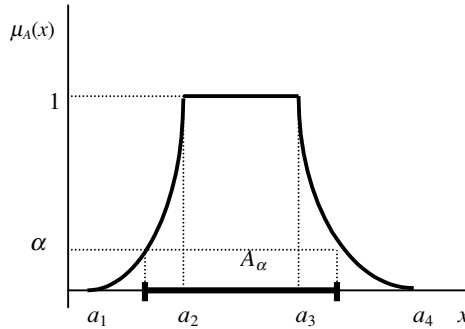
Fuzzy Sets Theory (FST) has been used successfully in insurance problems that require much actuarial subjective judgment and those for which measuring the embedded variables is difficult. Lemaire (1990) applied fuzzy logic to underwriting and reinsurance decisions whereas Cummins and Derrig (1993) used fuzzy decision to evaluate several econometric methods of claim cost forecasting. Derrig and Ostaszewski (1995) showed that fuzzy clustering methods are suitable for risk classification and Young (1996) applied fuzzy reasoning to insurance rate decisions. Therefore, and given that there are only vague data or data ill related with the behavior of future discount rates to predict them (e.g., the price of fixed-income securities or the opinions of "experts" about the future behavior of macroeconomic magnitudes), many authors think that it is often more suitable and realistic to make financial analyses in the long term with yields quantified with fuzzy numbers. For financial analysis, see Kaufmann (1986), Buckley (1987), or Li Calzi (1990), whereas in actuarial literature, see Lemaire (1990), Ostaszewski (1993), or Terceño et al. (1996) in a life insurance context, and articles by Cummins and Derrig (1997) and Derrig and Ostaszewski (1997) on the financial analysis of property-liability insurance. However, these articles do not explain in great detail how to estimate the discount rates with fuzzy sets. They usually suggest that "the rates are estimated subjectively by the experts using fuzzy numbers" but offer no more explanation.

In this article, we propose a solution to this problem. This involves estimating the TSIR with fuzzy sets, since the TSIR implicitly contains the expectations of the fixed income market agents (i.e., the experts) regarding the evolution of the future interest rates. Our results will be similar to those of Carriere (1999). His method obtains an estimate of the TSIR with probabilistic confidence intervals, whereas ours describes the yield curve as fuzzy confidence intervals. We also discuss how to use our fuzzy TSIR to price life-insurance contracts and property-liability policies. We would like to point out that using a fuzzy TSIR to price insurance policies was initially suggested in Ostaszewski (1993).

Another aim of this article is to suggest other actuarial applications of fuzzy regression. We have therefore developed the method for obtaining Incurred But Not Reported Reserves proposed by Benjamin and Eagles (1986) with fuzzy regression methods. We also discuss how fuzzy regression can help us with trending claim costs and with premium rating from the CAPM perspective.

The structure of the article is as follows. In the next section we describe some basic aspects of fuzzy arithmetic and fuzzy regression. In "Estimating the TSIR With Fuzzy Methods" we propose a method for obtaining a fuzzy TSIR based on fuzzy regression, apply our method to the Spanish public debt market, and compare our results with those of standard econometric methods. In "Using a Fuzzy TSIR for Financial and

FIGURE 1
A Fuzzy Number



Actuarial Pricing” we discuss how our fuzzy TSIR can be used in actuarial pricing. In “Discussing Further Actuarial Applications of Fuzzy Regression” we suggest further applications of fuzzy regression to insurance problems.

FUZZY ARITHMETIC AND FUZZY REGRESSION

Basics of FST and Fuzzy Numbers

FST is constructed from the concept of *fuzzy subset*. A fuzzy subset $\tilde{A}$ is a subset defined over a reference set X for which the level of membership of an element $x \in X$ to $\tilde{A}$ accepts values other than 0 or 1 (absolute nonmembership or absolute membership). A fuzzy subset $\tilde{A}$ can therefore be defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu_{\tilde{A}}(x)$ is called the membership function and is a mapping $\mu_{\tilde{A}}: X \rightarrow [0, 1]$. So, an element x has its image within $[0, 1]$, where 0 indicates nonmembership to the fuzzy subset $\tilde{A}$ and 1 indicates absolute membership. Alternatively, a fuzzy subset $\tilde{A}$ can be represented by its *level sets* α or α -cuts. An α -cut is an ordinary (crisp) set containing elements whose membership level is at least α . For a fuzzy subset $\tilde{A}$, we will name an α -cut with A_α being its mathematical expression: $A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$, $0 \leq \alpha \leq 1$.

A *fuzzy number* (FN) is a fuzzy subset $\tilde{A}$ defined over the real numbers (X is the set $\mathfrak{R}$). It is the main instrument of FST for quantifying uncertain or imprecise magnitudes (e.g., the discount rates in financial mathematics). Two other conditions are required for an FN. First, it must be a normal fuzzy set, i.e., it exists at least one $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$. Second, it must be convex (i.e., its α -cuts must be convex sets in the real numbers). Figure 1 shows the shape of an FN.

The most widely used FNs are triangular fuzzy numbers (TFNs) because they are easy to use and can be interpreted intuitively.¹ To construct a TFN named $\tilde{A}$, we must establish its center, a unique value a_C (i.e., $a_C = a_2 = a_3$ in Figure 1), and the deviations from there that we consider reasonable, i.e., its left spread, l_A ; and its right spread, r_A . A TFN² will be denoted as $\tilde{A} = (a_C, l_A, r_A)$. Its membership function, $\mu_{\tilde{A}}(x)$, is given

¹ The TFNs are a special case of a wider family of FNs called *L-R FNs*. For a detailed explanation see Dubois and Prade (1980).

² If $l_A = r_A = 0$, $\tilde{A}$ is the crisp number a_C .

by linear functions and its α -cuts, A_α , are confidence intervals where its extremes are also done by linear functions. So

$$\alpha = \mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{a_C - x}{l_A} & a_C - l_A < x \leq a_C \\ 1 - \frac{x - a_C}{r_A} & a_C < x \leq a_C + r_A; \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$A_\alpha = [A^1(\alpha), A^2(\alpha)] = [a_C - l_A(1 - \alpha), a_C + r_A(1 - \alpha)]. \quad (1)$$

In fuzzy regression, the symmetrical TFNs (STFNs) are widely used. These are TFNs where $l_A = r_A$ and we will denote them as $\tilde{A} = (a_C, a_R)$. The membership function and α -cuts of an STFN $\tilde{A}$ are

$$\alpha = \mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{|a_C - x|}{a_R} & a_C - a_R \leq x \leq a_C + a_R; \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$A_\alpha = [A^1(\alpha), A^2(\alpha)] = [a_C - a_R(1 - \alpha), a_C + a_R(1 - \alpha)]. \quad (2)$$

Figures 2 and 3 show the shape of a TFN and an STFN, respectively.

To develop our article we need to know the level of inclusion of an FN $\tilde{B}$ within another FN $\tilde{A}$, $\mu(\tilde{B} \subseteq \tilde{A})$. If the α -cuts of these FNs are $A_\alpha = [A^1(\alpha), A^2(\alpha)]$ and $B_\alpha = [B^1(\alpha), B^2(\alpha)]$, then $\mu(\tilde{B} \subseteq \tilde{A}) \geq \alpha$, if $B_\alpha \subseteq A_\alpha$, i.e., if

$$A^1(\alpha) \leq B^1(\alpha) \quad \text{and} \quad A^2(\alpha) \geq B^2(\alpha). \quad (3)$$

For example, in Figure 4, $\mu(\tilde{B} \subseteq \tilde{A}) \geq 0.4$.

FIGURE 2

The TFN $\tilde{A} = (a_C, l_A, r_A)$

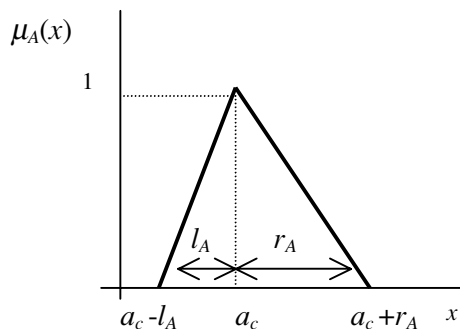
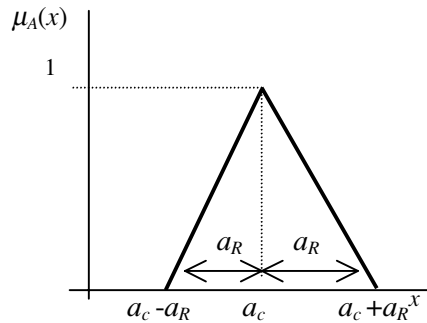
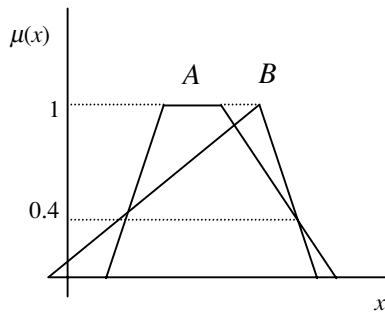


FIGURE 3

The STFN $\tilde{A} = (a_C, a_R)$

**FIGURE 4**

The Level of Inclusion and the “Greater or Equal” for Two Fuzzy Numbers



We also need to establish when one FN is greater than another.³ Ramik and Rimanek (1985) suggested that $\tilde{B}$ is greater or equal to $\tilde{A}$ with a membership level of at least α , $\mu(\tilde{B} \geq \tilde{A}) \geq \alpha$, if

$$A^1(\alpha) \leq B^1(\alpha) \quad \text{and} \quad A^2(\alpha) \leq B^2(\alpha). \quad (4)$$

For example, in Figure 4, $\mu(\tilde{B} \geq \tilde{A}) \geq 1$ and $\mu(\tilde{A} \geq \tilde{B}) \geq 0.4$.

In actuarial analysis we often need to evaluate functions (e.g., the net present value), which in a general way we shall symbolize as $y = f(x_1, x_2, \dots, x_n)$ —e.g., $x_1, x_2, \dots, x_{n-1}$ may be the cash flows and x_n the discount rate. Then, if $x_1, x_2, \dots, x_n$ are not given by crisp numbers but by the FNs $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$ (i.e., to calculate the net present value we know the cash flows and the discount rate imprecisely), when evaluating $f(\cdot)$ we will obtain an FN $\tilde{B}$, $\tilde{B} = f(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$. To determine the membership function of $\tilde{B}$, $\mu_{\tilde{B}}(y)$, we must apply the *Zadeh's extension principle*, exposed in the seminal

³ Many methods for ordering fuzzy numbers are proposed in the literature. Obviously, the choice of method depends on the problem. For our purpose we have chosen Ramik and Rimanek's criteria.

paper by Zadeh (1965). In a similar way as when handling arithmetically random variables, we obtain the membership function of the result when operating with FNs by convoluting membership functions of the FNs $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$ (for random variables, the density functions). The difference is that random variables are convoluted using operators sum-product, whereas with FN they are convoluted using operators max-min (max instead of sum and min instead of product). Mathematically,

$$\mu_{\tilde{B}}(y) = \max_{y=f(x_1, x_2, \dots, x_n)} \min(\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2), \dots, \mu_{\tilde{A}_n}(x_n)). \quad (5)$$

Unfortunately, it is often impossible to obtain a closed expression for the membership function of $\tilde{B}$ (this problem often arises when handling random variables). However, we may be able to obtain its α -cuts, B_α , from $A_{1\alpha}, A_{2\alpha}, \dots, A_{n\alpha}$ as

$$B_\alpha = f(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)_\alpha = f(A_{1\alpha}, A_{2\alpha}, \dots, A_{n\alpha}). \quad (6)$$

