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Fair Valuation of a Guaranteed Life Insurance Participating Contract Embedding a Surrender Option

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Abstract

In this article we deal with the problem of pricing a guaranteed life insurance participating policy, sold in the Italian market, which embeds a surrender option. This feature is an American-style put option that enables the policyholder to sell back the contract to the insurer at the cash surrender value. Employing a recursive binomial formula patterned after the Cox, Ross, and Rubinstein (1979) discrete option pricing model we compute, first of all, the total price of the contract, which also includes a compensation for the participation feature ("participation option," henceforth). Then this price is split into the value of three components: the *basic contract*, the *participation option*, and the *surrender option*. The numerical implementation of the model allows us to catch some comparative statics properties and to tackle the problem of suitably fixing the contractual parameters in order to obtain the premium computed by insurance companies according to standard actuarial practice.

Introduction

Life insurance contracts and pension plans are often very complex contingent claims that embed several financial options, both of European and of American style. According to Smith (1982) and Walden (1985), they can even be seen as option packages.

A typical example of European (put) option is implied by the maturity guarantees present in most types of equity-linked life insurance products. An accurate analysis of such guarantees is undoubtedly one of the main challenges in the life insurance business, more and more forced to broaden the range of its own products in order to force the increasing competition with financial markets. Large contributions in this direction have been made by financial and actuarial researchers since

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the pioneering work by Brennan and Schwartz (1976, 1979a, 1979b) and Boyle and Schwartz (1977).[1]

Another example of European (call) option is implied by the participation mechanism that characterizes policies with profits. Such a mechanism applies when "dividends" are credited to the policy reserve, thus producing an increase in the insurer's liabilities (benefits). This special feature has been studied, for instance, by Briys and de Varenne (1997), Grosen and Jørgensen (2000, 2002), Miltersen and Persson (2000), and Bacinello (2001), and in Europe is often referred to as "bonus."

In particular, Bacinello (2001) analyses a life insurance endowment policy with a minimum interest rate guaranteed in which both the benefit and the periodical premiums are annually adjusted according to the performance of a special investment portfolio. This contract is actually offered in the Italian market. Under the Black and Scholes (1973) and Merton (1973) framework, Bacinello (2001) expresses, first of all, the fair price of such a policy in terms of one-year call options, and then derives a very simple closed-form relation that characterizes fair contracts. However, as a concluding remark, Bacinello (2001) points out that an important issue connected with participating policies, which has not been dealt with in the article, is constituted by the presence of a surrender option. A surrender option is an American-style put option that entitles its owner (the *policyholder*) to sell back the contract to the issuer (the *insurer*) at the *cash surrender value*. Fair valuation of such an option, as well as an accurate assessment of the cash surrender values, are clearly crucial topics in the management of a life insurance company, both on the solvency and on the competitiveness sides.

The aim of the present article is just to fill this gap. We consider the single-premium version of the contract analyzed by Bacinello (2001) and define, first of all, a rule for computing the cash surrender values, which introduces an additional contractual parameter in the model. Then, by modeling the assets according to Cox, Ross, and Rubinstein (1979), we obtain a recursive algorithm for computing the fair price of the whole contract. Of course, this algorithm explicitly uses death and survival probabilities, since the contract can be surrendered only if it has not been surrendered yet and the insured is still alive. As in Bacinello (2001), the fair price of the corresponding participating contract without the surrender option is expressible in closed form, so that the value of the surrender option can be obtained residually. Then the total price is split into the values of three components: the *basic contract* (i.e., without profits and surrender), the *participation option*, and the *surrender option*. Although these embedded options are not sold separately from the other elements of the contract, we believe that such a decomposition can be very useful to an insurance company since it allows it to understand the incidence of the various components on the premium and, if necessary, to identify possible changes in the design of the policy. Moreover, the problem of choosing a set of contractual parameters that lead to a given level for the premium emphasized by Bacinello (2001) can be numerically solved within the same model.

The problem of valuing the surrender option embedded in life insurance products has already been tackled, under different assumptions and by various methodolo-

[1] For a categorization of the literature on equity-linked life insurance contracts with minimum guarantees see, e.g., Bacinello and Persson (2002).

gies, by Albizzati and Geman (1994), Grosen and Jørgensen (1997, 2000), and Jensen, Jørgensen, and Grosen (2001).

Albizzati and Geman (1994) consider a financial contract with a guaranteed interest rate ("contrat à taux garanti") proportional to the initial yield on a zero-coupon bond with the same maturity. Taking into account both initial expenses and taxes, Albizzati and Geman (1994) compare, at any given future date, the (deterministic) final value of the contract with the final value of a new one, having the same maturity and acquired by reinvesting the (guaranteed) cash surrender value at the prevailing market conditions. Financial uncertainty is then given by the evolution of the price of a zero-coupon bond with a fixed maturity. In particular, under the Heath, Jarrow, and Morton (1992) model and with the specification of a deterministic volatility structure, Albizzati and Geman (1994) derive a closed-form expression for the price of a European-style surrender option (i.e., of an option exercisable only at a fixed date). Then they use pooling arguments for "averaging" this price with respect to all possible exercise dates.

Grosen and Jørgensen (1997) instead consider a unit-linked contract with a minimum interest rate guaranteed. This contract can be surrendered at any time before its maturity, and the minimum guarantee is effective also in the case of early termination. Under the Black and Scholes (1973) and Merton (1973) framework, Grosen and Jørgensen (1997) express the total value of the minimum guarantee and the surrender option as the price of a standard American put option in an adjusted Black-Merton-Scholes economy in which the market rate is replaced by its spread over the minimum guaranteed interest rate.

Finally, a participating contract embedding a surrender option is also analyzed, in the Black-Merton-Scholes framework, and priced by means of a "Monte Carlo + binomial lattice" approach in Grosen and Jørgensen (2000), by a finite difference approach in Jensen, Jørgensen, and Grosen (2001).

The present article is organized as follows. In the next section we describe the structure of the contract and define all the liabilities that the insurer has to face. Then we introduce our valuation framework. After that, we derive the fair value of the contract and of all its components describing, in particular, our recursive algorithm. Next, we present some numerical results that allow us (i) to catch the comparative statics properties of the model, (ii) to discuss the possibility of suitably choosing the contractual parameters in order to obtain the premium computed by insurance companies according to standard actuarial practice. Finally, we conclude the article by hinting at some problems involved by the extension of the model to periodical premium contracts.

THE STRUCTURE OF THE CONTRACT

Consider a single-premium life insurance endowment policy issued at time 0 and maturing T years later (at time T). As is well known, under this contract the insurer is obliged to pay a specified amount of money (*benefit* or *sum insured*) to the beneficiary if the insured dies within the term of the contract or survives the maturity date. More precisely, we assume that, in case of death during the tth year of contract, the benefit is paid at the end of the year, i.e., at time t ($t = 1, \dots, T$); otherwise it is paid at maturity T.

We denote by x the age of the insured at time 0, by C_1 the "initial" sum insured, payable in case of death during the first year of contract, and by C_t the benefit payable at time t ($t = 2, \ldots, T$). As we will see in a moment, while C_1 is given, for $t > 1$ C_t is contingent on the performance of a special portfolio of assets (*reference portfolio*, henceforth). The insurer directly manages this portfolio and shares the profits with the policyholders.

Italian insurance companies price this contract exactly as a standard endowment policy with constant benefit C_1. Therefore, the (net) single premium, which we denote by U, is computed as follows:

$$U = C_1 A^{(i)}_{x:\overline{T}|} = C_1 \left[\sum_{t=1}^{T-1} (1+i)^{-t} {}_{t-1|}q_x + (1+i)^{-T} {}_{T-1}p_x \right]. \tag{1}$$

Here i (≥ 0) represents an annual compounded interest rate that, from now on, we will call *technical* rate, ${}_{t-1|}q_x$ denotes the probability that the insured dies within the tth year of contract (i.e., between times $t-1$ and t), and ${}_{T-1}p_x$ is the probability that he is still alive at time $T-1$. As usual, these probabilities depend on the age of the insured x, and are extracted from a mortality table that constitutes, together with i, the so-called *first-order bases*.

Observe that such premium U is expressed as an expected value of the benefit C_1 discounted from the random time of payment to time 0 with the technical rate i. Then, this premium makes the contract "fair" at inception with regard to the first order bases. Moreover, it implies that a return at the technical rate i is preassigned to the policy.

However, the participation feature forces the benefit to change, year by year. To see how this happens, we first introduce the following notation: g_t represents the rate of return on the reference portfolio during the tth year of contract, and η, between 0 and 1, identifies a participation coefficient. At the end of each policy year (except the last one), if the insured is still alive, the (prospective) policy reserve is "adjusted" at a rate δ_t defined as

$$\delta_t = \max\left\{ \frac{\eta g_t - i}{1+i}, 0 \right\}, \quad t = 1, \ldots, T-1. \tag{2}$$

This means that the reserve is proportionally increased, at the rate δ_t, since a "dividend" is credited to the policy.

