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Pricing Catastrophe Insurance Futures Call Spreads: A Randomized Operational Time Approach

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ABSTRACT

Actuaries value insurance claim accumulations using a compound Poisson process to capture the random, discrete, and clustered nature of claim arrival, but the standard Black (1976) formula for pricing futures options assumes that the underlying futures price follows a pure diffusion. Extant jump-diffusion option valuation models either assume diversifiable jump risk or resort to equilibrium arguments to account for jump risk premiums. We propose a novel randomized operational time approach to price options in information-time. The time change transforms a compound Poisson process to a more trackable pure diffusion and leads to a parsimonious option pricing formula as a risk-neutral Poisson sum of Black's prices in information-time with only two unobservable variables—the information arrival intensity and the information-time futures volatility.

INTRODUCTION

Amid much fanfare, the Chicago Board of Trade (CBOT) launched catastrophe insurance futures contracts (CAT futures) in 1992 and catastrophe insurance futures call spreads (CAT spreads) in 1993, with contract values linked to the loss ratios compiled by the Insurance Services Office (ISO). The CBOT replaced the ISO

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loss ratios by new loss ratios compiled by Property Claims Services (PCS) in September 1995. These derivative contracts provide underwriters and risk managers an effective alternative to hedge and trade *systematic* catastrophic losses. Insurers traditionally seek protection against underwriting risk through the mechanism of reinsurance. Because of worldwide capacity shortage and the increasing number of natural catastrophes in recent years, however, the introduction of a viable substitute to reinsurance is viewed by the insurance industry to be both timely and desirable.

Unlike reinsurance negotiations that are costly, lengthy, and irreversible, insurance futures and call spreads—because they are standardized and exchange-traded—have the advantages of reversibility, transactional efficiency, and anonymity.[1] Besides underwriters, mainstream institutional investors may also enjoy the diversification effect of these derivatives since catastrophic losses show no correlation with the price movements of stocks and bonds. However, trading of CAT futures has dwindled to nearly nil according to a report in *Institutional Investor* (November 1994). Investment managers who are supposed to sell the contracts believe the risk involved is too great.[2] Meanwhile, insurers and risk managers appear to be more interested in buying CAT spreads, resembling layers or excess-of-loss reinsurance agreements, for high-risk coverage. They also like both the cheap price (typically a few thousand dollars per contract) and the much lower risk that the potential payout is limited to a fraction of the loss ratios. At the end of 1993, there was open interest of only 300 contracts, but by December 1994 the figure was over 6,000, with the most popular contract being the "third quarter eastern," which covers the U.S. east coast's hurricane season (July, August, and September).[3]

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[1] CAT futures contract value is $25,000 times the ratio of a quarter's catastrophic loss, reported to the ISO from a representative national pool of catastrophe policies, to the same quarter's estimated property premium. It is capped at $50,000 or 200 percent to limit the seller's exposure. Since the premium base is known and constant throughout the life of the contract, the only unknown variable is the incurred quarterly loss. Thus, as losses increase during the quarter, so will the value of the CAT futures contract. In 1994, the ISO index contains a pool of 26 companies. See Chicago Board of Trade (1995) for detailed contract specifications.

Like other call spreads, the CAT spread involves buying a call at one strike price and selling an otherwise identical call at a higher strike price. These calls have the same maturities as their underlying asset, the CAT futures contract.

However, insurers can tailor their reinsurance contracts to seek precise risk protection. Reinsurers also provide a banking function to finance a ceding company's growth by paying a reinsurance commission and by assuming a portion of the unearned premium reserve that the ceding company is required to maintain by state regulators. For more comparisons between reinsurance and CAT futures, see D'Arcy and France (1992, 1993).

[2] See Cummins and Geman (1995) for a discussion on other possible reasons.

[3] In fact, trading of the tailor-made over-the-counter CAT options, introduced in 1994 from such firms as Index Reinsurance Brokers and Guy Carpenter and Center Reinsurance, has experienced even more success than their exchange counterparts because of the flexibility offered.