In actuarial mathematics, many functional relationships are continuously increasing or decreasing with respect to every variable in such a way that it is easy to evaluate the α -cuts of $\tilde{B}$. Buckley and Qu (1990b) demonstrated that if the function $f(\cdot)$ that induces $\tilde{B}$ is increasing with respect to the first m variables, where $m \leq n$, and decreasing with respect to the last $n-m$ variables, B_α is

$$\begin{aligned} B_\alpha &= [B^1(\alpha), B^2(\alpha)] \\ &= [f(A_1^1(\alpha), \dots, A_m^1(\alpha), A_{m+1}^2(\alpha), \dots, A_n^2(\alpha)), \\ &\quad f(A_1^2(\alpha), \dots, A_m^2(\alpha), A_{m+1}^1(\alpha), \dots, A_n^1(\alpha))]. \end{aligned} \quad (7)$$

Some operations with STFNS are easy to solve. For instance, if we multiply $\tilde{A} = (a_C, a_R)$ by a real number k , $\tilde{B} = k\tilde{A}$, the result is $\tilde{B} = (b_C, b_R) = (ka_C, |k|a_R)$. The sum of two⁴ STFNS, $\tilde{C} = \tilde{A} + \tilde{B}$, is also an STFNS. Concretely, $\tilde{C} = (c_C, c_R) = (a_C, a_R) + (b_C, b_R) = (a_C + b_C, a_R + b_R)$. So, if the FN $\tilde{B}$ is obtained from a linear combination of the STFNS $\tilde{A}_i = (a_{iC}, a_{iR})$, $i = 1, \dots, n$, i.e., $\tilde{B} = \sum_{i=1}^n k_i \tilde{A}_i$, where $k_i \in \Re$, $\tilde{B}$ will be an STFNS, $\tilde{B} = (b_C, b_R)$ where

$$(b_C, b_R) = (k_1 \cdot a_{1C} + k_2 \cdot a_{2C} + \dots + k_n \cdot a_{nC}, |k_1| \cdot a_{1R} + |k_2| \cdot a_{2R} + \dots + |k_n| \cdot a_{nR}). \quad (8)$$

Unfortunately, the result of a nonlinear operation with STFNS is not an STFNS. So, if we evaluate $\tilde{B} = f(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$, where we suppose that $\tilde{A}_i = (a_{iC}, a_{iR}) \forall i$, $\tilde{B}$ is often not an STFNS despite the characteristics of $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$. Even so, Dubois and Prade (1993) showed that if $f(\cdot)$ is increasing with respect to the first m variables, where $m \leq n$, and decreasing with respect to the others, $\tilde{B}$ can be estimated well by $\tilde{B}' = (b_C, b_R)$,

⁴ Clearly, the subtraction of two STFNS is an STFNS because the subtraction is the sum of the first FN with the second one multiplied by -1 .

where

$$\begin{aligned} b_C &= f(a_C), \\ b_R &= \sum_{i=1}^m \frac{\partial f(a_C)}{\partial x_i} a_{iR} - \sum_{i=m+1}^n \frac{\partial f(a_C)}{\partial x_i} a_{iR}, \quad \text{being} \\ a_C &= (a_{1C}, \dots, a_{mC}, a_{(m+1)C}, \dots, a_{nC}). \end{aligned} \quad (9)$$

So, for $\tilde{B} = \tilde{A}^k$ where $k \in \mathbb{R}$, if $\tilde{A} = (a_C, a_R)$, from (9) we obtain:

$$\tilde{B} \approx (b_C, b_R) = ((a_C)^k, |k|(a_C)^{k-1} a_R) \quad (10)$$

while for $\tilde{C} = \tilde{A} \cdot \tilde{B}$ where $k \in \mathbb{R}$, if $\tilde{A} = (a_C, a_R)$ and $\tilde{B} = (b_C, b_R)$, from (9) we obtain

$$\tilde{C} \approx (c_C, c_R) = (a_C \cdot b_C, a_C \cdot b_R + b_C \cdot a_R). \quad (11)$$

Tanaka and Ishibuchi's Fuzzy Regression Model

The fuzzy regression model developed in Tanaka (1987) and Tanaka and Ishibuchi (1992) is one of the most widely used models in fuzzy literature for economic applications.⁵ Like any regression technique, the aim of fuzzy regression is to determine a functional relationship between a dependent variable and a set of independent ones. Fuzzy regression allows to obtain functional relationships when independent variables, dependent variables, or both, are not crisp values but confidence intervals.

As in econometric linear regression, we shall suppose that the explained variable is a linear combination of the explanatory variables. This relationship should be obtained from a sample of n observations $\{(Y_1, X_1), (Y_2, X_2), \dots, (Y_j, X_j), \dots, (Y_n, X_n)\}$ where X_j is the j th observation of the explanatory variable, $X_j = (X_{0j}, X_{1j}, X_{2j}, \dots, X_{ij}, \dots, X_{mj})$. Moreover, $X_{0j} = 1 \forall j$, and X_{ij} is the observed value for the i th variable in the j th case of the sample. Y_j is the j th observation of the explained variable, $j = 1, 2, \dots, n$. The j th observation may either be a crisp value or a confidence interval. In either case, it can be represented through its center and its spread or radius as $Y_j = \langle Y'_{jC}, Y'_{jR} \rangle$, where Y'_{jC} is the center and Y'_{jR} is the radius.

Similarly, we suppose that the j th observation for the dependent variable is an α^* -cut of the FN it arises from where α^* may be stated previously by the decision maker. Also, the FN that quantifies the j th observation of the dependent variable is an STFNN that we will write as $\tilde{Y}_j = (Y_{jC}, Y_{jR})$. Therefore, since the α^* -cut of $\tilde{Y}_j, Y_{j\alpha^*}$ is (see (2)):

$$Y_{j\alpha^*} = \langle Y'_{jC}, Y'_{jR} \rangle = [Y_{jC} - Y_{jR}(1 - \alpha^*), Y_{jC} + Y_{jR}(1 - \alpha^*)], \quad j = 1, 2, \dots, n. \quad (12)$$

the center and spread of $\tilde{Y}_j$ can be obtained from its α^* -cut taking into account (2) as

$$Y'_{jC} = Y_{jC} \quad \text{and} \quad Y'_{jR} = Y_{jR}(1 - \alpha^*) \Rightarrow Y_{jR} = Y'_{jR} / (1 - \alpha^*) \quad (13)$$

⁵ Fedrizzi, Fedrizzi, and Ostasiewicz (1993) initially suggested fuzzy regression methods for economics. Some economic applications can be found in Ramenazi and Duckstein (1992) or Profillidis, Papadopoulos, and Botzoris (1999).

and we must estimate the following fuzzy linear function

$$\tilde{Y}_j = \tilde{A}_0 + \tilde{A}_1 X_{1j} + \cdots + \tilde{A}_m X_{mj}. \quad (14)$$

In this model of fuzzy regression, the disturbance is not introduced as a random addend in the linear relation but is incorporated into the coefficients $\tilde{A}_i, i = 0, 1, \dots, m$. Of course, the final objective is to adjust the fuzzy numbers $\tilde{A}_i$, which estimate $\tilde{A}_i$ from the available sample. Given the characteristics of $\tilde{Y}_j$, the parameters $\tilde{A}_i, i = 0, 1, 2, \dots, m$ must be STFNS. These parameters can therefore be written as $\tilde{A}_i = (a_{iC}, a_{iR}), i = 0, 1, \dots, m$. The main objective is to estimate every FN $\tilde{A}_i, \hat{A}_i = (\hat{a}_{iC}, \hat{a}_{iR})$. When we have obtained $\hat{A}_i$, the estimates of, $\hat{Y}_j = (\hat{Y}_{jC}, \hat{Y}_{jR})$, will be

$$\hat{Y}_j = \hat{A}_0 + \hat{A}_1 X_{1j} + \cdots + \hat{A}_m X_{mj}. \quad (15)$$

Therefore, $\hat{Y}_j$ is obtained from (8):

$$\hat{Y}_j = (\hat{Y}_{jC}, \hat{Y}_{jR}) = \sum_{i=0}^m (\hat{a}_{iC}, \hat{a}_{iR}) X_{ij} = \left(\sum_{i=0}^m \hat{a}_{iC} X_{ij}, \sum_{i=0}^m \hat{a}_{iR} |X_{ij}| \right) \quad (16)$$

whose α -cuts for a level α^* are

$$\begin{aligned} \hat{Y}_{j\alpha^*} &= [\hat{Y}_{jC} - \hat{Y}_{jR}(1 - \alpha^*), \hat{Y}_{jC} + \hat{Y}_{jR}(1 - \alpha^*)] \\ &= \left[\sum_{i=0}^m \hat{a}_{iC} X_{ij} - (1 - \alpha^*) \sum_{i=0}^m \hat{a}_{iR} |X_{ij}|, \sum_{i=0}^m \hat{a}_{iC} X_{ij} + (1 - \alpha^*) \sum_{i=0}^m \hat{a}_{iR} |X_{ij}| \right]. \end{aligned} \quad (17)$$

The parameters $\hat{a}_{iC}$ and $\hat{a}_{iR}$, must minimize the spreads of $\hat{Y}_j$, and simultaneously maximize the congruence of $\hat{Y}_j$ with $\tilde{Y}_j$, which is measured as $\mu(\tilde{Y}_j \subseteq \hat{Y}_j)$. Specifically, we must solve the following multiple objective program:

$$\text{Minimize } z = \sum_{j=1}^n \hat{Y}_{jR} = \sum_{j=1}^n \sum_{i=0}^m \hat{a}_{iR} |X_{ij}|, \quad \text{Maximize } \alpha \quad (18a)$$

subject to

$$\mu(\tilde{Y}_j \subseteq \hat{Y}_j) \geq \alpha \quad j = 1, 2, \dots, n; \quad \hat{a}_{iR} \geq 0 \quad i = 0, 1, \dots, m, \quad \alpha \in [0, 1]. \quad (18b)$$

If for the second objective we require a minimum accomplishment level α^* , i.e., the level that the decision maker considers that $\langle Y'_{jC}, Y'_{jR} \rangle, j = 1, 2, \dots, n$, has been obtained, the above program is transformed into the following linear one

$$\text{Minimize } z = \sum_{j=1}^n \sum_{i=0}^m \hat{a}_{iR} |X_{ij}| \quad (19a)$$

subject to

$$Y_{j\alpha^*} \subseteq \hat{Y}_{j\alpha^*} \quad j = 1, 2, \dots, n; \quad \hat{a}_{iR} \geq 0 \quad i = 0, 1, \dots, m. \quad (19b)$$

The first block of constraints in Equation (19b) is a consequence of the requirement that $\mu(\tilde{Y}_j \subseteq \hat{Y}_j) \geq \alpha^*$, which must be implemented by taking into account (3), (12), and (17). With the last block of constraints in Equation (19b) we ensure that $\hat{a}_{iR} \forall i$ will be nonnegative.

In our opinion, fuzzy regression techniques have a number of advantages over traditional regression techniques:

- (a) The estimates obtained after adjusting the coefficients are not random variables, which are difficult to manipulate in arithmetical operations, but fuzzy numbers, which are easier to handle arithmetically using α -cuts. So, when starting from magnitudes estimated by random variables (e.g., from a least squares regression), these random variables are often reduced to their mathematical expectation (which may or may not be corrected by its variance) to make them easier to handle. As we have already pointed out, this loss of information does not necessarily take place when we operate with FNs.
- (b) When investigating economic or social phenomena, the observations are a consequence of the interaction between the economic agents' beliefs and expectations, which are highly subjective and vague. A good way to treat this kind of information is therefore with FST. For example, the asset prices that are determined in the markets are due to the agents' expectations of future inflation and the issuers' credibility. We think that it is more realistic to consider that the bias between the observed value of the dependent variable and its theoretical value (the error) is not random but fuzzy. At least in this way we assume that the analyzed phenomena have a large subjective component.
- (c) The observations are often not crisp numbers but confidence intervals. For instance, the price of one financial asset throughout one session often oscillates within an interval and is rarely unique (e.g., it can oscillate within [\$100, \$105]). To be able to use econometric methods, the observations for the explained variable and/or the explanatory variable must be represented by a single value (e.g., \$102.5 for [\$100, \$105]), which involves losing a great deal of information. However, fuzzy regression does not necessarily reduce each variable to a crisp number, i.e., all the observed values can be used in the regression analysis.