We denote by F_t^- (F_t^+) the prospective policy reserve at time t ($t = 1, \ldots, T-1$), just before (respectively after) the adjustment. F_t^- is computed as the expected value of the current benefit C_t discounted from the random time of payment to time t with the technical rate i:

$$\begin{aligned} F_t^- &= C_t A^{(i)}_{x+t:\overline{T-t}|} = C_t \left[\sum_{h=1}^{T-t} (1+i)^{-h} {}_{h-1|}q_{x+t} + (1+i)^{-(T-t)} {}_{T-t}p_{x+t} \right], \\ t &= 1, \ldots, T-1. \end{aligned} \tag{3}$$

Here ${}_{h-1|}q_{x+t}$ represents the probability that the insured dies within the $(t+h)$th year of contract (i.e., between times $t+h-1$ and $t+h$) conditioned on the event that he is alive at time t, and ${}_{T-t}p_{x+t}$ is the probability that the insured is still alive at time T conditioned on the same event. Then

$$F_t^+ = F_t^-(1+\delta_t), \quad t = 1, \ldots, T-1. \tag{4}$$

The "dividend" $\delta_t F_t^-$ is used for purchasing an additional standard endowment policy with constant benefit $C_{t+1} - C_t$ and maturity T. The price of this additional policy is also computed by means of the first-order bases, i.e.,

$$(C_{t+1} - C_t)\, A^{(i)}_{x+t:\overline{T-t|}} = \delta_t F_t^-, \quad t = 1, \ldots, T-1.$$

Exploiting relations (3) and (4), it is easy to verify that the benefit is adjusted at the same rate of the reserve, i.e.,

$$C_{t+1} = C_t(1+\delta_t), \quad t = 1, \ldots, T-1, \tag{5}$$

and thus

$$F_t^+ = C_{t+1} A^{(i)}_{x+t:\overline{T-t|}}, \quad t = 1, \ldots, T-1. \tag{6}$$

It is also useful to express the benefit directly, so that its path dependence is immediately perceptible:

$$C_t = C_1 \prod_{k=1}^{t-1} (1+\delta_k), \quad t = 2, \ldots, T. \tag{7}$$

Taking into account the adjustment mechanism and the preassigned rate of return at the technical rate i, and disregarding the surrender possibility, we observe that the total return granted to the policyholder during the tth year of contract (except the year at the end of which the benefit is paid) is given by

$$(1+i)(1+\delta_t) - 1 = \max\{i, \eta g_t\}\,;$$

hence i can be interpreted as a minimum interest rate guaranteed. Moreover, if we consider the surrender option and the fact that the premium implicitly includes a compensation for it, we argue that the minimum interest rate guaranteed is even greater than i. However, it must be pointed out that such interpretation of i is correct only if the premium paid by the policyholder is expressed by relation (1). If, instead, it is different, for instance, greater, we can still state that there is a minimum guarantee provision in the contract since δ_t cannot be negative, but no more that the total rate of return on the policy in a given year is bounded from below by i. In particular, in the following sections we will neglect any kind of interpretation for i and simply consider it a contractual parameter that intervenes in the definition of the liabilities. We will then compute the "fair"[2] value of all the liabilities, and only in the numerical section will we discuss the problem of suitably choosing the contractual parameters in order that this value equals U.

We observe that the mechanism of dividend distribution described here is usually referred to as the *Compound Reversionary Bonus* method in the UK, where the dividend does not include only the excess of the actual investment return over the anticipated return at the technical rate i, but also the excess of expected mortality costs over actual costs, and the excess of expense loadings over actual expenses.[3] Our product is also very similar to the *Fully Variable Life Insurance* policies commonly sold in the United States,[4] and can be traced back to the "New York Life Model," from the name of the insurance company that developed it in 1969.[5] The main difference is that, in our model, the dividend cannot be negative because of the minimum guarantee provision. Moreover, the U.S. policies are usually paid by periodical premiums.

Coming now to the surrender conditions, we assume that surrender takes place (if the contract is still in force) at the beginning of the year, just after the announcement of the benefit for the coming year. The cash surrender value of an endowment life insurance policy is almost never given by the full policy reserve. There are several reasons to justify this, one of which is the need to avoid financial losses to the insurance company. A traditional and very simple way of computing this value was to apply a percentage, often decreasing with time-to-maturity, to the policy reserve suitably modified in order to take expenses into account.[6] Another "empirical" rule applied by some Italian insurance companies to periodical premium contracts is to discount the proportionate paid-up value of the policy[7] from maturity to the surrender date with a suitable rate, usually greater than i. As illustrative examples, in the numerical section we will consider two cases. In the first case we will follow the empirical rule, and replace the proportionate paid-up value with the current benefit in order to adapt it to a single premium contract. In the second case we follow the traditional rule and apply a constant percentage to the policy reserve.[8] In both cases the cash surrender value depends on a contractual parameter (the discount rate to apply to the benefit or, respectively, the percentage to apply to the policy reserve). We denote this parameter

[2] Based on a set of assumptions that we will introduce in the next section.
[3] See Booth et al. (1999).
[4] See Bowers et al. (1997).
[5] See Pitacco (2000).
[6] For a discussion of the rules and rationales concerning surrender conditions see, e.g., Black and Skipper (2000), Bowers et al. (1997), Lumsden (1991), and Pitacco (2000).
[7] This value, given by the current level of the benefit times the ratio between the number of premiums actually paid and the total number of premiums payable throughout the policy term, is the reduced benefit often granted to periodical premium endowment policies when the policyholder stops paying premiums.
[8] We do not need to modify this reserve because we are dealing with a single premium contract and, moreover, in this article we disregard expenses and relative loadings of the premiums.

by ρ, so that the cash surrender value at the beginning of the $(t+1)$th policy year (i.e., at time t) can be represented as

$$R_t = f(\rho, C_{t+1}, T - t, \ldots), \quad t = 0, \ldots, T-1, \tag{8}$$

where the function f will be specified in the numerical section.[9]

We remark that according to our assumptions surrender can also take place at time 0, just after the payment of the single premium. However, if the contract is fairly priced (in particular, as we will see in the next section, if arbitrage opportunities are ruled out of the market), R_0 is obviously less than (or, at the most, equal to) the premium, so that this is not an actual possibility for a rational and nonsatiated policyholder. Moreover, the surrender rule f and the contractual parameter ρ can be fixed in such a way that the cash surrender values are penalizing (to different degrees) or not, but the marketability of the policy could be seriously jeopardized when the cash surrender values are too low, even if this fact very likely implies a zero-value for the surrender option and hence a cheaper contract. Then, as we have already stated in the first section, the problem of choosing an adequate level for the contractual parameter ρ (given the surrender rule f) is also a crucial topic in the design of the product under scrutiny.

As a final and, indeed, very important remark concerning cash surrender values, we note that in the United States these values also play a crucial role in the qualification of life insurance in order to benefit by favorable tax treatment. In fact the U.S. Internal Revenue Code, Section 7702, establishes that life insurance policies must be considered life insurance under applicable state law when one of two alternative quantitative tests are met during the entire life of the contract.[10] In particular, the former test, called *cash-value accumulation test*, states that the cash surrender value cannot exceed the net single premium required to fund future benefits, that is, the prospective policy reserve of a paid-up contract such as a single premium contract. The interest rate used to compute this boundary is the maximum between the technical rate i and 4%. Therefore, this is another relevant issue to take into account in the design of the U.S. analogous products.

THE VALUATION FRAMEWORK

The contract described in the previous section is a typical example of a contingent claim, since it is affected by both mortality and financial risk. Whereas the mortality risk determines the moment in which the benefit is due, the financial risk affects the amount of the benefit and the surrender decision. We assume, in fact, that financial and insurance markets are perfectly competitive, frictionless,[11] and free of arbitrage opportunities. Moreover, all the agents are supposed to be rational and nonsatiated,

[9] Also in the second case, the cash surrender value depends (indirectly) on the current benefit C_{t+1} and on the time-to-maturity $T - t$ through the policy reserve. We explicitly emphasize such dependencies in relation (8) because we will exploit them later.

[10] See Black and Skipper (2000) and the references therein.

[11] In particular, there are no taxes, no transaction costs such as, e.g., expenses and relative loadings of the insurance premiums, and short sale is allowed.

and to share the same information. Therefore, in this framework, the surrender decision can only be the consequence of a rational choice, taken after comparison, at any time, between the total value of the policy (including the option of surrendering it in the future) and the cash surrender value.

As is standard in actuarial practice, we assume that mortality does not affect (and is not affected by) the financial risk, and that the mortality probabilities introduced in the previous section are extracted from a *risk-neutral* mortality measure, i.e., that all insurance prices are computed as expected values with respect to this specific measure. If, in particular, the insurance company is able to extremely diversify its portfolio in such a way that mortality fluctuations are completely eliminated, then the above probabilities coincide with the "true" ones. Otherwise, if mortality fluctuations do occur, then the "true" probabilities are "adjusted" in such a way that the premium, expressed as an expected value, is implicitly charged by a *safety loading*, which represents a compensation for accepting the mortality risk. In this case the adjusted probabilities derive from a *change of measure*, as often occurs in the financial economics environment; that is why we have called them "risk neutral."