The success of the CAT call spread contract warrants accurate pricing. However, despite the fact that actuaries have long modeled insurance claim accumulation using a compound Poisson process, the insurance industry employs the standard Black (1976) futures option pricing model (see Chicago Board of Trade, 1995) to price the contract. Black's model is an extension of the well-known Black-Scholes (1973) European option pricing model by assuming the underlying futures price process follows a Brownian motion.[4] The diffusion assumption ignores the sporadic nature of catastrophes and the subsequent concentration of (or jump in) the number of claims, and thus may lead to significant pricing errors. Whaley (1986) tests an American version of the Black model on SP-500 index futures options and finds that, during the 1983 testing period, there was a "moneyness" bias as well as a "maturity" bias. Since Whaley, no significant empirical study has reported on other futures options contracts. Similar pricing errors are found in tests of the Black-Scholes model using spot options and currency options. For example, Rubinstein (1985) finds that the Black-Scholes formula consistently undervalues out-of-the-money short-maturity options, because it ignores the fact that a significant jump may bring an option back into the money.[5] Bodurtha and Courtadon (1987) find that, for currency options with maturities less than 30 days, the degree of underpricing for out-of-the-money options was as high as 28.63 percent, and the overpricing for at-the-money options was as high as 9.61 percent. (However, the overpricing for in-the-money options was only 1.02 percent.)[6]

These results may be explained partly by the fact that the option's underlying asset's risk stems from random jumps, or surprises, rather than diffusion changes, or that the distribution of the underlying asset is leptokurtic. As a matter of fact, Jorion (1988) finds that 96 percent of the total currency risk and 36 percent of the total stock risk are jump risks. Thus, option valuation models based only on pure diffusions may not be adequate to explain actual prices. To verify whether a model that considers jumps is better suited for pricing currency options, Jorion shows that Merton's diffusion-jump formula (1976) can indeed explain 17.15 percent of the 28.63 percent and 4.62 percent of the 9.61 percent biases found in Bodurtha and Courtadon. Compared to the Black-Scholes model, the added jump component gives rise to fatter tails for the underlying asset return distribution, resulting in larger values for out-of-the-money and in-the-money options and smaller

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[4] Only when the arrival frequency approaches infinity and the jump size simultaneously approaches zero will a compound Poisson process approach a pure diffusion.

[5] Rubinstein (1985) carries out an extensive study on stock options by using a transaction data set. He looks at all reported trades on the 30 most active Chicago Board Options Exchange option classes between August 23, 1976, and August 31, 1978.

[6] Bodurtha and Courtadon (1987) apply an American version of the Black-Scholes formula to price currency options and find a definite maturity bias between February 28, 1983, and March 26, 1985. The formula underprices out-of-the-money options relative to at-the-money and in-the-money options. As time to maturity increases, the degree of mispricing dissipates.

values for at-the-money options. The shorter the maturity, the lower the jump intensity, and the deeper the moneyness, the larger the differences.

Since catastrophes arrive even less sporadically than, for example, the arrival of external shocks that cause exchange rates to jump, and usually with much larger jump sizes, one may reasonably argue that the use of the Black formula for pricing CAT options may produce significant pricing biases, and a pricing formula that considers the jump nature of catastrophe arrivals will be more accurate.[7] As a matter of fact, the compound Poisson process and its variations play an important role in traditional actuarial literature for its feasibility in computation and reasonable description of claims generated by a large and fairly homogeneous insurance portfolio. For example, see Beard, Pentikäinen, and Pesonen (1984) for risk theory applications.

Option pricing based on a compound Poisson process has also been well-developed in the financial literature. Merton (1976) first prices options in a jump-diffusion framework by assuming that jump risk is diversifiable. Later, Naik and Lee (1990), Ahn (1992), and Amin and Ng (1993) relax the assumption to permit systematic jump risk. Since jump risk cannot be hedged, they resort to equilibrium pricing to derive utility-dependent option pricing formulas. In insurance applications, Cummins and Geman (1994, 1995) price CAT call spreads using a jump-diffusion no-arbitrage framework. Recognizing that the option payoff is related to claim accumulation over the life of the option, they propose an Asian option approach to price the option with respect to the underlying nontraded CAT loss index. Since the CAT contracts only cover those catastrophes that arrive during an option's event-quarter such that the nonevent quarter covers the claims runoff, they add a jump component to the claim arrival diffusion process during the event-quarter only. However, they assume a given constant equilibrium market price of the jump risk as in Shimko (1992), in order to derive valuation results.

The purpose of this study is to price CAT call spreads with jump risk using a novel "randomized operation time" approach. Departing from previous approaches, we price the option not in the usual calendar-time but in an operation-time dimension such that the resulting isomorphic option pricing problem is easier to solve. The intuition is that a stochastic time change allows us to remove the jump stochastic volatility, and thus we can derive risk-neutral valuation results. Our approach differs from the Asian option approach proposed by Cummins and Geman (1994, 1995) in the sense that we price the option directly with respect to the traded CAT futures contracts, whose current prices reflect the market's aggregate expectation of the cumulative catastrophic claims over the life of the option.