ESTIMATING THE TSIR WITH FUZZY METHODS

Estimating the TSIR With Conventional Econometric Methods

Estimating the TSIR for a concrete date and a given market is fairly straightforward if there are many sufficiently liquid zero coupon bonds and their price can be observed without perturbations. However, fixed-income markets rarely enjoy these conditions simultaneously.

This subsection describes the essence of a family of methods for estimating the discount function associated to the TSIR with econometric methods. They can be used if

the sample is made up entirely of zero coupon bonds, entirely of bonds with coupon or, as is usual, if it is made up of both types of bonds.

These methods start from the fact that the r th bond, where $r = 1, 2, \dots, k$, in which k is the number of available bonds, provides several cash flows (coupons and principal). These are denoted by $\{(C_1^r, t_1^r); (C_2^r, t_2^r); \dots; (C_{n_r}^r, t_{n_r}^r)\}$, where C_i^r is the amount of the i th cash flow and t_i^r is its maturity in years. If we suppose that the default-free bonds do not include any option (i.e., they are not convertible or callable, etc.), the price of the r th bond is therefore the sum of the discounted value of every coupon and the principal with the corresponding spot rate

$$P^r = \sum_{i=1}^{n_r} C_i^r f_{t_i^r}, \quad (20)$$

where $f_{t_i^r}$ is the discounted value of one dollar with maturity t_i^r years.

Of course, subsequently we should define a form for the discount function to specify the econometric equation to be estimated. Our proposal is based on the methods that use splines (piecewise functions) to model the discount function. The best known methods are those in McCulloch's articles (1971, 1975) (quadratic splines and cubic splines) and in Vasicek and Fong's article (1982) (exponential splines). These methods suppose that the discount function is a linear combination of $m + 1$ functions of time. Therefore

$$f_t = \sum_{j=0}^m a_j g_j(t). \quad (21)$$

From (20) and (21) we can deduce that the following linear equation must be estimated

$$P^r = \sum_{i=1}^{n_r} C_i^r \sum_{j=0}^m a_j g_j(t_i^r) + \varepsilon_r. \quad (22)$$

We assume that $g_j(t)$ are splines and not simply polynomial functions because with splines we can determine $g_j(t)$ according to the distribution of the maturity dates of the sample and, therefore, fit the discount function better for the most common maturities. Moreover, with splines we can obtain TSIR profiles that do not fluctuate very much and forward rates that do not behave explosively.

A random disturbance is justified because several factors disturb the formation of the prices of fixed-income securities. Chambers, Carleton, and Waldman (1984) point out some of them; for example, the coupon-bearing of bonds with maturities greater than one contain information about more than one present value coefficient; the bond portfolios are not continuously rebalanced, so at any moment each bond can deviate by some (presumably random) amount or there is no single price for each bond, which implies that there is some inherent imprecision to the concept of a single price.

We will now present our fuzzy method for estimating the TSIR. We will first establish a hypothesis on which to construct a method for estimating the TSIR that uses fuzzy

regression from the analyzed framework. We will then discuss how to estimate the TSIR and the forward rates using fuzzy numbers and make an empirical application. As we stated above, we will suppose that the bonds in our analysis are default-free and only produce a stream of payments (coupons and principal or only principal if they are zero coupon bonds) and that they do not have any embedded option. Also, as the price of one bond we will take all the prices traded on 1 day, rather than an average price.

Hypothesis

Hypothesis 1: The price of the r th bond in a session is an STFNN. This price will be written as $\tilde{P}^r$ where

$$\tilde{P}^r = (P_C^r, P_R^r) P_C^r, P_R^r \geq 0, \quad r = 1, 2, \dots, k. \quad (23)$$

This hypothesis considers the price of a bond on 1 day to be “approximately P_C ,” and not exactly P_C . We think that this is more suitable because over a session it is usual to negotiate more than one price for the same bond (one for each trade). However, if these prices are unique then $P_R^r = 0$.

Hypothesis 2: The observed price for each security is an α -cut of the FN that quantifies that price for a predefined α^* . Its inferior and upper extremes are the minimum and maximum price of the bond over the session. Therefore, this interval will be expressed through its center and spread as $\langle P_C^r, P_R^r \rangle$. For example, if the traded price for one bond on 1 day has fluctuated between 100 and 103, it will be expressed as $\langle 101.5, 1.5 \rangle$. Similarly, from these parameters we can obtain the center and the spread of its corresponding FN, (23), taking into account (13):

$$P_C^r = P_C^r \text{ y } P_R^r = (1 - \alpha^*) P_R^r \Rightarrow P_R^r = P_R^r / (1 - \alpha^*), \quad 0 \leq \alpha^* \leq 1, \quad r = 1, 2, \dots, k. \quad (24)$$

Hypothesis 3: The discount function is quantified via an FN that depends on the time. So, for a given maturity t , the discount function is the following STFNN:

$$\tilde{f}_t = (f_{tC}, f_{tR}), \quad t > 0, \quad 0 \leq f_{tC} - f_{tR} \leq f_{tC} + f_{tR} \leq 1. \quad (25)$$

The price of the r th bond, (23), can therefore be written from (20) as

$$\tilde{P}^r = \sum_{i=1}^{n_r} C_i^r \tilde{f}_{t_i^r} \quad (26)$$

and combining (25) and (26) we obtain

$$(P_C^r, P_R^r) = \sum_{i=1}^{n_r} C_i^r (f_{t_i^r C}, f_{t_i^r R}). \quad (27)$$

So, the observed α^* -cut for the price of the r th bond, is

$$\langle P_C^r, P_R^r \rangle = \sum_{i=1}^{n_r} C_i^r \langle f_{t_i^r C}^r, f_{t_i^r R}^r \rangle \quad (28)$$

with $f_{t_i^r C}^r = f_{t_i^r C}$ and $f_{t_i^r R}^r = (1 - \alpha^*)f_{t_i^r R}$, $0 \leq \alpha^* \leq 1$, $r = 1, 2, \dots, k$.

Hypothesis 4: The discount function, (25), can be approximated from a linear combination of $m + 1$ functions $g_j(t)$, $j = 0, 1, \dots, m$ with image in $\mathfrak{N}^+$ that are continuously differentiable, and whose parameters are given by STFNN. In this way, these parameters can be represented as

$$\tilde{a}_j = (a_{jC}, a_{jR}), \quad a_{jR} \geq 0, \quad j = 0, 1, \dots, m \quad (29)$$

and so the discount function is obtained from (21), (29) and using (8):

$$\tilde{f}_t = \sum_{j=0}^m \tilde{a}_j g_j(t) = \sum_{j=0}^m (a_{jC}, a_{jR}) g_j(t) = \sum_{j=0}^m (a_{jC} g_j(t), a_{jR} |g_j(t)|). \quad (30)$$

Then, using (27) and (30), the price of the r th bond can be expressed by

$$\tilde{P}^r = (P_C^r, P_R^r) = \sum_{i=1}^{n_r} C_i^r \sum_{j=0}^m (a_{jC}, a_{jR}) g_j(t_i^r) \quad (31)$$

and the α^* -cut of the r th bond price, (28), is now

$$\langle P_C^r, P_R^r \rangle = \sum_{i=1}^{n_r} C_i^r \sum_{j=0}^m \langle a'_{jC}, a'_{jR} \rangle g_j(t_i^r) \quad (32)$$

with $a'_{jC} = a_{jC}$ and $a'_{jR} = (1 - \alpha^*)a_{jR}$, $0 \leq \alpha^* \leq 1$, $r = 1, 2, \dots, k$.

Adjusting the Discount Function Using a Fuzzy Regression Model

Since the value of the discount function for $t = 0$ should be 1, $\tilde{f}_0 = (f_{0C}, f_{0R}) = (1, 0)$. As McCulloch stated in (1971), this condition is met if $\tilde{a}_0 = (a_{0C}, a_{0R}) = (1, 0)$, $g_0(t) = 1$, and $g_j(0) = 0$, $j = 1, 2, \dots, m$. So, from (31), we can express the price of the r th bond as

$$\begin{aligned} \tilde{P}^r &= (P_C^r, P_R^r) = \sum_{i=1}^{n_r} C_i^r \left[(1, 0) + \sum_{j=1}^m (a_{jC}, a_{jR}) g_j(t_i^r) \right] \\ &= \sum_{i=1}^{n_r} C_i^r (1, 0) + \sum_{i=1}^{n_r} C_i^r \sum_{j=1}^m (a_{jC}, a_{jR}) g_j(t_i^r) \end{aligned} \quad (33)$$

and, using fuzzy arithmetic, we finally write

$$(P_C^r, P_R^r) - \sum_{i=1}^{n_r} C_i^r(1, 0) = \sum_{j=1}^m (a_{jC}, a_{jR}) \sum_{i=1}^{n_r} C_i^r g_j(t_i^r). \quad (34)$$

In this way, by identifying in Equation (34) $\tilde{Y}^r = (Y_C^r, Y_R^r) = (P_C^r, P_R^r) - \sum_{i=1}^{n_r} C_i^r(1, 0)$, we obtain

$$Y_C^r = P_C^r - \sum_{i=1}^{n_r} C_i^r \quad \text{and} \quad Y_R^r = P_R^r. \quad (35)$$

It is easy to verify in Equation (34) that the value of the j th explanatory variable for the r th bond is the crisp value $X_j^r = \sum_{i=1}^{n_r} C_i^r g_j(t_i^r)$. Therefore, from (34) and (35) we can write

$$(Y_C^r, Y_R^r) = \sum_{j=1}^m (a_{jC}, a_{jR}) X_j^r. \quad (36)$$

Then, after obtaining the estimate for every $\tilde{a}_j$, $\hat{a}_j = (\hat{a}_{jC}, \hat{a}_{jR})$, the fitted value of the explained variable of the r th observation, $\hat{\tilde{Y}}^r$, is

$$\hat{\tilde{Y}}^r = (\hat{Y}_C^r, \hat{Y}_R^r) = \sum_{j=1}^m (\hat{a}_{jC}, \hat{a}_{jR}) X_j^r. \quad (37)$$

Bearing in mind that the dependent variable and the parameters are actually quantified via their α^* -cut, the expression of the α^* -cut of $\hat{\tilde{Y}}^r$ (that of FN (37)) is

$$\langle \hat{\tilde{Y}}_C^r, \hat{\tilde{Y}}_R^r \rangle = \sum_{j=1}^m \langle \hat{a}_{jC}^r, \hat{a}_{jR}^r \rangle X_j^r = \sum_{j=1}^m \langle \hat{a}_{jC}^r X_j^r, \hat{a}_{jR}^r X_j^r | X_j^r | \rangle \quad (38)$$

where we must estimate the center and the spread of the α^* -cut for $\hat{a}_j$, $j = 1, \dots, m$. After estimating these parameters, the center and the radius $\hat{a}_{jC}$ and $\hat{a}_{jR}$ are estimated using (2) as

$$\hat{a}_{jC} = \hat{a}_{jC}' \quad \text{and} \quad \hat{a}_{jR} = \hat{a}_{jR}' / (1 - \alpha^*). \quad (39)$$