Coming now to the financial set-up, we assume that the rate of return on risk-free assets is deterministic and constant, and denote by r the annual compounded riskless rate. The financial risk that affects the policy under scrutiny is then generated by a stochastic evolution of the rates of return on the reference portfolio. In this connection, we assume that it is a well-diversified portfolio, split into *units*, and that any kind of yield is immediately reinvested and shared among all its units. Therefore the reinvested yields increase only the unit price of the portfolio but not the total number of units that changes when new investments or withdrawals are made.

These assumptions imply that the rates of return on the reference portfolio are completely determined by the evolution of its unit price. Denoting by G_τ this unit price at time τ (≥ 0), we have then

$$g_t = \frac{G_t}{G_{t-1}} - 1, \quad t = 1, \ldots, T-1. \tag{9}$$

For describing the stochastic evolution of G_τ, we choose the discrete model by Cox, Ross, and Rubinstein (1979), universally acknowledged for its important properties. In particular, it may be seen either as an "exact" model under which "exact" values for both European- and American-style contingent claims can be computed, or as an approximation of the Black and Scholes (1973) and Merton (1973) model to which it asymptotically converges.

In more detail, we divide each policy year into N subperiods of equal length, let $\Delta = 1/N$, fix a volatility parameter $\sigma > \sqrt{\Delta}\ln(1+r)$, set $u = \exp(\sigma\sqrt{\Delta})$ and $d = 1/u$. Then we assume that G_τ can be observed at the discrete times $\tau = t + h\Delta$, $t = 0, 1, \ldots$; $h = 0, 1, \ldots, N-1$ and that, conditionally on all relevant information available at time τ, $G_{\tau+\Delta}$ can take only two possible values: uG_τ ("up" value) and dG_τ ("down" value).

As is well known, in this discrete setting absence of arbitrage is equivalent to the existence of a risk-neutral probability measure under which all financial prices, discounted by means of the risk-free rate, are martingales. Under this risk-neutral measure, the probability of the event $\{G_{\tau+\Delta} = uG_\tau\}$ conditioned on all information available at time τ (i.e., in particular, on the knowledge of the value taken by G_τ), is given by

$$q = \frac{(1+r)^\Delta - d}{u-d}, \tag{10}$$

whereas

$$1 - q = \frac{u - (1+r)^\Delta}{u-d}$$

represents the risk-neutral (conditioned) probability of $\{G_{\tau+\Delta} = dG_\tau\}$. We observe that, in order to prevent arbitrage opportunities, we have fixed σ in such a way that $d < (1+r)^\Delta < u$, which implies a strictly positive value for both q and $1-q$.

The above assumptions imply that g_t, $t = 1, 2, \ldots, T-1$, are i.i.d. and take one of the following $N+1$ possible values

$$\gamma_j = u^{N-j} d^j - 1, \quad j = 0, 1, \ldots, N \tag{11}$$

with (risk-neutral) probability

$$Q_j = \binom{N}{j} q^{N-j} (1-q)^j, \quad j = 0, 1, \ldots, N. \tag{12}$$

Moreover, the adjustment rates of the benefit, δ_t, $t = 1, 2, \ldots, T-1$, are i.i.d., and can take $n+1$ possible values, given by

$$\mu_j = \frac{\eta\gamma_j - i}{1+i}, \quad j = 0, 1, \ldots, n-1 \tag{13}$$

with probability Q_j, and 0 with probability $1 - \sum_{j=0}^{n-1} Q_j$. Here

$$n = \left[\frac{N}{2} + 1 - \frac{\ln(1 + i/\eta)}{2\ln(u)} \right], \tag{14}$$

with $[y]$ the integer part of a real number y, represents the minimum number of "downs" such that a call option on the rate of return on the reference portfolio in a given year with exercise price i/η does not expire in-the-money.

THE FAIR VALUE OF THE CONTRACT AND ITS COMPONENTS

Under the assumptions described in the previous section, and in particular, taking into account that all the probabilities introduced so far are risk neutral and that the mortality uncertainty is independent of the financial one, the fair values of the European-style components of the contract can be computed in two separate stages: in the first stage the market value at time 0 of the benefit due at time t in case of death of the insured during the tth year of contract is computed for all $t = 1, 2, \ldots, T-1$, along with the market value of the benefit due at maturity T; in the second stage all these values are "averaged" with the probabilities of payment at each possible date. We recall that, for $t = 1, 2, \ldots, T-1$, these probabilities are given by ${}_{t-1|}q_x$, whereas the probability that the benefit is due at maturity T is given by ${}_{T-1|}q_x + {}_{T}p_x = {}_{T-1}p_x$.

The fair value of the basic contract: U^B. Recalling that we have called "basic contract" a standard endowment policy with benefit C_1 (without profits and without the surrender option), we have

$$U^B = C_1 A^{(r)}_{x:\overline{T}|} = C_1 \left[\sum_{t=1}^{T-1} (1+r)^{-t} {}_{t-1|}q_x + (1+r)^{-T} {}_{T-1}p_x \right]. \tag{15}$$

The fair value of the nonsurrendable participating contract: U^P. To compute this value we need, first of all, to compute the market price at time 0 of the benefit C_t, due at time $t = 1, 2, \ldots, T$. We denote this price by $\pi(C_t)$. Whereas

$$\pi(C_1) = C_1(1+r)^{-1}, \tag{16}$$

for $t > 1$

$$\pi(C_t) = E^Q\big[(1+r)^{-t} C_t\big],$$

where E^Q denotes the expectation taken with respect to the (financial) risk-neutral measure introduced in the previous section.

Recalling relation (7), and exploiting the stochastic independence of δ_k, $k = 1, 2, \ldots, T-1$, we have, first of all

$$\pi(C_t) = C_1(1+r)^{-t} \prod_{k=1}^{t-1} E^Q[1+\delta_k].$$

Then, taking into account that δ_k, $k = 1, 2, \ldots, T-1$, are also identically distributed, we have

$$\pi(C_t) = C_1(1+r)^{-t}(1+\mu)^{t-1} = \frac{C_1}{1+\mu}\left(\frac{1+r}{1+\mu}\right)^{-t}, \qquad t = 2, 3, \ldots, T, \tag{17}$$

where

$$\mu = E^Q[\delta_k] = \sum_{j=0}^{n-1} \mu_j Q_j, \qquad k = 1, 2, \ldots, T-1,$$

with Q_j, μ_j, and n defined in relations (12) to (14).

Observe that

$$\frac{\mu(1+i)}{\eta(1+r)} = \frac{1+i}{\eta} E^Q\big[(1+r)^{-1}\delta_k\big] = E^Q\bigg[(1+r)^{-1}\max\bigg\{g_k - \frac{i}{\eta}, 0\bigg\}\bigg]$$

represents the market price, at the beginning of each year of contract, of a European call option on the rate of return on the reference portfolio with maturity at the end of the year and exercise price i/η.

Finally, the fair value U^P is given by

$$U^P = \sum_{t=1}^{T-1} \pi(C_t)_{t-1|}q_x + \pi(C_T)\,{}_{T-1}p_x = \frac{C_1}{1+\mu} A^{(\lambda)}_{x:T\rceil}, \tag{18}$$

where

$$\lambda = \frac{r-\mu}{1+\mu} \quad \text{and} \quad A^{(\lambda)}_{x:T\rceil} = \sum_{t=1}^{T-1}(1+\lambda)^{-t}{}_{t-1|}q_x + (1+\lambda)^{-T}{}_{T-1}p_x.$$