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[7] By definition, a catastrophe is an event which causes in excess of $5 million in insured property damages and affects a significant number of insured and insurers.

As mentioned previously, a compound Poisson process becomes a pure diffusion when the arrival frequency approaches infinity and the jump size simultaneously approaches zero. Thus, the Black formula is actually a limiting case of the jump formula. In this regard, we may argue that, theoretically, the jump formula ought to outperform the diffusion formula.

The "randomized operational time" concept in probability theory (Feller, 1971) dictates that a simple change of time scale will frequently reduce a general nonstationary process to a more tractable stationary one. For example, as stated in Feller (1971), claims will often occur at an accelerated rate in a growing insurance business; however, this departure from stationarity can be removed by the simple expedient of introducing an operational time scale that measures the frequency of claims. In practice, the choice of the time scale is usually dictated by the nature of things. This concept actuates the subordinated processes (SP hereafter) model to relate a general nonstationary process in the usual calendar-time scale to its stationary counterpart in the new time scale.

An SP is obtained by randomizing the time clock of a stationary process {X(t)} using a new time clock {T(t)}, where {T(t)} is a process with nonnegative independent increments. The resulting nonstationary process {X(T(t))} is subordinated to {X(t)}, called the parent process, and is directed by {T(t)}, called the directing process or the subordinator. Therefore, by changing the time scale from t to T(t), {X(T(t))} is transformed to a stationary process.[8] The compound Poisson modeling used in the insurance literature can be considered as an SP application (a jump SP) where the Poisson arrival is the subordinator, and the claim amount per arrival is the parent process distributed as gamma, lognormal, or Pareto. The actuaries' valuation of insurance claim accumulations as random sums is consistent with the construct of an SP model. However, actuarial literature has yet to employ the more interesting implication of the model that a simple time change can reduce the original nonstationary process in calendar-time to a stationary counterpart in the new-time scale. This operation removes unobservable stochastic volatility that often leads to a utility-dependent pricing result, and thus makes pricing more straightforward because one is now working with the friendlier pure diffusion processes.

Since the price of a CAT futures contract, the underlying contract of a CAT spread, reflects the market's aggregate expectation of the catastrophic cumulative

[8] In finance, Clark (1973) tests the predictions of the SP model that asset returns follow a subordinated process in calendar-time and are nonstationary with stochastic volatility but follow the stationary parent process in information-time with constant volatility. Testing the SP hypothesis empirically, he finds that the SP model outperforms the normal and the stable Paretian hypotheses in that (1) the leptokurtosis of daily asset returns is significantly reduced, (2) either transaction counts or cumulative trading volume is a good proxy for the information arrival process, and (3) both the stochastic mean and variance of the asset return process in calendar-time are deterministic functions of these proxies. His finding is widely supported for describing not only spot but also futures price changes by Epps and Epps (1976), Westerfield (1977), Kon (1984), Harris (1986), and Callaway (1989) using daily stock prices, Harris (1987) using transaction data, and Tauchen and Pitts (1983) and Hall, Brorsen, and Irwin (1989) using futures. These results are commonly known as tests of asset returns on mixture of normals.

On the theoretical front, Huffman (1987) lends some support to the SP modeling by arguing that the stochastic process governing transaction volume should be an important source of information that traders face, and thus should affect asset prices. He derives a dynamic general equilibrium model of capital asset pricing that determines trading volume endogenously and relates it positively to the magnitude of changes in stock prices under certain conditions.

claims relative to the total earned premium, it changes with the arrival of information signals with bearing on the expected claims, such as weather forecasts, increased seismic activities, larger than expected catastrophes, number of claim arrivals, etc. The total price change of the contract in any fixed calendar-time interval, be it one day or one week, reflects the accumulation of a random number of independent information arrivals. This random number causes the stochastic volatilities we commonly observe in financial markets. During less eventful periods, prices evolve slowly, while prices evolve faster with higher trading volume as more information arrives. If prices are not measured on a calendar-time basis, but from information to information such that the "time" elapsed between any two trades is uniformly distributed, then the stochastic volatility problem will no longer exist. This explains the reason for a time change under the operational time approach.