Adding certain constraints, which are related to the properties of the discount function, to obtain $\hat{a}_{jC}'$ and $\hat{a}_{jR}'$, we have to solve the following linear program

$$\text{Minimize } z = \sum_{r=1}^k \hat{\tilde{Y}}_R^r = \sum_{r=1}^k \sum_{j=1}^m \hat{a}_{jR}' \sum_{i=1}^{n_r} C_i^r |g_j(t_i^r)| = \sum_{j=1}^m \hat{a}_{jR}' \sum_{r=1}^k \sum_{i=1}^{n_r} C_i^r |g_j(t_i^r)| \quad (40a)$$

subject to

$$\begin{aligned}\hat{Y}_C^r - \hat{Y}_R^r &= \sum_{j=1}^m \hat{a}'_{jC} \sum_{i=1}^{n_r} C_i^r g_j(t_i^r) - \sum_{j=1}^m \hat{a}'_{jR} \sum_{i=1}^{n_r} C_i^r |g_j(t_i^r)| \\ &\leq Y_C^r - Y_R^r \quad r = 1, 2, \dots, k,\end{aligned}\quad (40b)$$

$$\begin{aligned}\hat{Y}_C^r + \hat{Y}_R^r &= \sum_{j=1}^m \hat{a}'_{jC} \sum_{i=1}^{n_r} C_i^r g_j(t_i^r) + \sum_{j=1}^m \hat{a}'_{jR} \sum_{i=1}^{n_r} C_i^r |g_j(t_i^r)| \\ &\geq Y_C^r + Y_R^r \quad r = 1, 2, \dots, k\end{aligned}\quad (40c)$$

$$\begin{aligned}\sum_{j=1}^m \hat{a}'_{jC} (g_j(sP) - g_j((s+1)P)) - \sum_{j=1}^m \hat{a}'_{jR} (|g_j(sP)| - |g_j((s+1)P)|) &\geq 0 \\ s = 1, \dots, u-1\end{aligned}\quad (40d)$$

$$\begin{aligned}\sum_{j=1}^m \hat{a}'_{jC} (g_j(sP) - g_j((s+1)P)) + \sum_{j=1}^m \hat{a}'_{jR} (|g_j(sP)| - |g_j((s+1)P)|) &\geq 0 \\ s = 1, \dots, u-1\end{aligned}\quad (40e)$$

$$\sum_{j=1}^m \hat{a}'_{jC} g_j(uP) - \sum_{j=1}^m \frac{\hat{a}'_{jR}}{L^{-1}(\alpha^*)} |g_j(uP)| \geq -1 \quad (40f)$$

$$\sum_{j=1}^m \hat{a}'_{jC} g_j(P) + \sum_{j=1}^m \frac{\hat{a}'_{jR}}{L^{-1}(\alpha^*)} |g_j(P)| \leq 0 \quad (40g)$$

$$\hat{a}'_j R \geq 0 \quad j = 1, 2, \dots, m. \quad (40h)$$

The constraints (40b), (40c), and (40h) correspond to Tanaka's regression model (see (18b) or (19b)). Similarly, (40d) and (40e) ensure that the discount function is decreasing, and that it is accomplished for an arbitrary periodicity P (in years). Therefore uP is the greatest maturity that we will use in a later analysis. It is reasonable to suppose that uP is close to the expiration of the bond with the greatest maturity. In Equations (40d) and (40e) we use the Ramik and Rimanek's criteria for ordering FNs (see (4)). Finally, constraints (40f) and (40g) ensure that the discount function is within $[0, 1]$.

Estimating the Spot Rates and the Forward Rates Using Fuzzy Numbers

The discount function in t , f_t , is obtained from its corresponding spot rate i_t by $f_t = (1 + i_t)^{-t}$, and then

$$i_t = (f_t)^{-1/t} - 1 \quad (41)$$

If the discount function in t is an FN, the spot rate will be an FN $\tilde{i}_t$. Its membership function can be obtained applying the extension principle (5) to the relation (41) as

follows:

$$\mu_{\tilde{f}_t}(y) = \max_{y=x^{-1/t}-1} \mu_{\tilde{f}_t}(x) = \mu_{\tilde{f}_t}[(1+y)^{-t}].$$

Unfortunately, despite using a discount function quantified via an STFNN, the spot rate is not an STFNN because it is not a linear function of $\tilde{f}_t$. However, applying (10) in (41) we obtain

$$i_t \approx (i_{tC}, i_{tR}) = \left(\frac{1}{(f_t)^{\frac{1}{t}}} - 1, \frac{f_t}{t \cdot (f_t)^{\frac{t+1}{t}}} \right). \quad (42)$$

To obtain the forward rate for the t th year, $\tilde{r}_t$, we should solve the following fuzzy equation:

$$\tilde{f}_{t-1} (1 + \tilde{r}_t)^{-1} = \tilde{f}_t. \quad (43)$$

The α -cuts of $\tilde{r}_t$, $r_{t\alpha}$ are (see Appendix):

$$\begin{aligned} r_{t\alpha} &= [r_t^1(\alpha), r_t^2(\alpha)] \\ &= \left[\frac{f_{(t-1)C} + f_{(t-1)R}(1-\alpha)}{f_{tC} + f_{tR}(1-\alpha)} - 1, \frac{f_{(t-1)C} - f_{(t-1)R}(1-\alpha)}{f_{tC} - f_{tR}(1-\alpha)} - 1 \right]. \end{aligned} \quad (44)$$

Then, although $\tilde{r}_t$ is not an STFNN, it can be approximated reasonably well by this type of FN. In the Appendix we demonstrate that its approximation by means of an STFNN is

$$\tilde{r}_t \approx (r_{tC}, r_{tR}) = \left(\frac{f_{(t-1)C}}{f_{tR}} - 1, \frac{f_{(t-1)C} \cdot f_{tR} - f_{tC} \cdot f_{(t-1)R}}{(f_{tC})^2} \right). \quad (45)$$

Notice that, although we take annual periods, calculating implied rates for any other periodicity is not a problem because the discount function is a continuous function of the maturity.

Empirical Application

In this subsection, we used our method to estimate the TSIR in the Spanish public debt market on June 29, 2001. Table 1 shows the bonds included in our sample and their characteristics.

To fit the TSIR in this date, we formalized the discount function using McCulloch's quadratic splines.⁶ So we took $m = 5$, and the knots that we used to construct the splines were $d_1 = 0$ years, $d_2 = 1.58$ years, $d_3 = 3.83$ years, $d_4 = 8.96$ years, and

⁶ For a detailed explanation of how we constructed our $g_j(t)$ and chose our knots, see McCulloch (1971).

TABLE 1

Prices of the Bonds Negotiated in the Spanish Debt Market on June 29, 2001

k	Asset	Coupon (annual)	Maturity (days)	Maturity (years)	$P_{\min}^k$	$P_{\max}^k$
1	T-Bill	0.00%	18	0.05	99.779	99.779
2	BOND	5.35%	163	0.45	103.258	103.313
3	T-Bill	0.00%	382	1.05	95.758	95.758
4	BOND	4.25%	391	1.07	103.907	103.947
5	T-Bill	0.00%	521	1.43	94.220	94.220
6	BOND	5.25%	576	1.58	103.555	103.669
7	STRIP	0.00%	576	1.58	93.579	93.749
8	BOND	3.00%	576	1.58	99.337	99.376
9	STRIP	0.00%	756	2.07	91.540	91.540
10	BOND	4.60%	757	2.07	104.670	104.917
11	BOND	4.50%	1,122	3.07	104.017	104.166
12	BOND	4.65%	1,216	3.33	98.466	98.702
13	BOND	3.25%	1,307	3.58	97.026	97.200
14	BOND	4.95%	1,490	4.08	105.407	105.918
15	BOND	10.15%	1,673	4.58	126.340	126.340
16	BOND	4.80%	1,945	5.33	97.785	98.385
17	BOND	7.35%	2,098	5.75	113.539	113.539
18	BOND	6.00%	2,402	6.58	107.400	108.206
19	STRIP	0.00%	2,775	7.60	68.412	68.412
20	BOND	5.15%	2,948	8.08	104.101	104.307
21	BOND	4.00%	3,134	8.59	92.679	93.473
22	BOND	5.40%	3,680	10.08	97.716	98.923
23	BOND	5.35%	3,771	10.33	96.966	97.749
24	BOND	6.15%	4,229	11.59	108.098	108.168
25	STRIP	0.00%	4,230	11.59	53.357	53.357
26	BOND	4.75%	4,774	13.08	96.506	97.567
27	BOND	6.00%	10,073	27.60	103.722	105.194
28	BOND	5.75%	11,351	31.10	93.954	94.777

$d_5 = 31.1$ years. Then, the functions $g_j(t)$, $j = 1, \dots, 5$ are

$$\begin{aligned}
 g_1(t) &= \begin{cases} -t^2/(2 \cdot 1.58) + t & 0 \leq t < 1.58 \\ 1.58/2 & 1.58 \leq t < 31.1 \end{cases}; \\
 g_j(t) &= \begin{cases} 0 & 0 \leq t < d_{j-1} \\ (t - d_{j-1})^2 / [2(d_j - d_{j-1})] & d_{j-1} \leq t < d_j \\ -\frac{(t - d_j)^2}{2(d_{j+1} - d_j)} + (t - d_j) + \frac{(d_j - d_{j-1})}{2} & d_j \leq t \leq d_{j+1} \\ 1/2 \cdot (d_{j+1} - d_{j-1}) & d_{j+1} \leq t \leq d_5 \end{cases}, \quad j = 2, 3, 4. \\
 g_5(t) &= \begin{cases} 0 & 0 \leq t < 8.96 \\ (t - 8.96)^2 / [2 \cdot (31.1 - 8.96)] & 8.96 \leq t < 31.1 \end{cases}
 \end{aligned}$$

We used OLS regression, taking for the price of the k th bond $(P_{\min}^k + P_{\max}^k)/2$. Its determination coefficient is $R^2 = 99.98\%$ whereas the discount function is

$$f_t = 1 - 0.04394g_1(t) - 0.03672g_2(t) - 0.04875g_3(t) - 0.03572g_4(t) - 0.00730g_5(t).$$

To fit the TSIR by using fuzzy regression we took for the price of the k th bond the interval $P^k = \langle P_C^k, P_R^k \rangle$, where $P_C^k = (P_{\min}^k + P_{\max}^k)/2$ and $P_R^k = (P_{\max}^k - P_{\min}^k)/2$. To build the constraints (40d), (40e), (40f), and (40g) in fuzzy regression we assumed an annual periodicity. The final value of the objective function (40a) is $z = 23.77$. To interpret this value, we should remember that the size of our sample was 28 assets. When taking the level of congruence $\alpha^* = 0.5$, our fuzzy discount function is

$$\begin{aligned} \tilde{f}_t = & (1, 0) + (-0.04280, 0.00422)g_1(t) + (-0.03874, 0.00043)g_2(t) \\ & + (-0.04675, 0.00397)g_3(t) + (-0.03841, 0.00069)g_4(t) + (-0.00255, 0)g_5(t) \end{aligned}$$

and requiring $\alpha^* = 0.75$, the discount function is

$$\begin{aligned} \tilde{f}_t = & (1, 0) + (-0.04280, 0.00843)g_1(t) + (-0.03874, 0.00086)g_2(t) \\ & + (-0.04675, 0.00793)g_3(t) + (-0.03841, 0.00139)g_4(t) + (-0.00255, 0)g_5(t). \end{aligned}$$

Table 2 shows the spot and forward rates for the next 15 years with OLS and fuzzy regressions (in the last case, for $\alpha^* = 0.5, 0.75$). We can see that the parameter α^* can be interpreted as an indicator of the perceived uncertainty in the market. If α^* increases, uncertainty of the observations for the explained variable (price of the bonds) also increases (see Wang and Tsaur, 2000) and the spread of the subsequent estimates of the spot rates and the forward rates will be wider.

To compare the difference between the OLS estimates and the estimates of fuzzy regression for discount rates, Table 2 includes coefficient $D = 100 \times |V^E - V^F|/V^E$, where V^E is the value of a yield rate obtained with econometric methods and V^F is the center of the fuzzy estimate for this rate. We can see that the estimates of the spot rates with each method are quite similar. This similarity decreases when estimating forward rates.