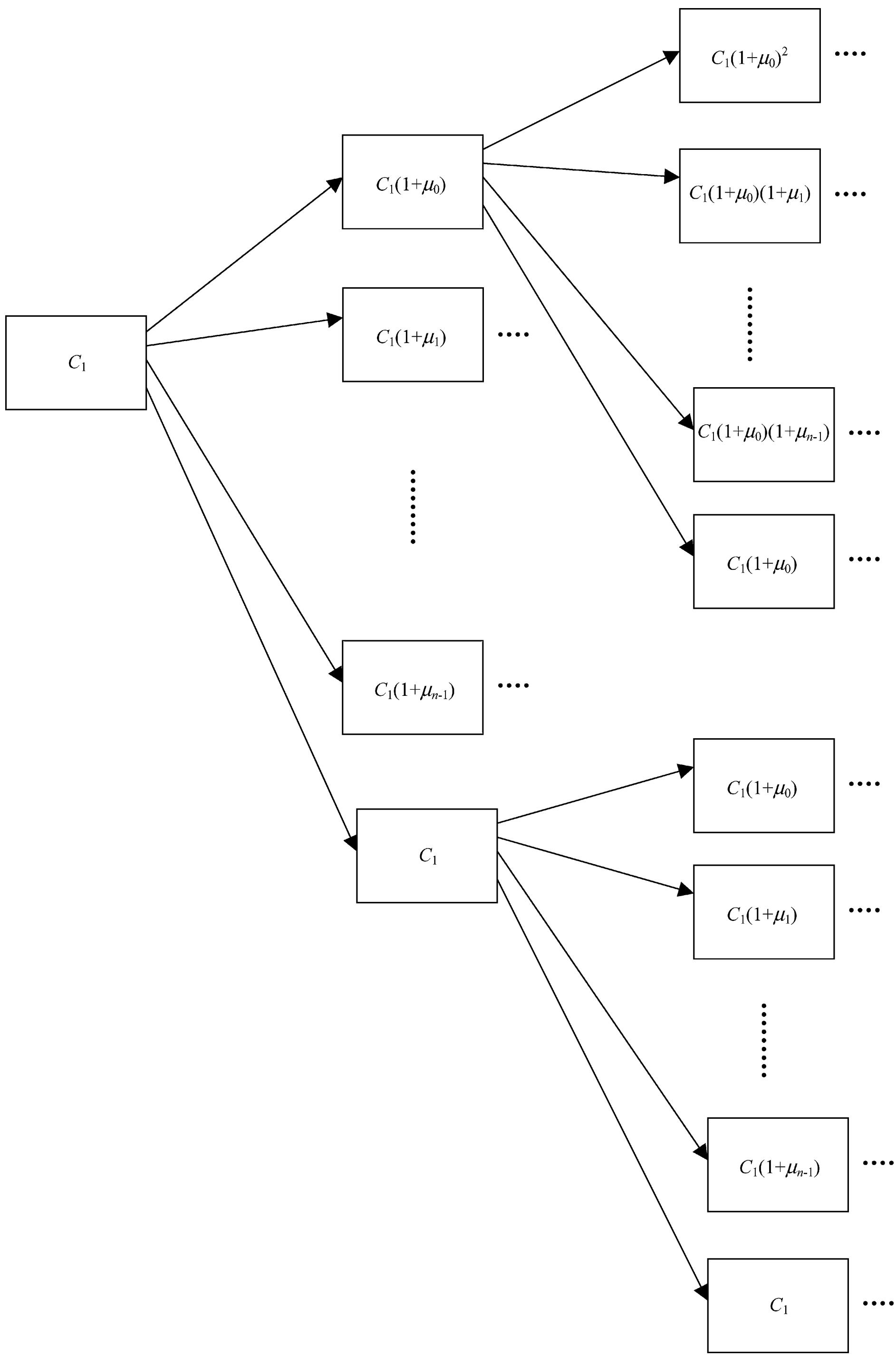
The fair value of the participation option: B. The value of this option is simply given by the difference between U^P and U^B:

$$\begin{aligned} B &= U^P - U^B \\ &= C_1\Bigg\{\sum_{t=2}^{T-1}(1+r)^{-t}[(1+\mu)^{t-1}-1]\,{}_{t-1|}q_x \\ &\qquad + (1+r)^{-T}[(1+\mu)^{T-1}-1]\,{}_{T-1}p_x\Bigg\}. \end{aligned} \tag{19}$$

The fair value of the whole contract: U^T. Under our assumptions, the stochastic evolution of the benefit $\{C_t, t = 1, 2, \dots, T\}$ can be represented by means of an $(n+1)$-nomial tree. In the root of this tree we represent the initial benefit C_1 (given); then each node of the tree has $n+1$ branches that connect it to $n+1$ following nodes (see Figure 1). In the nodes at time t we represent the possible values of C_{t+1}. The possible trajectories that the stochastic process of the benefit can follow from time 0 to time t ($t = 1, 2, \dots, T-1$) are $(n+1)^t$, but not all these trajectories lead to different nodes. The tree is, in fact, *recombining*, and the different nodes (or *states of nature*) at time t are $\binom{n+t}{n}$.

In the same tree we can also represent the cash surrender values defined by relation (8), the fair price of the whole contract, and a *continuation* price that we are going to define immediately. The last two prices can be computed by means of a backward recursive procedure operating from time $T-1$ to time 0. In particular, in each step and node the fair price of the whole contract is given by the maximum between the cash surrender value and the continuation price.

FIGURE 1
The Stochastic Evolution of the Benefit



To see this we denote, first of all, by $\{V_t, t = 0, 1, \ldots, T-1\}$ and $\{W_t, t = 0, 1, \ldots, T-1\}$ the stochastic processes composed by the fair values of the whole contract, and the continuation values, respectively, at the beginning of the $(t+1)$th year of contract (time t), and let $U^T = V_0$. Then, observing that in each node at time $T-1$ (if the insured is alive) the continuation value is given by

$$W_{T-1} = (1+r)^{-1} C_T \tag{20}$$

since in case of continuation the benefit C_T is due with certainty at time T, we have

$$V_{T-1} = \max\{W_{T-1}, R_{T-1}\}. \tag{21}$$

Now assume that, at time $t < T-1$, the insured is still alive and the benefit has reached a given node of the tree, denoted by K. In order to catch the link between values at time t and values at time $t+1$, we denote by C_{t+1}^K, R_t^K, V_t^K, and W_t^K the benefit, the cash surrender value, the fair price of the whole contract, and the continuation price at time t in the node K, and by $V_{t+1}^{K(j)}$, $W_{t+1}^{K(j)}$ $j = 0, 1, \ldots, n$, the fair value of the whole contract and the continuation value at time $t+1$ in each node following K. In more detail, $V_{t+1}^{K(j)}$ ($W_{t+1}^{K(j)}$ respectively) , $j = 0, 1, \ldots, n-1$, represent the value when $\delta_{t+1} = \mu_j$ (with risk-neutral probability Q_j), whereas $V_{t+1}^{K(n)}(W_{t+1}^{K(n)})$ represents the value corresponding to $\delta_{t+1} = 0$ (with probability $1 - \sum_{j=0}^{n-1} Q_j$) (see Figure 2).

We observe that in the node K to continue the contract means to receive, at time $t+1$, the benefit C_{t+1}^K if the insured dies within 1 year, or to be entitled to a contract whose total random value (including the option of surrendering it in the future) equals V_{t+1} if the insured survives. The continuation value at time t (in the node K) is then given by the risk-neutral conditional expectation of this payoff, discounted for 1 year with the risk-free rate

$$W_t^K = (1+r)^{-1}\left\{q_{x+t} C_{t+1}^K + p_{x+t}\left[\sum_{j=0}^{n-1} V_{t+1}^{K(j)} Q_j + V_{t+1}^{K(n)}\left(1 - \sum_{j=0}^{n-1} Q_j\right)\right]\right\}, \quad t = 0, 1, \ldots, T-2. \tag{22}$$

Here q_{x+t} denotes the probability that the insured, alive at time t, dies within 1 year, and $p_{x+t} = 1 - q_{x+t}$.

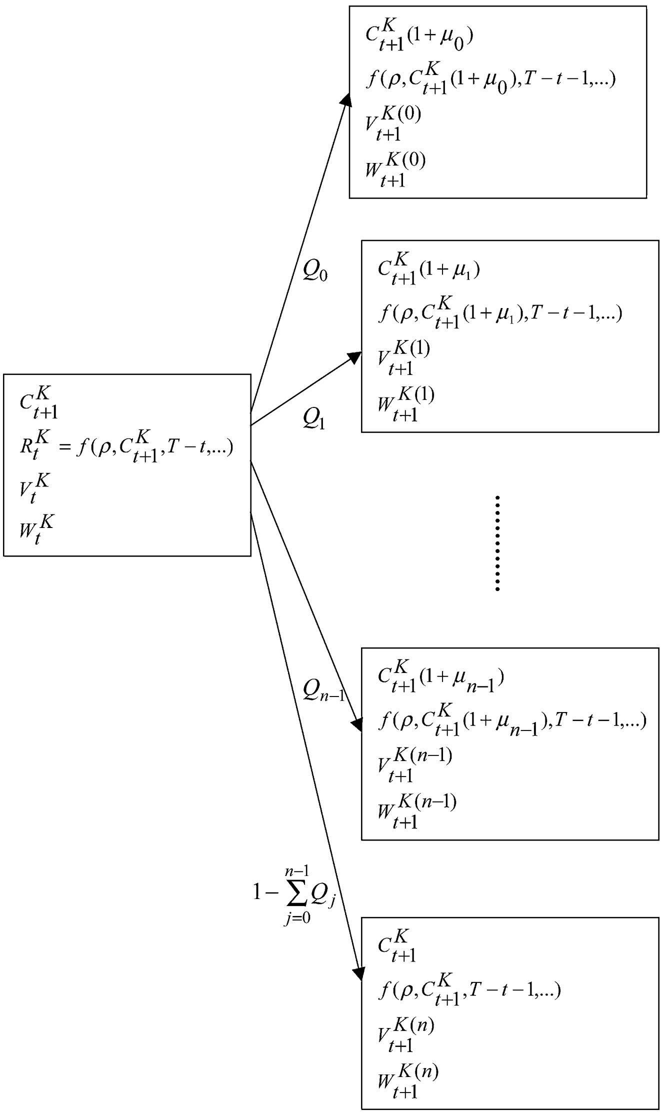
To conclude, we have then

$$V_t^K = \max\{W_t^K, R_t^K\}, \quad t = 0, 1, \ldots, T-2. \tag{23}$$

The fair value of the surrender option: S. The fair price at time 0 of the surrender option is given by the difference between U^T and U^P,

$$S = U^T - U^P. \tag{24}$$

FIGURE 2
Link between Values at Time t and Values at Time $t + 1$



NUMERICAL RESULTS

In this section we present some numerical results for the fair value of the contract and of all its components. To obtain these results we have extracted the mortality probabilities from the Italian Statistics for Females Mortality in 1991, fixed $C_1 = \$10{,}000$, $T = 5$, $N = 250$, and considered different values for the remaining parameters.