By letting the futures price follow a jump SP in calendar-time, where the information arrival process is the randomizing subordinator and the price change process per informational arrival the parent process, we first demonstrate how to make a time change to transform the calendar-time jump SP on to an information-time lognormal diffusion. As a result, the isomorphic information-time option has stationary underlying asset return but random maturity, because the number of information arrivals prior to its calendar-time expiration date is random. Thus, in SP modeling, the option pricing with systematic jump risk problem in calendar-time is isomorphic to an option pricing problem with constant volatility but random maturity.

Next, we price the option in information-time using Dynkin's (1965) version of the Feynman-Kac formula that allows for a random terminal date to solve the option pricing partial differential equation. The resulting information-time formula is a *risk-neutral* Poisson sum of information-time Black's (1976) prices over the option's maturity domain. It contains two underlying prices—the matching bond and the futures prices—and two unobservable variables—the information arrival intensity parameter and the information-time futures volatility. Because the time change preserves the simultaneity between the option's and the matching bond's expiration, the bond exists implicitly as part of another option with a different strike. Consequently, the market is complete with two sources of uncertainty, the price and the maturity risk, in the triplet consisting of the option to be priced, the underlying futures, and another option on the same futures with the same maturity.

The remainder of this article is organized so that the next two sections contain the stochastic time change operation and the actual pricing of CAT futures options. Then, we numerically compare the resulting formula to the Black formula. Finally, we draw a conclusion and suggest possible future research directions.

THE STOCHASTIC TIME CHANGE

Since the compound Poisson modeling prevalent in the insurance literature for the claim accumulation process is an application of the SP concept (a jump SP) as discussed above, we assume that the catastrophe futures price changes follow a jump

SP in calendar-time to reflect the sporadic nature of catastrophe arrival.[9] We assume that the parent process is a lognormal diffusion and the directing process is a homogeneous Poisson as follows:[10]

$$dX_t/X_t = \mu_{Xt}dt + \sigma_{Xt}dW_{Xt}, \tag{1}$$

where μ_{Xt} and σ^2_{Xt} are the *stochastic* calendar-time instantaneous mean and variance, respectively, and

$$\mu_{Xt}dt = \mu_X \, dn(t) \tag{2}$$

$$\sigma^2_{Xt}dt = \sigma^2_X \, dn(t), \tag{3}$$

where

$$\begin{aligned} &dn(t) = 1 \text{ if jump occurs once in dt with probability j dt, otherwise} \\ &dn(t) = 0 \text{ with probability } 1 - j \text{ dt.} \end{aligned} \tag{4}$$

Equations (2) and (3) specify the empirical relationships that the instantaneous mean and variance of calendar-time futures returns, $\mu_{Xt}dt$ and $\sigma^2_{Xt}\,dt$, are linear to (or more generally a deterministic function of) random information arrival, and thus μ_X and σ^2_X, the information-time proportional factors, are constants. If no information arrives in dt, dn(t) = 0, then $\mu_{Xt}dt = \sigma^2_{Xt}\,dt = 0$; that is, the futures price stays put. On the other hand, if one information arrives in dt, then $\mu_{Xt}dt = \mu_X$ and $\sigma^2_{Xt}\,dt = \sigma^2_X$, and the futures price jumps instantaneously according to a lognormal distribution with constant mean and variance μ_X and σ^2_X. The probability of more than one arrival in dt is zero. In other words, *the instantaneous mean and variance of calendar-time futures returns are stochastic due to random information arrivals.* According to the construct of the SP, a parent process can take on other specifications as long as it is stationary.

Denoting the calendar-time clock as dt and the information-time clock as dn, the information-time stationary parent process is obtained by the following sto-

[9] Shimko (1992) values multiple claim insurance policies in calendar-time by assuming that the claim arrival follows a jump SP with stochastic claim arrival frequency.

[10] In practice, the choice as to the specification of the directing process in SP modeling is dictated by the nature of things as pointed out by Feller (1971). In Praetz (1972) and in Blattberg and Gonedes (1974), the information arrivals follow an inverted gamma distribution; in Clark (1973), it follows a lognormal distribution; and in Mandelbrot and Taylor (1967), it follows a non-normal stable law.

chastic time change; that is, by substituting equations (2) and (3) into equation (1), the calendar-time return process:

$$dX_n / X_n = \mu_X dn + [\sigma_X \sqrt{dn(t)/dt}]\ [\sqrt{dt/dn(t)}\ dW_{Xn}] \tag{5}$$

$$= \mu_X dn + \sigma_X dW_{Xn}. \tag{6}$$

Equation (6) is a lognormal diffusion representing the stationary parent process in information-time, where both μ_X and σ_X are constant. In other words, the futures return per information arrival follows a lognormal distribution. Since information is being revealed only as a countable number of Poisson events, strictly speaking, the price process should not be defined over a time continuum. However, since the total price change in any fixed calendar-time interval reflects an accumulation of many small, independent bits of information, proxied by transaction counts or volume, equation (6) can be viewed as an approximation to a diffusion. This is analogous to the continuous trading assumption made by Black and Scholes in justifying the employment of a diffusion as the asset return process.