USING A FUZZY TSIR FOR FINANCIAL AND ACTUARIAL PRICING

Calculating the Present Value of an Annuity

Buckley (1987) determines the present value of a stream of amounts when these amounts and the discount rate are given by FNs. Buckley supposes that the discount rate to be applied throughout the evaluation horizon was a unique FN $\tilde{i}$. It implies that the reference TSIR is flat. If we also suppose that amounts are given by means of nonnegative FNs, in which the t th cash flow is the FN $\tilde{C}_t$, and that they are an immediate postpayable annuity with an annual periodicity, then the net present value of that annuity is the following FN, $\tilde{V}$:

$$\tilde{V} = \sum_{t=1}^n \tilde{C}_t(1 + \tilde{i})^{-t}. \quad (46)$$

TABLE 2
Comparison of the Estimates of OLS With the Fuzzy Estimates of the TSIR in the Spanish Public Debt Market on June 29, 2001

<i>T</i>	Econometric Methods		Fuzzy Regression $\alpha^* = 0.5$			Fuzzy Regression $\alpha^* = 0.75$			<i>D</i>	
	$\hat{i}_t$	r_t	$\tilde{i}_t$	$\tilde{r}_t$	$\hat{i}_t$	$\tilde{r}_t$	$\hat{i}_t$	r_t		
1	0.0435	0.0435	(0.0433, 0.0033)	(0.0433, 0.0033)	(0.0433, 0.0066)	(0.0433, 0.0066)	0.44%	0.44%	0.44%	
2	0.0423	0.0412	(0.0431, 0.0023)	(0.0430, 0.0012)	(0.0431, 0.0045)	(0.0430, 0.0025)	1.92%	1.92%	4.41%	
3	0.0440	0.0473	(0.0447, 0.0023)	(0.0479, 0.0025)	(0.0447, 0.0047)	(0.0479, 0.0049)	1.70%	1.70%	1.31%	
4	0.0470	0.0563	(0.0472, 0.0029)	(0.0547, 0.0047)	(0.0472, 0.0058)	(0.0547, 0.0094)	0.38%	0.38%	2.75%	
5	0.0496	0.0599	(0.0494, 0.0034)	(0.0581, 0.0055)	(0.0494, 0.0068)	(0.0581, 0.0109)	0.41%	0.41%	2.92%	
6	0.0513	0.0600	(0.0510, 0.0037)	(0.0594, 0.0052)	(0.0510, 0.0074)	(0.0594, 0.0103)	0.52%	0.52%	1.01%	
7	0.0525	0.0599	(0.0524, 0.0039)	(0.0606, 0.0048)	(0.0524, 0.0077)	(0.0606, 0.0096)	0.23%	0.23%	1.27%	
8	0.0534	0.0595	(0.0536, 0.0039)	(0.0619, 0.0043)	(0.0536, 0.0078)	(0.0619, 0.0086)	0.35%	0.35%	4.01%	
9	0.0540	0.0589	(0.0547, 0.0039)	(0.0632, 0.0037)	(0.0547, 0.0078)	(0.0632, 0.0074)	1.19%	1.19%	7.30%	
10	0.0545	0.0592	(0.0556, 0.0039)	(0.0645, 0.0035)	(0.0556, 0.0077)	(0.0645, 0.0070)	2.02%	2.02%	8.88%	
11	0.0551	0.0606	(0.0566, 0.0039)	(0.0658, 0.0038)	(0.0566, 0.0077)	(0.0658, 0.0076)	2.67%	2.67%	8.61%	
12	0.0557	0.0619	(0.0574, 0.0039)	(0.0670, 0.0042)	(0.0574, 0.0078)	(0.0670, 0.0083)	3.19%	3.19%	8.25%	
13	0.0562	0.0633	(0.0582, 0.0039)	(0.0682, 0.0045)	(0.0582, 0.0079)	(0.0682, 0.0090)	3.58%	3.58%	7.79%	
14	0.0568	0.0647	(0.0590, 0.0040)	(0.0693, 0.0049)	(0.0590, 0.0080)	(0.0693, 0.0098)	3.87%	3.87%	7.22%	
15	0.0574	0.0660	(0.0598, 0.0041)	(0.0703, 0.0053)	(0.0598, 0.0082)	(0.0703, 0.0106)	4.07%	4.07%	6.52%	

It is often impossible to obtain the membership function of $\tilde{V}$ by means of the extension principle. However, it is fairly straightforward to obtain a closed expression of the α -cuts of the present value. If we bear in mind that the function of present value is continuously decreasing (increasing) with respect to the discount rate (the amounts), we can calculate the upper and lower extremes of its α -cuts immediately from (7). If the amounts and the cash flows are given by STFNs, $\tilde{V}$ will not be an STFN but it can be approximated by an STFN from (9).

It is well known that it is quite unrealistic to suppose a flat TSIR. Moreover, in the previous section we proposed a method for obtaining an empirical fuzzy TSIR and discussed how to obtain its spot and implied rates. So, (46) can be generalized to any shape of the TSIR, using the spot rates, the forward rates, or the discount function

$$\tilde{V} = \sum_{t=1}^n \tilde{C}_t \cdot (1 + \tilde{r}_t)^{-t} = \sum_{t=1}^n \tilde{C}_t \cdot \prod_{j=1}^t (1 + \tilde{r}_j)^{-1} = \sum_{t=1}^n \tilde{C}_t \cdot \tilde{f}_t. \quad (47)$$

Of course, from the α -cuts of the amounts and the discount rates (or alternatively the discount function), we can easily obtain $\tilde{V}$ from (7). Moreover, if the amounts are crisp (e.g., if they are the amounts paid by a bond), and to obtain their present value we use a discount function quantified via an STFN like (25), (47) can be written as

$$\tilde{V} = \sum_{t=1}^n C_t \tilde{f}_t = \sum_{t=1}^n C_t (f_{tC}, f_{tR}) = \left(\sum_{t=1}^n C_t \cdot f_{tC}, \sum_{t=1}^n C_t \cdot f_{tR} \right). \quad (48)$$

Example: Suppose an investor is going to buy bonds of the Spanish public debt market on June 29, 2001. The maturity of these bonds is 5 years, and they offer a 5 percent annual coupon. The values of 1 monetary unit with maturities from 1 to 5 years obtained from the regression with $\alpha^* = 0.5$ are given in Table 3. Therefore, the investor obtains the following preliminary price for one of these bonds from (48):

$$\begin{aligned} \tilde{P} &= 5 \cdot (0.9585, 0.0030) + 5 \cdot (0.9190, 0.0040) + 5 \cdot (0.8770, 0.0059) \\ &\quad + 5 \cdot (0.8315, 0.0093) + 105 \cdot (0.7858, 0.0128) \\ &= (100.44, 1.46). \end{aligned}$$

Pricing Life-Insurance Contracts

In this subsection, we show how to obtain the net single premium for some life-insurance contracts (n -year pure endowments, n -year term life-insurance contracts,

TABLE 3
Free Default Discount Factors for the Next 5 Years

$\tilde{f}_1$	$\tilde{f}_2$	$\tilde{f}_3$	$\tilde{f}_4$	$\tilde{f}_5$
(0.9585, 0.0030)	(0.9190, 0.0040)	(0.8770, 0.0059)	(0.8315, 0.0093)	(0.7858, 0.0128)

and n -year endowments—the combination of the two first contracts) from our fuzzy TSIR. To simplify the analysis, we will suppose that the insured amounts are fixed beforehand and that they are annual and payable at the end of each year of the contract. Taking into account that standard life-insurance mathematics establishes that the net single premium for a policy is the discounted value of the mathematical expectation of the guaranteed amounts, and naming as C_n the amount payable at the end of an n -year pure endowment, the premium for an individual aged x for this contract, $\tilde{\Pi}_1$, is

$$\tilde{\Pi}_1 = C_n \cdot (1 + \tilde{i}_t)^{-n} \cdot {}_n p_x = C_n \cdot \tilde{f}_n \cdot {}_n p_x, \quad (49)$$

where ${}_n p_x$ stands for the probability that an individual aged x attains at the age $x + n$.

If we suppose that for an n -year term life insurance the amounts are payable at the end of the year of death, for an insured person aged x , the premium, $\tilde{\Pi}_2$, is

$$\tilde{\Pi}_2 = \sum_{t=0}^{n-1} C_t \cdot (1 + \tilde{i}_t)^{-(t+1)} \cdot {}_t|q_x = \sum_{t=0}^{n-1} C_t \cdot \tilde{f}_{t+1} \cdot {}_t|q_x \quad (50)$$

where ${}_t|q_x$ stands for the probability of death at age $x + t$ and C_t the insured amount for this event.

The net single premium for an n -year endowment, $\tilde{\Pi}_3$, is obtained from (49) and (50) as:

$$\tilde{\Pi}_1 = \tilde{\Pi}_1 + \tilde{\Pi}_2. \quad (51)$$

So, if the discount function estimated by an STFNN, the net single premium for an n -year pure endowment, (49), is reduced to

$$\begin{aligned} \tilde{\Pi}_1 &= (\Pi_1, l_{\Pi_1}) = C_n \cdot \tilde{f}_n \cdot {}_n p_x = C_n \cdot (f_{nC}, f_{nR}) \cdot {}_n p_x \\ &= (C_n \cdot f_{nC} \cdot {}_n p_x, C_n \cdot f_{nR} \cdot {}_n p_x) \end{aligned} \quad (52)$$

and then, for an n -year term life insurance, (50), we obtain

$$\begin{aligned} \tilde{\Pi}_2 &= (\Pi_2, l_{\Pi_2}) = \sum_{t=0}^{n-1} C_t \cdot \tilde{f}_{t+1} \cdot {}_t|q_x = \sum_{t=0}^{n-1} C_t \cdot (f_{(t+1)C}, f_{(t+1)R}) \cdot {}_t|q_x \\ &= \left(\sum_{t=0}^{n-1} C_t \cdot f_{(t+1)C} \cdot {}_t|q_x, \sum_{t=0}^{n-1} C_t \cdot f_{(t+1)R} \cdot {}_t|q_x \right). \end{aligned} \quad (53)$$

Then, from (51), (52), and (53) we obtain the price of an n -year endowment, $\tilde{\Pi}_3$:

$$\tilde{\Pi}_3 = (\Pi_3, l_{\Pi_3}) = (\Pi_1, l_{\Pi_1}) + (\Pi_2, l_{\Pi_2}) = (\Pi_1 + \Pi_2, l_{\Pi_1} + l_{\Pi_2}). \quad (54)$$

TABLE 4

Fuzzy Pure Single Premiums for Several Kinds of Policies

	35 Years	45 Years	55 Years	65 Years
$\tilde{\Pi}_1$	(779.69, 12.72)	(770.86, 12.58)	(752.42, 12.27)	(711.67, 11.61)
$\tilde{\Pi}_2$	(6.76, 0.06)	(16.50, 0.14)	(36.92, 0.31)	(81.92, 0.69)
$\tilde{\Pi}_3$	(786.46, 12.78)	(787.37, 12.72)	(789.34, 12.59)	(793.59, 12.30)

Table 4 shows the fuzzy pure single premiums of the three types of policies for people of several ages and with the value of the discount function in Table 3. In all cases the duration of the contracts is 5 years. We suppose that the amounts of the insured events are 1,000 monetary units and for the calculations we have taken the Swiss GRM-82 mortality tables.