We observe that our choice for N implies a daily change in the unit price of the reference portfolio since there are about 250 trading days in a year. Moreover, this choice guarantees a very good approximation to the Black and Scholes (1973) and Merton (1973) model. In fact, if we assumed that the unit price of the reference portfolio follows a geometric Brownian motion with volatility parameter σ, then the market value, at the beginning of each year of contract, of a European call option with underlying variable the rate of return on the reference portfolio, exercise price i/η and maturity at the end of the year, would be given by

$$\phi(a) - \frac{1+i/\eta}{1+r}\phi(b),$$

where

$$a = \frac{\ln(1+r) - \ln(1+i/\eta)}{\sigma} + \frac{\sigma}{2}, \qquad b = a - \sigma,$$

and ϕ denotes the cumulative distribution function of a standard normal variate. In a very large number of numerical experiments carried out with different sets of parameters we have found that the difference between this Black and Scholes (1973) price and the one obtained in our model with $N = 250$ is less than 1 basis point (bp). However, this large number of steps in each year requires a large amount of CPU time; that is why we have not fixed a high value for T in our illustrative examples.

As already mentioned in the second section, we have specified two alternative rules for computing the cash surrender values. According to the former, the cash surrender value at the beginning of each year of contract is given by the current benefit discounted from maturity to the surrender date with an annual compounded rate ρ_1:

$$R_t = C_{t+1}(1+\rho_1)^{-(T-t)}, \quad t = 0, 1, \ldots, T-1. \tag{25}$$

According to the latter rule, the cash surrender value is a rate ρ_2 of the policy reserve F_t^+ defined by relation (6) for $t = 1, \ldots, T-1$, and of the premium U defined by relation (1) for $t = 0$:

$$\begin{aligned} R_t &= \rho_2 C_{t+1} A^{(i)}_{x+t:\overline{T-t}|} \\ &= \rho_2 C_{t+1}\left[\sum_{h=1}^{T-t}(1+i)^{-h}\,{}_{h-1|}q_{x+t} + (1+i)^{-(T-t)}\,{}_{T-t}p_{x+t}\right], \\ &\qquad\qquad t = 0, 1, \ldots, T-1. \end{aligned} \tag{26}$$

In order to get some numerical feeling and to catch some comparative statics properties of the model, we have fixed a basic set of values for the parameters x, r, i, η, σ, ρ_1, ρ_2, and then we have moved each parameter one at a time. For comparison, we have also computed the premium U defined by relation (1). As already discussed in the second section, if this is the single premium paid by the policyholder, then i can be interpreted as a minimum interest rate guaranteed.

The basic set of parameters, fixed in such a way that the fair price of the whole contract U^T is very close to U, is as follows:

$$x = 50, r = 0.05, i = 0.02, \eta = 0.5, \sigma = 0.15, \rho_1 = 0.035, \rho_2 = 0.985.$$

With these parameters we have obtained the following results:

$$U^B = \$7,845, B = \$1,084, U^P = \$8,930, U = \$9,062.$$

Moreover, if the cash surrender values are computed according to relation (25), then the fair price of the surrender option $S_{(1)} = \$128$ and that of the whole contract $U^T_{(1)} = \$9{,}058$. If, instead, the cash surrender values are expressed by relation (26), then $S_{(2)} = \$123$ and $U^T_{(2)} = \$9{,}053$.

Even without the aid of numerical results it is quite obvious that the fair value of the basic contract U^B increases with respect to the age of the insured x, decreases with the market rate r, and is constant with respect to the remaining parameters $i, \eta, \sigma, \rho_1, \rho_2$. As for the participation option B, it increases with the participation coefficient η and the volatility parameter σ, decreases with the age of the insured x and the technical rate i, is constant with respect to the surrender parameters ρ_1 and ρ_2; instead, the behavior of B is *a priori* undetermined with respect to the market rate r. The fair value of the nonsurrendable participating contract U^P increases with respect to η and σ, decreases with i, is constant with respect to ρ_1 and ρ_2, and undetermined with respect to x and r. The single premium U increases with x, decreases with i, and is constant with respect to $r, \eta, \sigma, \rho_1, \rho_2$. Finally, the fair value of the surrender option $S_{(j)}$ and that of the whole contract $U^T_{(j)}$ are *a priori* undetermined with respect to all the parameters except ρ_j ($j = 1, 2$). More precisely, if the cash surrender values are expressed by relation (25), then $S_{(1)}$ and $U^T_{(1)}$ both decrease with ρ_1; if, instead, relation (26) holds, then $S_{(2)}$ and $U^T_{(2)}$ increase with ρ_2. This behavior is summarized in Table 1.

From this behavior we can argue that, when the fair value of the nonsurrendable participating contract U^P is not greater than U, it is possible to find (numerically) a value of the surrender parameter ρ_j such that $U^T_{(j)} = U$. As we will see from the following tables, the remaining parameters can also be chosen in such a way that $U^T_{(j)} = U$. In more detail, in Table 2 we present the results obtained when x varies between 40 and 60 and in Table 3 are those obtained when r varies between 2% and 10% with step 0.5%. In Table 4 i varies between 0% and 5% with step 0.5%; in Table 5 η varies between 5% and 100% with step 5%; in Table 6 σ varies between 5% and 50% with step 5%. Finally, in Tables 7 and 8 we move the surrender parameters ρ_1 and ρ_2, from 0% to 5% and from 97% to 100% respectively, with step 0.5%. To better convey the numerical behavior of the premiums with respect to the various parameters, we also plot some of these results in Figures 3 to 8.

From the results reported in Table 2 and plotted in Figure 3 we can notice that the age of the insured seems to have a very small influence on the premiums, at least in the range of values considered here. The basic premium U^B is about 78% of the initial benefit C_1, the participation option is rather expensive (about 11% of this benefit), whereas the surrender option is very cheap (between 1.2% and 1.3% of C_1). Moreover,

TABLE 1
The Expected Behavior of the Premiums with Respect to the Various Parameters

	x	r	i	η	σ	ρ_1	ρ_2
U^B	↗	↘	=	=	=	=	=
B	↘	?	↘	↗	↗	=	=
U^P	?	?	↘	↗	↗	=	=
$S_{(1)}$	?	?	?	?	?	↘	=
$U^T_{(1)}$	?	?	?	?	?	↘	=
$S_{(2)}$	?	?	?	?	?	=	↗
$U^T_{(2)}$	?	?	?	?	?	=	↗
U	↗	=	↘	=	=	=	=

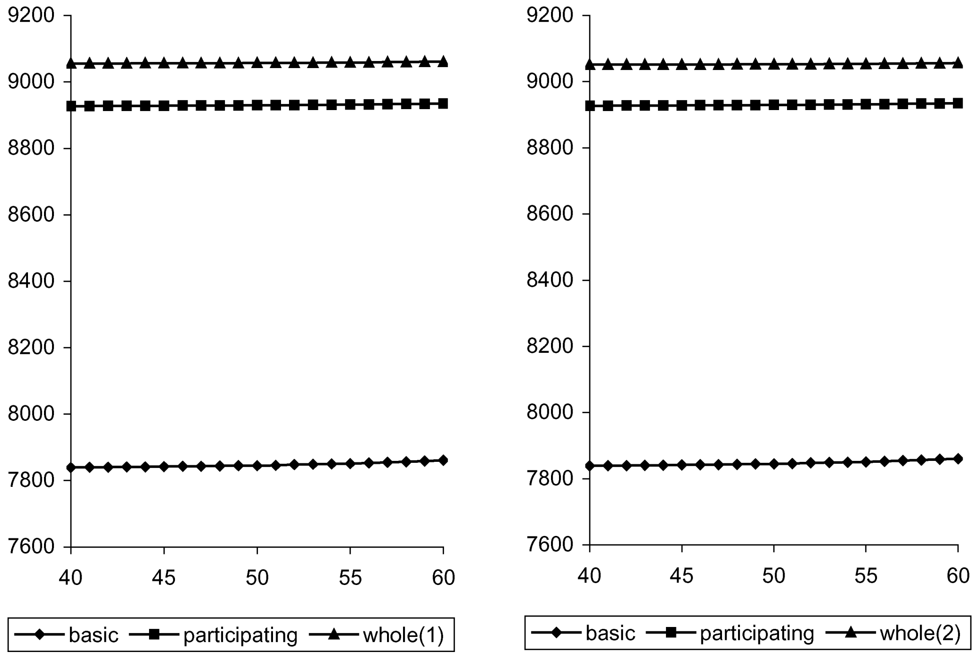
Note: Basic contract U^B, the participation option B, nonsurrendable participating contract $U^P = U^B + B$, surrender option $S_{(j)}$ $(j = 1, 2)$, whole contract $U^T_{(j)} = U^P + S_{(j)}$, single premium U; age of the insured x, market rate r, technical rate i, participation coefficient η, volatility coefficient σ, surrender parameters ρ_j $(j = 1, 2)$.