However, note that the new stationary process has a random stopping time for two reasons. First, the futures, having a fixed maturity in calendar-time, now have random maturity in information-time. The number of claim arrivals prior to its calendar-time expiration date is random and is dictated by the Poisson directing process. Second, because the CAT futures contracts have a cap of \$50,000 (a loss ratio of 200 percent), as soon as the futures price hits \$50,000, an absorbing barrier, the futures price stays at \$50,000 and thus the price process is "killed." Likewise, the corresponding CAT spread's maturity is also random. Given the compound Poisson assumption and conditional upon a given informational arrival, the probability for the futures price to hit the cap is given by a lognormal distribution. The probability of a given number of information arrivals within a given calendar-time maturity is governed by a Poisson distribution.

INFORMATION-TIME PRICING: THE FEYNMAN-KAC APPROACH

Since a long capped call is equivalent to a standard long call spread—that is, a long call plus a short call with a strike at the cap—we first price a call without the cap and then value a capped call as a call spread. This derivation leads to a standard call spread formula. We first price a European call and then discuss the American extension. Since the call has a random maturity, we solve the option pricing partial differential equation with a random terminal condition by using Dynkin's (1965, Theorem 13.16) version of the Feynman-Kac formula that solves a type of parabolic partial differential equation with a random terminal condition, called the Cauchy problem. We briefly introduce the Cauchy problem and Dynkin's solution. We then show that our partial differential equation is a Cauchy problem and consequently can be solved using Dynkin's solution.

Define a family $\{X^{Xn}: X \in R^K, n \in [0, \tilde{N}]\}$ of Ito processes made up almost surely with respect to a filtered probability space (Ω, f, F, P), where X^{Xn} denotes an

Ito process starting at time n at the point X with drift $\mu(X, n)$ and volatility $\sigma(X, n)$, and $\tilde{N}$ denotes a random stopping time in the time span $[0, \infty)$ made up almost surely with respect to a separated filtered probability space (Ω', f', F', P'). Next define a function $V: R^K x[0, \tilde{N}] \rightarrow R$ by

$$V(X, n) = E_{X, \tilde{N}} \left[\int_n^{\tilde{N}} e^{-\Phi(s)} u(X, s)ds + e^{-\Phi(\tilde{N})} g(X, \tilde{N})\right], \tag{7}$$

where $E_{X,\tilde{N}}$ is the expectation operator with respect to X and $\tilde{N}$, $\Phi(s) = n\int_n^s \theta(X, v)\, dv$, and $\theta(X,s)$, $u(X,s)$, and $g(X, \tilde{N})$ are functions defined over $R^K x\, [0, \tilde{N}] \rightarrow R$. In finance terminology, $\theta(X,s)$, $u(X,s)$, and $g(X, \tilde{N})$ can be thought of as the instantaneous discount rate, the cash flows, and the terminal condition, respectively.

Dynkin's Feynman-Kac formula states that, under certain regularity conditions, V is well defined by equation (7) on $R^k x[0, \tilde{N}] \rightarrow R$, is $C^{2,1}(R^K x[0, \tilde{N}])$, satisfies a growth condition, and solves the following partial differential equation, that is, the Cauchy problem:

$$DV(X, n) - \theta(X, n)\, V(X, n) + u(X, n) = 0,\ (X, n) \in R^k x[0, \tilde{N}] \tag{8}$$

with a random boundary condition dictated by the *same* stopping time $\tilde{N}$ as X^{Xn}, that is,

$$V(X, \tilde{N}) = g(X, \tilde{N}),\ X \in R^K,$$

where D is the characteristic operator defined as

$$DV(X, n) = V_n(X, n) + V_X\mu(X, N) + \tfrac{1}{2}tr[\sigma(X, n)^T V_{XX}(X, n)\, \sigma(X, n)],$$

and where the subscripts of V(X, n) denote partial derivatives. Moreover, V is the unique function with these properties. The regularity conditions are that the given functions σ, θ, and u are independent of time n and satisfy a Hölder condition and that the function $[\sigma\sigma^T]^{-1}$ exists and is bounded.