The maturities of the bonds used to estimate the TSIR, and obviously the temporal horizon covered with this TSIR may not be large enough to discount all the cash flows (e.g., when pricing a whole life annuity). To complete the interest rates for the maturities of the TSIR that are not covered in the regression, de Andrés (2000) suggested two solutions:

- (1) The first involves estimating subjectively a unique nominal interest rate for the longer maturities by Fisher's relationship. In this way, Devolder (1988) suggested obtaining the discount rate for long-term insurance policies by using Fisher's relationship as: nominal interest rate = real interest rate + $\lambda \times$ anticipated inflation, where $0 \leq \lambda < 1$. Regarding the real interest rate, Devolder stated that "generally it must be quantified between the 2 and 3 percent" and the anticipated inflation "must be reasonable in the long term." Clearly, these sentences allow a fuzzy quantification even though this is probably not the aim of the author. If we call $\tilde{i}$ the FN that quantifies the discount rate, we can obtain this, e.g., as $\tilde{i} = (0.025, 0.01) + \lambda \tilde{\pi}$, where $\tilde{\pi}$ stands for the anticipated inflation.
- (2) From financial logic, the shape of the TSIR must be asymptotic. So, a second solution involves taking the latest forward (or spot) rate of our estimated TSIR as a reference for the rate to discount the amounts whose maturities are not covered by the TSIR.

Fuzzy Financial Pricing of Property-Liability Insurance

In this subsection we will show how to apply our fuzzy TSIR when pricing property-liability insurance. For a wide discussion on this topic, consult Myers and Cohn (1987) (MC) and Cummins (1990) under a nonfuzzy environment or Cummins and Derrig (1997) (CD) under a fuzzy environment. To simplify our explanation, we will suppose only a three-period model. The MC model states that the present value of the premiums must compensate the cost of the liabilities and the taxes for the insurer. Supposing a single premium, only two periods (years) in claiming, but that the TSIR can have any shape, the CD formulation can be transformed into

$$P = L(1 - \tau) \sum_{t=1}^2 c_t f_t^{(L)} + \tau P f_1 + \tau r_1(1 + \delta) P f_1 + \tau r_2(1 + \delta) P f_2 c_2, \quad (55)$$

where P = pure single premium. The parameter δ ($\delta > 0$) indicates the proportion of the fair premium corresponding to the surplus and τ is the tax rate that we suppose the same for the underwriting profit and the investment income. L is the total amount of the claim cost whereas c_t is the proportion of the liabilities payable at the t th year. Therefore $c_1 + c_2 = 1$. Notice that we have supposed that the proportion of the claims cost in the t th year deductible from income taxes is equal to c_t .

If f_t is the present value of 1 unitary unit payable at t with the spot rate free of risk of failure for that maturity (i_t), then $f_t = (1 + i_t)^{-t}$. Similarly, $f_t^{(L)}$ is the value of the discount function for one monetary unit of liability payable at the t th year. This can be obtained from the spot rate of the liabilities at time t , $i_t^{(L)}$. Therefore, $f_t^{(L)} = (1 + i_t^{(L)})^{-t}$. To simplify our discussion, we will suppose that $i_t^{(L)}$ is obtained by applying over i_t the risk loading k_t , by doing $i_t^{(L)} = (1 - k_t)(1 + i_t) - 1$, where $1 > k_t > 0$. From this relation, the following connection between $f_t^{(L)}$ and f_t arises

$$f_t^{(L)} = (1 - k_t)^{-t} f_t. \quad (56)$$

Finally, r_t , $t = 1, 2$ is the return obtained by investing the premium within the t th year of the contract. If we assume, as is usual, that this return corresponds to the risk-free rate, within a pure expectations framework, r_t can be quantified by the forward rate for the t th year of the contract. Then, taking into account that r_t can be obtained from the values of spot discount function f_{t-1} and f_t as

$$r_t = (f_{t-1}/f_t) - 1. \quad (57)$$

Then, (55) can be rewritten from (56), (57) and remembering that $c_1 + c_2 = 1$ as

$$P = L(1 - \tau) \sum_{t=1}^2 \frac{c_t}{(1 - k_t)^t} f_t + \tau P \{ (1 + \delta) + [1 - (1 + \delta)c_1] f_1 - (1 + \delta)c_2 f_2 \} \quad (58)$$

and so the value of the pure single premium is

$$P = \frac{L(1 - \tau) \sum_{t=1}^2 \frac{c_t}{(1 - k_t)^t} f_t}{1 - \tau \{ (1 + \delta) + [1 - (1 + \delta)c_1] f_1 - (1 + \delta)c_2 f_2 \}}. \quad (59)$$

We wish to show how to introduce fuzziness into the future behavior of interest rates when pricing a property-liability contract. The fuzziness of this behavior is introduced by the discount function, which is fuzzy. Moreover, we will allow the total amount of the liabilities, L , to be estimated by an STF, i.e., $\tilde{L} = (L_C, L_R)$. This is easy to interpret from an intuitive point of view: the actuary estimates that the total cost of the claims will be "around L_C ." We will suppose that c_t , k_t , τ , and δ are crisp parameters.

If the discount factor and the total cost of the claims are done by FNs, we will actually obtain a fuzzy premium, which must be obtained by solving the fuzzy version of

Equation (55):

$$\tilde{P} = \tilde{L}(1 - \tau) \sum_{t=1}^2 \frac{c_t}{(1 - k_t)^t} \tilde{f}_t + \tau \tilde{P} \{(1 + \delta) + [1 - (1 + \delta)c_1] \tilde{f}_1 - (1 + \delta)c_2 \tilde{f}_2\}. \quad (60)$$

Buckley and Qu (1990c) suggested obtaining the solution of a fuzzy equation from the solution of its crisp version. So, we will obtain $\tilde{P}$ from (59) as

$$\tilde{P} = \frac{\tilde{L}(1 - \tau) \sum_{t=1}^2 \frac{c_t}{(1 - k_t)^t} \tilde{f}_t}{1 - \tau \{(1 + \delta) + [1 - (1 + \delta)c_1] \tilde{f}_1 - (1 + \delta)c_2 \tilde{f}_2\}}. \quad (61)$$

To determine the α -cuts of $\tilde{P}$, P_α , we must bear in mind that in Equation (59) the premium is a function of the value of the liabilities and the free risk discount function (or, alternatively a function of the expected evolution of interest rates), i.e., the premium in Equation (59) can be denoted as $P = f(L, f_1, f_2)$. Clearly, $f(\cdot)$ is an increasing function of L . On the other hand, it is not easy to know whether the premium increases (or decreases) when the discount function increases (the interest rates decreases) from the partial derivatives of $P(L, f_1, f_2)$. However, we do know by financial intuition that the price of the insurance is basically related to the present value of the liabilities, and that it is clearly increasing (decreasing) with respect to the values of the discount function (spot rates). Therefore, if we start from the discount function that define our TSIR, $\tilde{f}_t = (f_{t_C}, f_{t_R})$, we can obtain P_α by applying (7) as

$$P_\alpha = [P^1(\alpha), P^2(\alpha)]$$

$$= \left[\frac{(L_C - L_R(1 - \alpha))(1 - \tau) \sum_{t=1}^2 \frac{c_t}{(1 - k_t)^t} (f_{t_C} - f_{t_R}(1 - \alpha))}{1 - \tau \{(1 + \delta) + [1 - (1 + \delta)c_1](f_{1_C} - f_{1_R}(1 - \alpha)) - (1 + \delta)c_2(f_{2_C} - f_{2_R}(1 - \alpha))\}}, \right.$$

$$\left. \frac{(L_C + L_R(1 - \alpha))(1 - \tau) \sum_{t=1}^2 \frac{c_t}{(1 - k_t)^t} (f_{t_C} + f_{t_R}(1 - \alpha))}{1 - \tau \{(1 + \delta) + [1 - (1 + \delta)c_1](f_{1_C} + f_{1_R}(1 - \alpha)) - (1 + \delta)c_2(f_{2_C} + f_{2_R}(1 - \alpha))\}} \right]. \quad (62)$$

However, from (9), we can approximate $\tilde{P}$ by a STBN, i.e., $\tilde{P} \approx (P_C, P_R)$ where

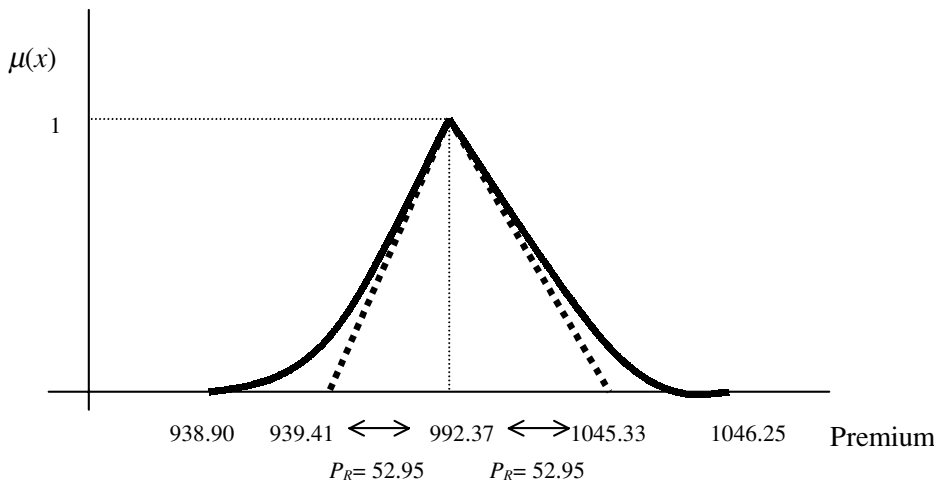
$$P_C = \frac{L_C(1 - \tau) \sum_{t=1}^2 \frac{c_t}{(1 - k_t)^t} f_{t_C}}{1 - \tau \{(1 + \delta) + [1 - (1 + \delta)c_1] f_{1_C} - (1 + \delta)c_2 f_{2_C}\}} \quad (63)$$

$$P_R = \frac{\partial P(L_C, f_{1_C}, f_{2_C})}{\partial L} L_R + \frac{\partial P(L_C, f_{1_C}, f_{2_C})}{\partial f_1} f_{1_R} + \frac{\partial P(L_C, f_{1_C}, f_{2_C})}{\partial f_2} f_{2_R}.$$

TABLE 5
Premiums for Several Payments of Losses

Pair (c_1, c_2)	(0.75, 0.25)	(0.5, 0.5)	(0.25, 0.75)
Premium	(1005.91, 53.54)	(992.37, 52.96)	(978.99, 52.35)

FIGURE 5
Fuzzy Premium of a Three Period Property-Liability Insurance With $c_1 = c_2 = 0.5$



In the following example we consider the following variables of a property-liability insurance: $k_1 = k_2 = 1\%$, $\delta = 5\%$, and $\tau = 34\%$ and the values of the discount function in Table 3. The duration of the contract is 2 years and the total amount of the liability for one contract is $\tilde{L} = (1000, 50)$. Table 5 shows the fuzzy premiums for several pairs (c_1, c_2) when using the approximating formula (63) of $\tilde{L}$.

Figure 5 represents the shapes of the true fuzzy premium (obtained from (62) and represented with a solid line) and our approximation (63), for the distribution of the claims $c_1 = 0.5$ and $c_2 = 0.5$. Clearly, the triangular approximation fits the real value of the premium well and is easier to interpret: the value of the fair premium must be 992.37 but there may be acceptable deviations no longer than 52.95.

DISCUSSING FURTHER ACTUARIAL APPLICATIONS OF FUZZY REGRESSION

In this section, we reflect on other applications of fuzzy regression to insurance problems. We will focus our discussion on the specific problem of estimating the incurred but not reported claims reserves, but we will also suggest other (in our opinion) promising applications.