TABLE 2
The Premiums versus the Age of the Insured x (in dollar terms)

x	U^B	B	U^P	$S_{(1)}$	$U^T_{(1)}$	$S_{(2)}$	$U^T_{(2)}$	U
40	7839	1088	8927	129	9056	124	9052	9059
41	7840	1088	8927	129	9056	124	9052	9059
42	7840	1088	8928	129	9056	124	9052	9059
43	7841	1087	8928	129	9056	124	9052	9060
44	7841	1087	8928	129	9057	124	9052	9060
45	7842	1086	8928	128	9057	124	9052	9060
46	7843	1086	8929	128	9057	124	9052	9061
47	7843	1086	8929	128	9057	124	9052	9061
48	7844	1085	8929	128	9057	124	9053	9061
49	7845	1085	8929	128	9057	124	9053	9062
50	7845	1084	8930	128	9058	123	9053	9062
51	7846	1084	8930	128	9058	123	9053	9062
52	7848	1083	8930	128	9058	123	9053	9063
53	7849	1082	8931	128	9058	123	9054	9063
54	7850	1081	8931	127	9059	123	9054	9064
55	7851	1080	8932	127	9059	123	9054	9065
56	7853	1079	8932	127	9059	122	9055	9065
57	7855	1078	8933	127	9060	122	9055	9066
58	7857	1077	8934	127	9060	122	9056	9067
59	7859	1076	8934	126	9061	122	9056	9068
60	7861	1074	8935	126	9061	121	9057	9069

Note: Basic contract U^B, participation option B, nonsurrendable participating contract $U^P = U^B + B$, surrender option $S_{(j)}$ $(j = 1, 2)$, whole contract $U^T_{(j)} = U^P + S_{(j)}$, the single premium U.

FIGURE 3
The Premiums versus the Age of the Insured x



in all the examples reported here the fair value of the whole contract is (slightly) less than U, so that some contractual parameter (for instance, ρ_j) should be modified in order that $U^T_{(j)} = U$. Finally, the increasing trend of the basic premium U^B "beats" the decreasing trend of the participation option B, so that $U^P = U^B + B$ increases with x. Also, the surrender options $S_{(j)}$, $j = 1, 2$, decrease in value with x, but not so strongly to capsize the behavior of $U^T_{(j)} = U^P + S_{(j)}$, increasing with x.

From Table 3 (and Figure 4) we note that all the results reported are very sensitive with respect to the market rate r, and this is not surprising at all. The value of the basic contract ranges from 90.62% of C_1 (when $r = i = 2\%$) to 62.26% (when $r = 10\%$), and that of the participation option from 9.55% of C_1 to 12.79%, thus exhibiting an increasing trend. However, once again this trend is beaten by the trend of U^B, so that $U^P = U^B + B$ decreases with r (from 100.17% of C_1 to 75.05%). Moreover, observe that when $r = i = 2\%$, the nonsurrendable participating contract is quoted "over par." The surrender options $S_{(j)}$, $j = 1, 2$, both increase in value with r, but this behavior does not capsize the decreasing trend of $U^T_{(j)} = U^P + S_{(j)}$. In particular, both $S_{(1)}$ and $S_{(2)}$ are valueless if $r \leq 3.5\%$, $S_{(1)}$ reaches the level of 9.15% of C_1 and $S_{(2)}$ that of 14.21% when $r = 10\%$. Finally, there exists a level of r, between 4.5% and 5%, such that $U^T_{(j)} = U$ for $j = 1, 2$.

From Table 4 (and Figure 5) we can observe that the technical rate i has a strong influence on the value of the participation option B (as expected), which ranges from 14.89% of C_1 (when $i = 0$) to 6.46% (when $i = r = 5\%$). The same happens for U and

TABLE 3
The Premiums versus the Market Rate r (in dollar terms)

r	U^B	B	U^P	$S_{(1)}$	$U^T_{(1)}$	$S_{(2)}$	$U^T_{(2)}$
0.020	9062	955	10017	0	10017	0	10017
0.025	8844	977	9821	0	9821	0	9821
0.030	8633	999	9631	0	9631	0	9631
0.035	8427	1020	9448	0	9448	0	9448
0.040	8228	1042	9270	44	9314	40	9309
0.045	8034	1063	9097	87	9184	82	9179
0.050	7845	1084	8930	128	9058	123	9053
0.055	7662	1105	8767	168	8935	163	8930
0.060	7484	1126	8610	206	8816	316	8926
0.065	7311	1146	8457	242	8700	469	8926
0.070	7143	1166	8309	278	8587	617	8926
0.075	6979	1185	8165	312	8477	761	8926
0.080	6820	1205	8025	394	8420	901	8926
0.085	6666	1224	7890	530	8420	1036	8926
0.090	6515	1243	7758	662	8420	1168	8926
0.095	6369	1261	7630	790	8420	1296	8926
0.100	6226	1279	7505	915	8420	1421	8926

Note: Basic contract U^B, participation option B, nonsurrendable participating contract $U^P = U^B + B$. Surrender option $S_{(j)} (j = 1, 2)$, whole contract $U^T_{(j)} = U^P + S_{(j)}$, single premium $U = 9062$.

FIGURE 4
The Premiums versus the Market Rate r

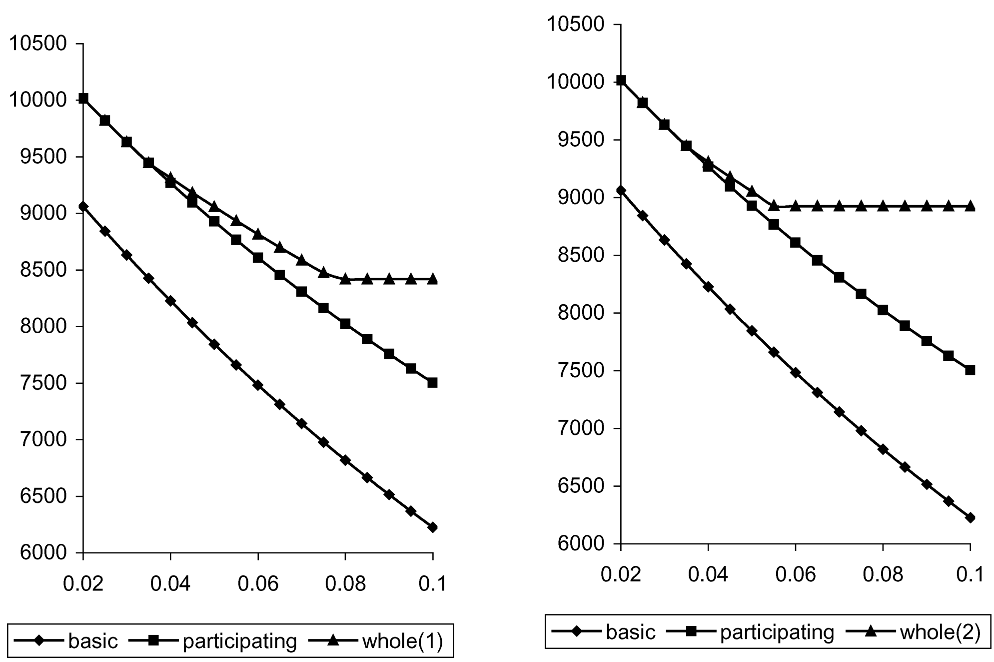
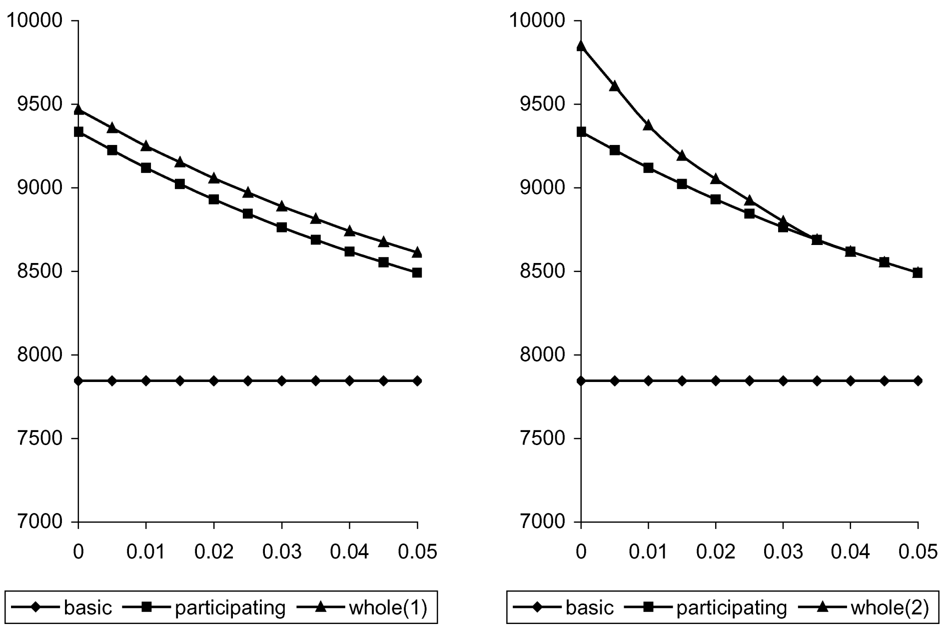