The Feynman-Kac formula employs integration evaluation as in equation (7), which requires lower computational complexity to solve the partial differential equation than using a direct numerical solution such as the finite-elements method. This is the principal claim for its importance in physics applications.[11]

[11] See Chorin (1971) for the numerical approximation of equation (8).

Next, we use Dynkin's solution to value an uncapped European CAT call futures option that has a random maturity in information-time.

Theorem 1

The value of an uncapped European CAT call futures option is the following Poisson sum of the Black's prices in information-time:

$$C(X, n) = \sum_{m=0}^{\infty} \Gamma(m, j) B [X N(d_1^*) - k N(d_2^*)], \tag{9}$$

where $\Gamma(m, j) = e^{-j(T-t)}[j(T-t)]^m/m!$ *is the Poisson probability mass function with intensity* j; $T - t$ *is the option's calendar-time maturity;* $d_1^* = [ln(X/k) + 0.5\sigma_X^2 m]/\sigma_X\sqrt{m}$; $d_2^* = d_1^* - \sigma_X\sqrt{m}$; *and* B, k, σ_X, *and* m *denote the price of the matching bond, the strike price, the futures volatility, and the information-time maturity index, respectively.*

Proof

According to equation (6), the changes of the futures price, $\{X^{X_n}: X \in R, n \in [0, \tilde{N}]\}$, follow a lognormal process in information-time with random stopping time $\tilde{N}$. For notational convenience, we use X here to denote the futures price in information-time in lieu of X_n as above. The futures price process has a random stopping time dictated by $\tilde{N}$, the last random information arrival prior to its calendar-time expiration date. The drift and volatility terms of X are denoted as $\mu(X, n) = \mu_X X$ and $\sigma(X, n) = \sigma_X X$, respectively, where μ_X and σ_X are constants.

Since futures trading requires no initial investment and the futures return follows a diffusion, the standard continuous-hedging or no-arbitrage argument should yield the following fundamental partial differential equation with a random terminal condition for the option price C(X, n):

$$C_n(X, n) - r^*C(X, n) + \tfrac{1}{2}\sigma_X^2 X^2 C_{XX}(X, n) = 0 \tag{10}$$

for all $(X, n) \in [0, \infty)x[0, \tilde{N}]$, with boundary conditions

$$C(X, \tilde{N}) = \mathrm{Max}\,[X(\tilde{N}) - k, 0], \tag{10a}$$

$$C(0, n) = 0, \tag{10b}$$

where the subscripts of C(X, n) denote partial derivatives, r* is the information-time riskless interest rate, k is the exercise price, and equation (10a) is a random terminal condition *also dictated by* $\tilde{N}$. This is because the CAT futures and futures option have the same maturity. A comparison of equations (10) and (8) shows that our option pricing problem is a Cauchy problem, with $\theta(X, n) = r^*$,

$u(X, n) = 0$, $g(X, \tilde{N}) = \text{Max}[X(\tilde{N}) - k, 0]$, $\mu(X, n) = 0$, and $\sigma(X, n) = \sigma_X X$. Assuming that the partial differential equation is well-behaved—that is, being $C^{2,1}$—and satisfies a Lipschitz and a growth condition, we can state its solution according to equation (7) as

$$C(X, n) = E_{\tilde{N},X}\{e^{-r^*(\tilde{N}-n)}\,\text{Max}\,[X(\tilde{N}) - k, 0]\}, \tag{11}$$

where X is the solution to the following linear Ito stochastic differential equation:

$$dX/X = \sigma_X\, dW_{Xn},\ n \in [0, \tilde{N}]. \tag{11a}$$

So the option value is its expected discounted payoff, replacing the drift term of the futures price X with a zero drift.