Calculating the *Incurred But Not Reported Claims Reserves* (IBNR reserves) is a classic topic in nonlife-insurance mathematics. Unfortunately, they may not be calculated from a wide statistical database. Straub (1997) states that taking into account experiences that are too far from the present can lead to unrealistic estimates. For example, if

TABLE 6
IBNR-Triangle

Development Years	Accident Years						
	1	2		j		$n - 1$	n
1	$Z_{1,1}$	$Z_{1,2}$	$\dots$	$Z_{1,j}$	$\dots$	$Z_{1,n-1}$	$Z_{1,n}$
2		$Z_{2,2}$	$\dots$	$Z_{2,j}$	$\dots$	$Z_{2,n-1}$	$Z_{2,n}$
$\vdots$						$\vdots$	$\vdots$
i							
$\vdots$						$\vdots$	$\vdots$
$n - 1$						$Z_{n-1,n-1}$	$Z_{n-1,n}$
n							$Z_{n,n}$

the claims are related to bodily injuries, the future losses for the company will depend on the behavior of the wage index grown that will be taken to determine the amount of indemnification, changes in court practices, and public awareness of liability matters.

To calculate the IBNR reserve we begin from the historical data ordered in a triangle like that in Table 6.

In Table 6, $Z_{i,j}$ is the accumulated incurred losses of accident year j at the end of development year i where $j = 1$ denotes the most recent accident year and $j = n$ the oldest accident year. Obviously, we do not know, for the j th year of occurrence, the accumulated losses in the development years $i = j + 1, \dots, n$ and therefore, these losses must be predicted. It is well known that the classical method of predicting these losses is the Chain Ladder (CL). However, as it is pointed out in the survey England and Verrall (2002) claims reserving methods, during the last years greater interest of actuarial literature has been focused not only in calculating the best estimate of claim reserves but also on determining its downside potential from a stochastic perspective. Obviously, the final objective is to provide the actuary a well-founded mathematical tool to determine solvency margins for the reserves.

One way to focus this problem consists of departing from the pure CL method and making statistical refinements over it. In this way, Benjamin and Eagles (1986) propose a slight generalization of the CL method, known as London Chain Ladder (LCL), which is based on the use of OLS regression over the accumulated claims. On the other hand, Mack (1993) and England and Verrall (1999) do not suppose a concrete structure of the underlying data. Concretely, Mack (1993) provides analytical expressions for the prediction errors in claims and reserve estimates whereas England and Verrall (1999) propose combining standard CL estimates and bootstrapping techniques to determine the variance of the error in the predicted reserves.

Another extended way to focus this problem consists of modeling the incremental claims as random variables with a predefined distribution function. So, while Wright (1990) and Renshaw and Verrall (1998) use the Poisson random variables, Kremer (1982), Renshaw (1989), and Verrall (1989) use a log-normal approach. Then, the subsequent question to answer is how to define the associated parameters to these

random variables with respect to the year of underwriting and delay period. There are several approaches in the literature to do it. The more extended way consists of using one separate parameter for each development period. However, to avoid over-parameterization, some articles propose using parametric expressions (e.g., the Hoerl curve) or nonparametric smoothing methods like the one given by England and Verrall (2001). It must be remarked that, as it is pointed out in Mack (1993), in spite of the fact that these approaches are based on CL "philosophy," some of them present fundamental differences with respect to the pure CL method.

Our fuzzy sets approach to determine the value and variability of IBNR will be based on combining the generalization of CL by Benjamin and Eagles (1986), the LCL, and fuzzy regression. So, let us briefly expose Benjamin and Eagle's method. This is built from the hypothesis that the evolution of the claims of the accidents occurred in the year j from the i th year to the $i+1$ th year of development can be approximated by the linear relation:

$$Z_{i+1,j} = b_i + c_i Z_{i,j} + \varepsilon_i, \quad (64)$$

where b_i is the intercept, c_i is the slope, and ε_i is the perturbation term. The coefficients b_i and c_i must be estimated by OLS. Of course, the estimates of b_i and c_i , $\hat{b}_i$ and $\hat{c}_i$, respectively, must be obtained from the observations in the IBNR-triangle, i.e., from the pairs $\{(Z_{i+1,j}; Z_{i,j})\}_{j \geq i}$. Notice that the CL method is a special case of the LCL method when we consider that $b_i = 0$.

It is easy to check that the amount of the whole claims for the accidents occurred in the j th year at the end of the n years of development, $\hat{Z}_{n,j}$, is

$$\hat{Z}_{n,j} = \hat{b}_{n-1} + \hat{c}_{n-1} \{ \dots \hat{b}_{j+2} + \hat{c}_{j+2} [\hat{b}_{j+1} + \hat{c}_{j+1} (\hat{b}_j + \hat{c}_j Z_{j,j})] \}. \quad (65)$$

If we suppose that the expansion of the claims produced by the accidents in a given year is done within n years, then $\hat{Z}_{n,j}$ is the estimate of the amount of all the claims corresponding to the year of occurrence j . Therefore, the IBNR reserves corresponding to the accidents of the year j , R_j , is obtained by calculating the difference between $\hat{Z}_{n,j}$ and the amount of the claims reported, $Z_{j,j}$, i.e., $R_j = \hat{Z}_{n,j} - Z_{j,j}$. So, the total IBNR reserve (R) (corresponding to all the accidents from year 1 to year n) is⁷

$$R = \sum_{j=1}^n R_j. \quad (66)$$

We think that this approach has several drawbacks. First, OLS is useful when we start from a wide sample, but it is not advisable for calculating IBNR reserves. Similarly, using all the information available in the IBNR-triangle requires estimating $\hat{Z}_{n,j}$ not

⁷ Notice that with this approach, we do not consider investment income (i.e., the investment income is assumed to be 0%). However, this is the traditional approach to IBNR reserves. To simplify our explanation, we will continue with this hypothesis.

by exact values but by means of probabilistic confidence intervals. Unfortunately, this requires a great computational effort. In any case, we think that these considerations can be extended to the statistical methods discussed above.

We will show that adapting LCL method to fuzzy regression can be a suitable alternative. It will allow us to use all the information provided by the IBNR-triangle more efficiently. So, let us assume that the evolution of the accumulated claims of the accidents happened in the year j from the i th to the $i+1$ th developing years can be adjusted using a fuzzy linear relation $\tilde{Z}_{i+1,j} = \tilde{b}_i + \tilde{c}_i Z_{i,j}$. If we state that $\tilde{b}_i$ and $\tilde{c}_i$ are the STFNS $(b_{iC}, b_{iR}), (c_{iC}, c_{iR})$, respectively, we can write

$$\begin{aligned}\tilde{Z}_{i+1,j} &= (Z_{(i+1,j)C}, Z_{(i+1,j)R}) = (b_{iC}, b_{iR}) + (c_{iC}, c_{iR})Z_{i,j} \\ &= (b_{iC} + c_{iC}Z_{i,j}, b_{iR} + c_{iR}Z_{i,j}).\end{aligned}\quad (67)$$

The estimates of $\tilde{b}_i$ and $\tilde{c}_i$, are symbolized as $\hat{b}_i = (\hat{b}_{iC}, \hat{b}_{iR})$ and $\hat{c}_i = (\hat{c}_{iC}, \hat{c}_{iR})$, respectively. Then, the prediction of the final cost of the accidents produced in year j , $\hat{\tilde{Z}}_{n,j}$, is obtained from (65) as

$$\hat{\tilde{Z}}_{n,j} = \hat{b}_{n-1} + \hat{c}_{n-1} \left\{ \dots \hat{b}_{j+2} + \hat{c}_{j+2} \left[\hat{b}_{j+1} + \hat{c}_{j+1} (\hat{b}_j + \hat{c}_j Z_{j,j}) \right] \right\}. \quad (68)$$

Clearly, $\hat{\tilde{Z}}_{n,j}$ is not an STFNS, but it can be approximated reasonably well by a TSFNS, i.e., $\hat{\tilde{Z}}_{n,j} \approx (\hat{\tilde{Z}}_{(n,j)C}, \hat{\tilde{Z}}_{(n,j)R})$. To do this, we must, in the fuzzy recursive calculation (68), use the approximating formula (11) for multiplication between two STFNS. Finally, we obtain the IBNR reserve for the j th year of occurrence as the FNS $\tilde{R}_j = (R_{jC}, R_{jR})$:

$$\tilde{R}_j = \hat{\tilde{Z}}_{n,j} - Z_{j,j} = (\hat{\tilde{Z}}_{(n,j)C} - Z_{j,j}, \hat{\tilde{Z}}_{(n,j)R}). \quad (69)$$

Therefore, the whole IBNR reserve is the STFNS $\tilde{R} = (R_C, R_R)$, which is obtained from (66) as

$$\tilde{R} = \sum_{j=1}^n \tilde{R}_j = \left(\sum_{j=1}^n R_{jC}, \sum_{j=1}^n R_{jR} \right). \quad (70)$$

To illustrate our proposal, we develop the following example, which is similar to the one in Straub (1997, p. 106). Table 7 shows the IBNR triangle we will use.

Table 8 shows the main results when nonfuzzy methods are used. The development coefficients when we use fuzzy regression with an inclusion level $\alpha = 0.5$ and the fuzzy IBNR reserves are given in Table 9.

Let us illustrate how we have used fuzzy regression. If we use the fuzzy LCL with intercept, to obtain the development coefficients from the year $i = 1$ to the year $i = 2$, i.e., the STFBS $\hat{b}_1 = (\hat{b}_{1C}, \hat{b}_{1R})$ and $\hat{c}_1 = (\hat{c}_{1C}, \hat{c}_{1R})$, and taking a level of inclusion

TABLE 7
IBNR Triangle in Our Analysis

Development Year	Occurrence Year					
	1 (Y-6)	2 (Y-5)	3 (Y-4)	4 (Y-3)	5 (Y-2)	6 (Y-1)
1	130	125	150	150	140	125
2		213	252	258	232	211
3			413	392	374	340
4				549	449	490
5					539	613
6						613

TABLE 8
Determining IBNR Reserves by Nonfuzzy Methods

Development Coefficients				IBNR Reserves		
Year of Development	LCL with b_i (A)		LCL without b_i (B)	Year of Occurrence	R_j from (A)	R_j from (B)
i	$\hat{b}_i$	$\hat{c}_i$	$\hat{c}_i$			
1	-1.619	1.702	1.690	Y-6	477.67	453.97
2	61.398	1.336	1.592	Y-5	384.62	353.18
3	46.844	1.233	1.359	Y-4	261.16	276.41
4	158.854	0.927	1.228	Y-3	118.68	125.14
5	0	1	1	Y-2/Y-1	0.00	0.00
				R	1242.13	1208.70

$\alpha \geq \alpha^* = 0.5$, we must take the data in the second row of Table 7 and solve the following linear program:

$$\text{Minimize } 5\hat{b}_{1R} + 690\hat{c}_{1R}$$

subject to :

$$\hat{b}_{1C} - 0.5\hat{b}_{1R} + 125\hat{c}_{1C} - 62.5\hat{c}_{1R} \leq 213 \leq \hat{b}_{1C} + 0.5\hat{b}_{1R} + 125\hat{c}_{1C} + 62.5\hat{c}_{1R}$$

$$\hat{b}_{1C} - 0.5\hat{b}_{1R} + 150\hat{c}_{1C} - 75\hat{c}_{1R} \leq 252 \leq \hat{b}_{1C} + 0.5\hat{b}_{1R} + 150\hat{c}_{1C} + 75\hat{c}_{1R}$$

$$\hat{b}_{1C} - 0.5\hat{b}_{1R} + 150\hat{c}_{1C} - 75\hat{c}_{1R} \leq 258 \leq \hat{b}_{1C} + 0.5\hat{b}_{1R} + 150\hat{c}_{1C} + 75\hat{c}_{1R}$$

$$\hat{b}_{1C} - 0.5\hat{b}_{1R} + 140\hat{c}_{1C} - 70\hat{c}_{1R} \leq 232 \leq \hat{b}_{1C} + 0.5\hat{b}_{1R} + 140\hat{c}_{1C} + 70\hat{c}_{1R}$$

$$\hat{b}_{1C} - 0.5\hat{b}_{1R} + 125\hat{c}_{1C} - 62.5\hat{c}_{1R} \leq 211 \leq \hat{b}_{1C} + 0.5\hat{b}_{1R} + 125\hat{c}_{1C} + 62.5\hat{c}_{1R}$$

$$\hat{b}_{1R}, \hat{c}_{1R} \geq 0$$

and solving $\hat{b}_{1C} = -12$; $\hat{b}_{1R} = 0$; $\hat{c}_{1C} = 1.771$; $\hat{c}_{1R} = 0.057$.