TABLE 4
The Premiums versus the Technical Rate i (in dollar terms)

i	B	U^P	$S_{(1)}$	$U^T_{(1)}$	$S_{(2)}$	$U^T_{(2)}$	U
0.000	1489	9335	134	9468	515	9850	10000
0.005	1380	9226	132	9358	383	9609	9755
0.010	1274	9120	131	9250	255	9374	9517
0.015	1178	9023	129	9153	169	9192	9286
0.020	1084	8930	128	9058	123	9053	9062
0.025	999	8845	127	8972	79	8924	8844
0.030	917	8763	126	8889	36	8799	8633
0.035	844	8689	125	8814	0	8689	8427
0.040	772	8618	124	8741	0	8618	8228
0.045	708	8554	123	8676	0	8554	8034
0.050	646	8492	122	8613	0	8492	7845

Note: Basic contract $U^B = 7845$, participation option B, nonsurrendable participating contract $U^P = U^B + B$, surrender option $S_{(j)}$ $(j = 1, 2)$, whole contract $U^T_{(j)} = U^P + S_{(j)}$, the single premium U.

FIGURE 5
The Premiums versus the Technical Rate i



the fair price of the surrender option $S_{(2)}$. Recall, in fact, that i negatively affects the cash surrender values when computed according to relation (26). The value of the surrender option $S_{(1)}$, instead, does not seem to be very sensitive with respect to i. Anyway, all the prices reported in Table 4 are decreasing with i and, in particular,

TABLE 5
The Premiums versus the Participation Coefficient η (in dollar terms)

η	B	U^P	$S_{(1)}$	$U^T_{(1)}$	$S_{(2)}$	$U^T_{(2)}$
0.05	3	7848	571	8420	1078	8926
0.10	53	7898	522	8420	894	8792
0.15	147	7993	427	8420	668	8660
0.20	261	8107	313	8420	423	8530
0.25	387	8232	188	8420	170	8402
0.30	520	8365	120	8485	116	8481
0.35	655	8501	122	8622	117	8618
0.40	796	8642	124	8766	119	8761
0.45	939	8785	126	8911	121	8906
0.50	1084	8930	128	9058	123	9053
0.55	1233	9078	130	9208	125	9204
0.60	1385	9230	132	9362	128	9358
0.65	1538	9384	135	9518	130	9513
0.70	1694	9539	137	9676	132	9671
0.75	1851	9697	139	9836	134	9831
0.80	2011	9856	141	9998	136	9993
0.85	2172	10018	144	10161	139	10156
0.90	2336	10181	146	10327	141	10322
0.95	2501	10347	148	10495	143	10490
1.00	2669	10514	151	10665	145	10659

Note: Basic contract $U^B = 7845$, participation option B, nonsurrendable participating contract $U^P = U^B + B$, surrender option $S_{(j)}\,(j = 1, 2)$, whole contract $U^T_{(j)} = U^P + S_{(j)}$, the single premium $U = 9062$.

$S_{(2)} = 0$ when $i \geq 3.5\%$. Finally, a value of i between 2% and 2.5% is such that $U^T_{(j)} = U$ for $j = 1, 2$.

As far as the participation coefficient η is concerned, we notice, from Table 5 and Figure 6, a very strong influence on the value of the participation option, that ranges from 0.03% of C_1 (when $\eta = 5\%$) to 26.69% (when $\eta = 100\%$). Observe, moreover, that the nonsurrendable participating contract is quoted over par when $\eta \geq 85\%$. Also, the values of the surrender options and, especially, $S_{(2)}$, are quite sensitive with respect to η. In particular, $S_{(1)}$, equal to 5.71% of C_1 when $\eta = 5\%$, decreases until 1.2% of C_1 when $\eta = 30\%$, then increases very slightly and reaches the value of 1.51% of C_1 when $\eta = 100\%$. Anyway, the nonmonotonicity of $S_{(1)}$ does not capsize the increasing trend of $U^T_{(1)} = U^P + S_{(1)}$. As for $S_{(2)}$, it decreases from 10.78% of C_1 (when $\eta = 5\%$) to 1.16% (when $\eta = 30\%$), and then slightly increases up to 1.45% of C_1 for $\eta = 100\%$. This behavior also influences the trend of $U^T_{(2)} = U^P + S_{(2)}$, that does not result in monotonicity as well. Finally, a value of η between 50% and 55% makes $U^T_{(j)} = U$ for $j = 1, 2$.

Most of the comments concerning the behavior of the premiums with respect to the participation coefficient η are still valid when referred to the volatility coefficient σ.

FIGURE 6
The Premiums versus the Participation Coefficient η

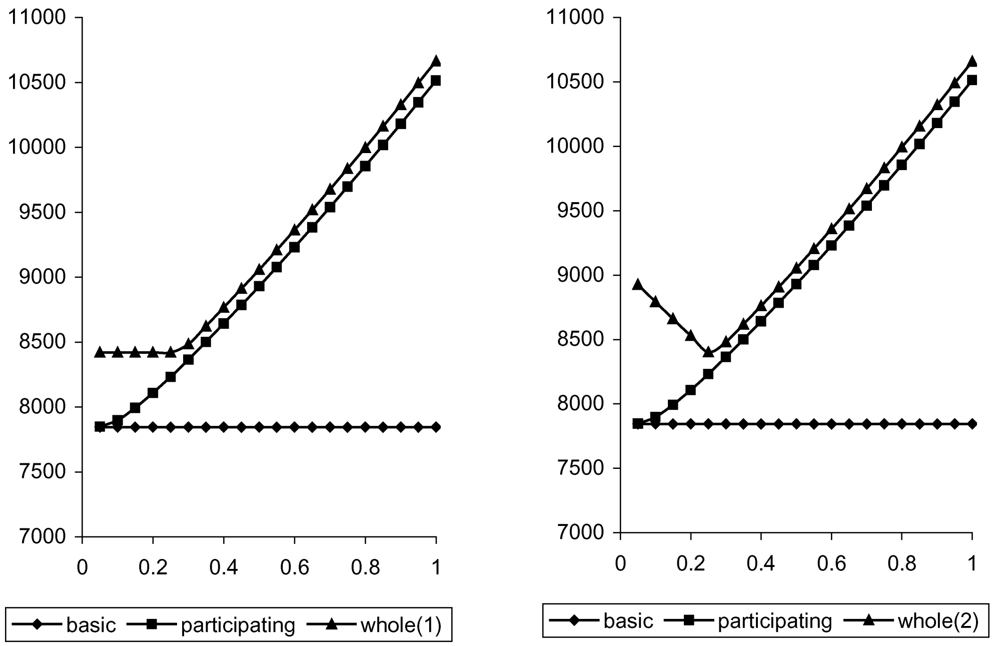


TABLE 6
The Premiums versus the Volatility Coefficient σ (in dollar terms)

σ	B	U^P	$S_{(1)}$	$U^T_{(1)}$	$S_{(2)}$	$U^T_{(2)}$
0.05	408	8253	166	8420	673	8926
0.10	740	8586	123	8709	206	8792
0.15	1084	8930	128	9058	123	9053
0.20	1440	9286	133	9419	128	9414
0.25	1804	9649	138	9788	133	9783
0.30	2175	10020	144	10164	139	10159
0.35	2559	10405	149	10554	144	10549
0.40	2953	10799	155	10954	149	10948
0.45	3356	11201	161	11362	155	11356
0.50	3767	11612	167	11779	161	11773

Note: Basic contract $U^B = 7845$, participation option B, nonsurrendable participating contract $U^P = U^B + B$, surrender option $S_{(j)}\,(j = 1, 2)$, whole contract $U^T_{(j)} = U^P + S_{(j)}$, single premium $U = 9062$.