Recall that the distribution of $\tilde{N} - n$ in the option's calendar-time maturity, $T - t$, is Poisson with intensity parameter j, and that $\tilde{N} - n$ and dX/X are uncorrelated. Then equation (11) becomes

$$\begin{aligned} C(X,n) &= E_{\tilde{N}} E_X\{e^{-r^*(\tilde{N}-n)}\,\text{Max}[X(\tilde{N}) - k, 0]\} \\ &= \sum_{m=0}^{\infty} \Gamma(m,j) E_X\{e^{-r^*m}\,\text{Max}[X(n+m) - k, 0]\}, \end{aligned} \tag{12}$$

where $E_X\{e^{-r^*m}\text{Max}[X(n + m) - k, 0]\}$ given equation (11a) is exactly the Black's (1976) price with a fixed-maturity m. Therefore,

$$C(X, n) = \sum_{m=0}^{\infty} \Gamma(m, j)\, C_B(X,\ n;\ r^*,\ \sigma_X^2,\ m,\ k), \tag{13}$$

where $\Gamma(m, j) = e^{-j(T-t)}[j(T-t)]^m/m!$ = the Poisson probability mass function with intensity j;

C_B = $e^{-r^*m}[X\ N(d_1^*) - kN(d_2^*)]$ = the Black operator, where $d_1^* = [\ln(X/k) + 0.5\,\sigma_X^2\, m]/\sigma_X\sqrt{m}$ and $d_2^* = d_1^* - \sigma_X\sqrt{m}$;

$T - t$ = the option's calendar-time maturity,

r^* = the information-time riskless rate,

σ_X = the futures volatility, and

m = information-time maturity index.

Since e^{-r^*m}, the information-time present value operator over the option's maturity, equals $e^{-r(T-t)}$, the corresponding calendar-time present value operator, the formula can be simplified to

$$C(X, n) = \sum_{m=0}^{\infty} \Gamma(m, j)\, B\,[X\, N(d_1^*) - k\, N(d_2^*)], \tag{14}$$

where $B = e^{-r(T-t)}$ is the price of a riskless matching bond with maturity $T - t$. Q.E.D.

A close look at equation (14) shows that the closed-form formula is risk neutral.[12] There are two underlying prices, B and X, and there are also two unobservable variables, j and σ_X, the information arrival intensity parameter and the information-time futures volatility. The intuition is that, because the time change preserves the simultaneity between the option's and the matching bond's expiration, the bond exists implicitly as part of another option with a different strike. Consequently, the market is complete with two sources of uncertainty, the price and the maturity risk, in the triplet consisting of the option to be priced, the underlying futures, and another option on the same futures with the same maturity.

Because a diffusion is a limiting case of a jump SP when the jump arrival intensity approaches infinity and the jump size simultaneously approaches zero, upon simple derivation, it is straightforward to see that the Black formula of equation (16) is a special case of equation (14) under these conditions. Comparing the calendar-time norm of the utility-dependent option pricing with stochastic volatility formulas, the information-time formula being risk neutral provides valuable simplification. With only two unobservable variables to be estimated, the formula is also parsimonious in implementation. So the "randomized operational time" concept materializes in that the time change from calendar-time to information-time has indeed simplified the option pricing problem in hand.

Next, since the value of a capped option equals the value of a call spread, the value of a capped call is

$$C^*(X, n) = C(X, n, k) - C(X, n, 200), \tag{15}$$

where $C^*(X, n)$ = the value of the capped call,
$C(X, n, k)$ = the value of uncapped calls with strike price k, and
$C(X, n, 200)$ = the value of uncapped calls with strike price \$50,000.

To extend to American futures options, we follow Barone-Adesi and Whaley (1987) for an analytic approximation of the American extension of the Black formula. This method is based on MacMillan's (1986) quadratic approximation of

[12] In general, Dynkin's Feynman-Kac formula provides an efficient numerical solution to the partial differential equation because of the employment of integration evaluation, and thus is robust to alternative specifications of stochastic processes and to the contractual details of the instrument being valued. However, in certain simpler cases, a closed-form solution can be obtained, such as ours when the futures price follows a simple pure diffusion and the option incurs no cash payout prior to its expiration. It is superior to a direct numerical solution, such as using the finite-elements method, in solving the partial differential equation due to lower computational complexity. This is the principal claim for its importance in physics applications.

the American put stock options and is considerably faster than either the finite difference or the compound option approximation methods. According to this approximation, the American extension of the Black formula, denoted as $C_B(X, n)$, is

$$\begin{aligned} C_B(X, n) &= c_B(X, n) + A(X/X^*)^q, \quad \text{where } X < X^*, \\ &= X - k, \qquad\qquad\qquad\quad \text{where } X \geq X^*, \end{aligned} \tag{16}$$

$$\begin{aligned} \text{where } A &= (X^*/q)\{1 - BN[d_1(X^*)]\}, \\ d_1(X^*) &= [\ln(X^*/X) + 0.5\sigma_X{}^2 m]/\sigma_X\sqrt{m}, \\ q &= (1 + \sqrt{1 + 4h})/2, \text{ and} \\ h &= 2r/[\sigma_X{}^2(1 - B)]. \end{aligned}$$

X^* is the critical futures price above which the American futures option should be exercised immediately and is determined iteratively by solving

$$X^* - k = c_B(X, n) + \{1 - B\, N[d_1(X^*)]\}X^*/q. \tag{16a}$$

Barone-Adesi and Whaley (1987) provide an algorithm for solving equation (16a) in five iterations or less. The approximation to American futures put options is similar.