TABLE 9
Determining IBNR Reserves With Fuzzy Regression

Year of Development i	Development Coefficients			IBNR Reserves		
	LCL with b_i (A)		LCL without b_i (B)	Year of Occurrence	$\tilde{R}_j$ from (A)	$\tilde{R}_j$ from (B)
1	$\hat{b}_i$	$\hat{e}_i$	$\hat{e}_i$	Y-6	(476.13, 80.48)	(439.62, 138.70)
2	(-12, 0)	(1.771, 0.057)	(1.689, 0.063)	Y-5	(385.49, 68.33)	(339.72, 114.01)
3	(106.553, 0)	(1.161, 0.11)	(1.579, 0.12)	Y-4	(259.12, 45.78)	(265.66, 88.63)
4	(-1.308, 0)	(1.344, 0.12)	(1.341, 0.12)	Y-3	(118.68, 0)	(123.93, 27.77)
5	(158.85, 0)	(0.927, 0)	(1.226, 0.051)	Y-2/Y-1	(0, 0)	(0, 0)
	(0, 0)	(1, 0)	(1, 0)	$\tilde{R}$	(1239.42, 194.59)	(1168.93, 369.1)

Finally, we would like to mention some other promising applications of fuzzy regression in actuarial science. One useful application may be in trending claims costs.⁸ When projecting future claims costs, standard actuarial practice uses time trend models. More academic approaches propose regressing the cost of the claims with respect to the value of an economic index (e.g., the consumer price index). So, to obtain the final prediction of the claims amount at a future moment we need to predict the value of the economic index at that moment. To do this, time trending may again be used. An alternative is to employ ARIMA time-series models.

Several instruments are derived from the fuzzy regression model in this article that may provide suitable solutions. Watada (1992) proposed a fuzzy model for time series analysis based on polynomial time trending with STFNs. Tseng et al. (2001) combined conventional ARIMA models with Tanaka and Ishibuchi's regression method. Obviously, in both cases these fuzzy methods will lead to forecasting with STFNs.

Another way of determining the discount rate of the liabilities is to use CAPM (see, e.g., Taylor, 1994, for a review of CAPM applied to insurance). With CAPM, the beta of the liabilities (β_L) is obtained from the beta of the asset portfolio (β_A) and the insurer's equity (β_E), i.e., $\beta_L = f(\beta_A, \beta_E)$ and $\beta_L < 0$. Cummins and Derrig (1997, p. 25) suggested *fuzzifying* the statistical estimates of the betas since, as they point out, there are several sources that disturb their quantification. Fuzzy regression provides a suitable way of doing this *fuzzification* and enables the decision maker to grade the level of uncertainty in the final estimates by choosing the appropriate level of congruency in fuzzy regression (α^*).

CONCLUSIONS

Actuarial pricing requires an estimate of the behavior of future interest rates that are unknown when the valuation is made. In the last few years, the analysis of temporal structure of interest rates has become an important topic in actuarial science. Likewise, several articles in the actuarial literature consider that estimating the discount rates using fuzzy numbers is a good alternative. With this in mind, we have attempted to make up for the uncertainty about estimating interest rates with fuzzy numbers, i.e., to develop the hypothesis "*an 'expert' subjectively estimates the yield rates.*" If we accept that the "experts" are the traders of the fixed income markets, these subjective estimates are implied in the price of the debt instruments and, therefore, in the yield curve of these instruments. Our method quantifies the experts' subjective estimates of the spot rates and spot interest rates for the future (the forward rates) with fuzzy numbers. This method, based on a fuzzy regression technique, uses all the prices of the bonds negotiated throughout one session in such a way that we do not lose any information. On the other hand, when we use an econometric method we must reduce these prices to representative ones, thus losing some information. We would like to remark that the results of our method are similar to those of Carriere (1999), but that he suggests fitting the yield curve using probabilistic confidence interval values whereas we fit the temporal structure of interest rates with fuzzy numbers. So, if we represent these fuzzy numbers by their α -cuts, we obtain a yield curve described by a direct analog of confidence intervals.

⁸ For an extensive exposition see Cummins and Derrig (1993).

We have adjusted the yield curve with symmetrical TFNs because the arithmetic is easy and interpreting the estimates is intuitive because they are well adapted to the way people make predictions. An interest rate given by $(0.03, 0.005)$ indicates that we expect an interest rate of about 3 percent, and that we do not expect deviations from it to be greater than 50 basis points. We then discussed how to use a fuzzy TSIR to calculate the present value of a stream of nonrandom amounts and the net single premium for some life and property-liability insurance contracts. We showed that to obtain the fuzzy prices is easy because our method fits the TSIR by a discount function that is described with symmetrical TFNs.

Finally, we discussed other actuarial problems where applying fuzzy regression is promising. We concentrated our discussion on calculating the IBNR reserves but also indicated other future areas of research such as trending cost claims and estimating the beta of liabilities.

Notice that when our initial data is given by fuzzy numbers, the estimate of the objective magnitude (the premiums, the IBNR reserve, etc.) is not an exact value but a fuzzy number. For example, in “Fuzzy Financial Pricing of Property-Liability Insurance” the value of the premium for property-liability insurance was $(992.37, 52.36)$. This can be understood as “the premium must be approximately 992.37.” To obtain the definitive value of the magnitude, it must be transformed into a crisp value. To do this we need to apply a *defuzzifying* method—see Zhao and Govind (1990) for a wide discussion of fuzzy mathematics, and Cummins and Derrig (1997) or Terceño et al. (1996) for applications in fuzzy-actuarial analysis. Another way of doing this, which is very consistent with practice in the real world, is to consider the fuzzy quantification as a first approximation that allows a margin for the “actuarial subjective judgment” or upper and lower bounds for acceptable market prices. Finally, the actuary must use his/her intuition and experience to establish the crisp value of the fuzzy estimate. For example, for premium $(992.37, 52.36)$, the actuary might decide that a final price 1017.37 is acceptable but 928 is not.

APPENDIX

Obtaining the Forward Rates From a Fuzzy TSIR

The forward rate for the t th year, $\tilde{r}_t$, can be obtained from the fuzzy equation

$$\tilde{f}_{t-1} (1 + \tilde{r}_t)^{-1} = \tilde{f}_t. \quad (\text{A1})$$

To solve this equation, we identify $(1 + \tilde{r}_t)^{-1} = \tilde{G}_t$, where $\tilde{G}_t$ is the value in $t-1$ of one monetary unit payable at t according to the TSIR. Bearing in mind that the value of the discount function for any t is an STFN $\tilde{f}_t = (f_{tC}, f_{tR})$, the above equation can therefore be written using the α -cuts as:

$$\begin{aligned} & [f_{(t-1)C} - f_{(t-1)R}(1 - \alpha), f_{(t-1)C} + f_{(t-1)R}(1 - \alpha)] \cdot [G_t^1(\alpha), G_t^2(\alpha)] \\ & = [f_{tC} - f_{tR}(1 - \alpha), f_{tC} + f_{tR}(1 - \alpha)]. \end{aligned} \quad (\text{A2})$$

It is easy to see that in Equation (A2):

$$G_t^1(\alpha) = (1 + r_t^2(\alpha))^{-1} \quad \text{and} \quad G_t^2(\alpha) = (1 + r_t^1(\alpha))^{-1}. \quad (\text{A3})$$

The solution of the α -cut Equation (A3) is:⁹

$$G_{t\alpha} = [G_t^1(\alpha), G_t^2(\alpha)] = \left[\frac{f_{tC} - f_{tR}(1 - \alpha)}{f_{(t-1)C} - f_{(t-1)R}(1 - \alpha)}, \frac{f_{tC} + f_{tR}(1 - \alpha)}{f_{(t-1)C} + f_{(t-1)R}(1 - \alpha)} \right]. \quad (A4)$$

Unfortunately, the solution (A4) might not exist (i.e., the expression (A4) does not correspond to a confidence interval). For example, consider that (A4) for a predefined α is given by $[0.9, 0.95]$. $[G_t^1(\alpha), G_t^2(\alpha)] = [0.875, 0.9]$. Then $[G_t^1(\alpha), G_t^2(\alpha)] = [0.972, 0.947]$. Clearly, it is not a confidence interval since $0.972 < 0.947$. From Buckley and Qu (1990a, p. 46, Theorem 3), the necessary and sufficient condition for the existence of $\tilde{G}_t$ is

$$\frac{f_{tR}}{f_{(t-1)R}} \geq \frac{f_{tC}}{f_{(t-1)C}} \Rightarrow f_{(t-1)C} \cdot f_{tR} \geq f_{tC} \cdot f_{(t-1)R}. \quad (A5)$$

If we define P in Equations (40d), (40e), (40f), and (40g) as being lower than or equal to the periodicity used to obtain the spot rates and forward rates, and $g_j(t)$ as nondecreasing, it is easy to check that $f_{tR} \geq f_{(t-1)R}$. Moreover, if we combine the constraints (40d) and (40e), we see that $f_{(t-1)C} \geq f_{tC}$. Therefore, $f_{(t-1)C} \cdot f_{tR} \geq f_{tC} \cdot f_{(t-1)R}$, and we can obtain the α -cuts of $\tilde{G}_t$ with (A4).

Then, from (A3) and (A4), we obtain the α -cuts of $\tilde{r}_t, r_{t\alpha}$ by making:

$$\begin{aligned} r_{t\alpha} &= [r_t^1(\alpha), r_t^2(\alpha)] \\ &= \left[\frac{f_{(t-1)C} + f_{(t-1)R}(1 - \alpha)}{f_{tC} + f_{tR}(1 - \alpha)} - 1, \frac{f_{(t-1)C} - f_{(t-1)R}(1 - \alpha)}{f_{tC} - f_{tR}(1 - \alpha)} - 1 \right]. \end{aligned} \quad (A6)$$

It is easy to check that the extremes of $r_{t\alpha}$ are not given as a linear expression of α . However, if we approximate the functions $r_t^1(\alpha), r_t^2(\alpha)$ using Taylor's expansion to the first grade from $\alpha = 1$, then:

$$\begin{aligned} r_t^1(\alpha) &\approx \frac{f_{(t-1)C}}{f_{tR}} - 1 - \frac{f_{(t-1)C} \cdot f_{tR} - f_{tC} \cdot f_{(t-1)R}}{(f_{tC})^2} (1 - \alpha) \quad \text{and} \\ r_t^2(\alpha) &\approx \frac{f_{(t-1)C}}{f_{tR}} - 1 + \frac{f_{(t-1)C} \cdot f_{tR} - f_{tC} \cdot f_{(t-1)R}}{(f_{tC})^2} (1 - \alpha). \end{aligned} \quad (A7)$$

In conclusion, from (A7) it is easy to check that $\tilde{r}_t$ can be approximated by an STFN $\tilde{r}_t \approx (r_{tC}, r_{tR})$ where:

$$r_{tC} = \frac{f_{(t-1)C}}{f_{tR}} - 1 \quad \text{and} \quad r_{tR} = \frac{f_{(t-1)C} \cdot f_{tR} - f_{tC} \cdot f_{(t-1)R}}{(f_{tC})^2} (1 - \alpha). \quad (A8)$$

⁹ In this case we apply the so-called solution "classical solution" in fuzzy sets literature. This solution does not always exist but as we will specify the functions g_j as nondecreasing, we can ensure that it does, as we will show.

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