From Table 6 (and Figure 7), in fact, we can observe that B is very sensitive with respect to σ, and ranges from 4.08% of C_1 (when $\sigma = 5\%$) to 37.67% (when $\sigma = 50\%$). Also, $S_{(2)}$ is quite sensitive, and not monotonic, with respect to σ, whereas $S_{(1)}$, not monotonic as well, does not seem to be very sensitive. The premium $U^T_{(1)} = U^P + S_{(1)}$ is increasing,

FIGURE 7
The Premiums versus the Volatility Coefficient σ

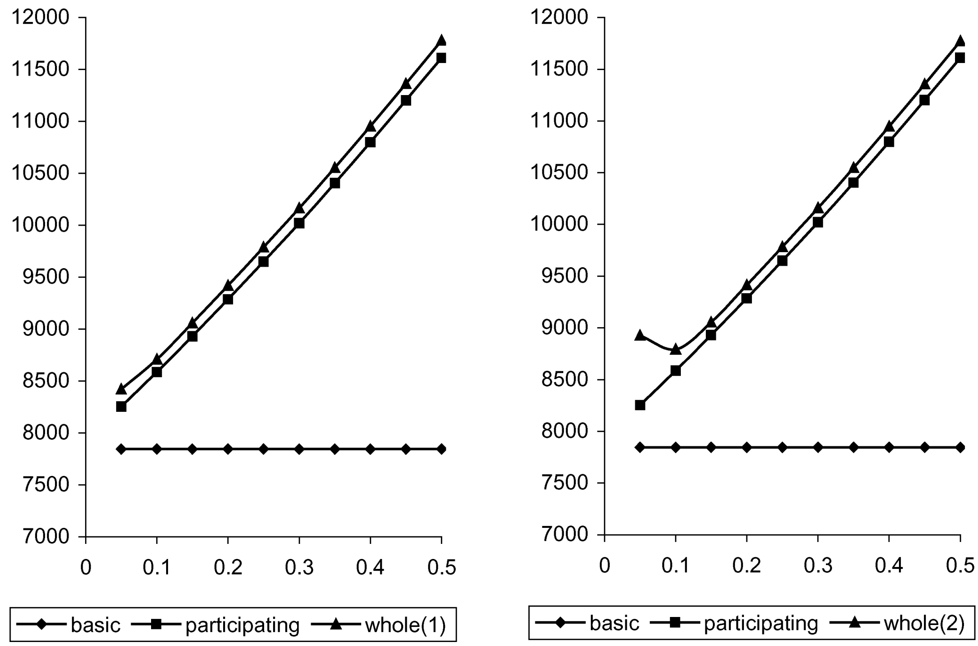


TABLE 7
The Premiums versus the Surrender Parameter ρ_1 (in dollar terms)

ρ_1	$S_{(1)}$	$U_{(1)}^T$
0.000	1070	10000
0.005	824	9754
0.010	585	9515
0.015	353	9283
0.020	260	9189
0.025	215	9145
0.030	172	9101
0.035	128	9058
0.040	85	9015
0.045	42	8972
$\geq$0.050	0	8930

Note: Basic contract $U^B = 7845$, participation option $B = 1084$, nonsurrendable participating contract $U^P = U^B + B$, surrender option $S_{(1)}$, whole contract $U_{(1)}^T = U^P + S_{(1)}$, single premium $U = 9062$.

FIGURE 8
The Premiums versus the Surrender Parameter ρ_j, $j = 1, 2$

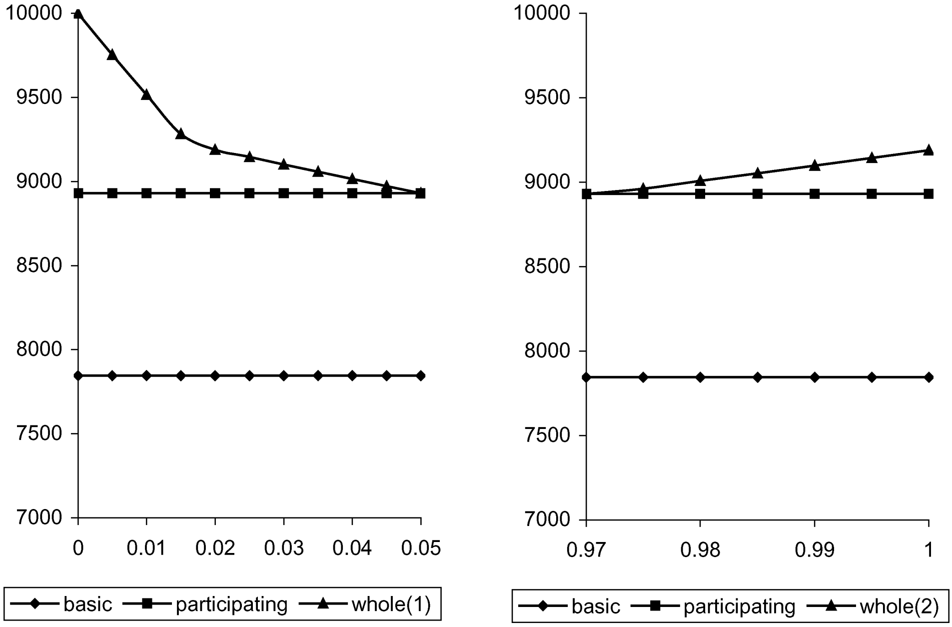


TABLE 8
The Premiums versus the Surrender Parameter ρ_2 (in dollar terms)

ρ_2	$S_{(2)}$	$U^T_{(2)}$
≤0.970	0	8930
0.975	32	8962
0.980	78	9008
0.985	123	9053
0.990	169	9098
0.995	214	9144
1.000	260	9189

Note: Basic contract $U^B = 7845$, participation option $B = 1084$, nonsurrendable participating contract $U^P = U^B + B = 8930$, surrender option $S_{(2)}$, whole contract $U^T_{(2)} = U^P + S_{(2)}$, single premium $U = 9062$.

whereas $U^T_{(2)} = U^P + S_{(2)}$ is not monotonic. The nonsurrendable participating contract is quoted over par when $\sigma \geq 30\%$, and there exists a value of σ, between 15% and 20%, such that $U^T_{(j)} = U$ for $j = 1, 2$.

From Table 7 and the first graph in Figure 8 we note that when the cash surrender values are computed according to relation (25), the influence of the discount rate ρ_1

is very strong, as expected. In particular, if $\rho_1 = 0$, i.e., if there are no penalties and surrender is treated as death, then the value of the surrender option $S_{(1)}$ is 10.7% of C_1 and the whole contract is quoted exactly at par. When instead $\rho_1 \geq 5\%$, then $S_{(1)} = 0$. Finally, there exists a value of ρ_1, between 3% and 3.5%, such that $U^T_{(1)} = U$.

When the cash surrender values are computed according to relation (26) the surrender option $S_{(2)}$, although being quite sensitive with respect to ρ_2, reaches the maximum value of only 2.6% of C_1 when $\rho_2 = 100\%$, and is valueless for $\rho_2 \leq 97\%$. Moreover, a value of ρ_2 between 98.5% and 99% makes $U^T_{(2)} = U$ (see Table 8 and the second graph in Figure 8).

CONCLUDING REMARKS

In this article we have analyzed a single premium life insurance endowment policy in which the benefit is annually adjusted according to the performance of a special investment portfolio. In addition to this participation mechanism, which is coupled with the provision of a minimum return guaranteed, the contract is also equipped with a surrender option, i.e., with an American-style option to sell it back before expiration at a price computed according to a predetermined formula (*cash surrender value*). Then this policy can be divided in three components: the *basic contract*, the *participation option*, and the *surrender option*. Assuming that the unit price of the reference portfolio follows the discrete model by Cox, Ross, and Rubinstein (1979), we obtain a closed-form expression for the fair value of the first two components, and present a recursive algorithm for computing the fair value of the third one. The numerical implementation of the model allows us to also address the problem of suitably choosing the contractual parameters in order that the fair price of the whole contract equals the premium computed by insurance companies according to standard actuarial practice.

The policy analyzed here is very often paid by annual premiums. However, the extension of the valuation model proposed here in order to compute the annual premium is not at all trivial. The fair price of the whole contract depends, in fact, on the value of the surrender option, which in turn depends on the annual premium. Moreover, even though this total price was given, in order to compute the annual premium it should be split into an annuity with installments paid only if the insured is still alive and the contract has not been surrendered yet. Then the annual premium also determines the value of this annuity through the surrender decision. A real vicious circle arises in this way and, what is more, the numerical solution of the problem, at least with a satisfactory level of accuracy, is thwarted by the high computational complexity of the model. This problem constitutes an important topic to be addressed in the near future.

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