Given equations (9), (16), and (16a), the information-time American futures call price is approximated by

$$C(X, n) = \sum_{m=0}^{\infty} \Gamma(m, j)\ C_B(X,\ n;\ r^*,\ \sigma_X^2,\ m,\ k). \tag{17}$$

The Information-Time and Black Formulas: A Numerical Comparison

The only difference between the information-time and the Black models is their distributional assumptions: the former assumes a jump SP, that is, a compound Poisson, while the latter assumes a pure diffusion. Compared to a diffusion, a jump process gives rise to fatter tails for the underlying asset return distribution, resulting in larger values for out-of-the-money and in-the-money options and smaller values for at-the-money options. The shorter the maturity, the lower the jump intensity, and the deeper the moneyness, the larger the differences. Only when the jump arrival intensity approaches infinity and the jump size simultaneously approaches zero will a jump process approach a diffusion. Thus, the Black formula can be considered as a limiting case of our information-time formula when $j = \infty$. In a similar note for another option whose underlying return is also marked with jumps, Bodurtha and Courtadon (1987) apply an American version of the Black-Scholes formula to currency options and find the formula significantly underprices out-of-the-money options and overprices at-the-money options. As time to maturity increases, the degree of mispricing dissipates. When Merton's diffusion-jump formula (1976) is applied, however, a large portion of the biases are eliminated (see Jorion, 1988).

To make preliminary observation on the usefulness of the information-time formula, we compare numerically values obtained from equation (15) to Black's (1976) values for European options, and from equation (17) to the Barone-Adesi and Whaley (1987) American extension of Black's values for American options.[13] We assume the futures price (X) is 20 to approximate the average loss ratios of the September Eastern contracts, and, additionally, we assume that the riskless rate (r) is 5 percent, the strike prices (k) are 10, 20, and 40, respectively, to represent deep-in-the-money, at-the-money, and deep-out-of-the-money, respectively. We report maturity ranges from 0.025 year to 0.5 year in order to examine the maturity effect. In accordance with the actuarial literature for catastrophe arrival in recent years, we choose an annual volatility of 60 percent, and consider three jump arrival intensities (j): 2, 15, and 30, to examine the intensity effect. We compute the information-time volatility (σ_X) using the formula $\sigma_X = 60\,\% / \sqrt{j}$ to ensure that the information-time and the Black formulas apply the same expected annual volatility. This application ensures that the Black value is a limiting case of the information-time value when the jump arrival intensity approaches infinity.

Table 1 reports the European values. Compared to the information-time values, the Black formula, based on the pure diffusion assumption, significantly underprices out-of-the-money options. The shorter the maturity and the lower the jump intensity, the larger the relative differences. The Black formula underestimates the possibility that a significant jump may bring an out-of-the-money option back into the money. For at-the-money and in-the-money options, the results are also consistent with the prediction that the Black formula overprices at-the-money options and underprices in-the-money options, but the degree of mispricing is significantly smaller than the case of out-of-the-money options, especially for the in-the-money cases. For the in-the-money case, one may also note that, except when $j = 2$, the longer the maturity, the lower the option values. This happens when the option is so deep in the money that the buyer is actually better off by locking in the profit as soon as possible. When $j = 2$, however, the possibility of a large enough forthcoming jump may induce the buyer to prefer to wait out, that is, for a large

[13] Empirical testing is not yet feasible since the PCS options have been traded only since September 1995, while the futures volume has been very thin. Nonetheless, actuarial empirical studies have used the compound Poisson process to make estimation and prediction. For example, Norberg (1991) applies it to predict the total liability of a non–life insurance company, Jorgensen and Paes De Souza (1994) apply it to analyze the claim data of Brazilian care insurance portfolios, and Waldmann (1994) applies it to the exact calculation of the aggregate claims distribution in the individual life model. Additionally, empirical evidence in the financial literature has supported the compound Poisson specification of asset price and exchange rate processes. For example, Jorion (1988) shows empirically that a diffusion-jump process outperforms both a pure diffusion and a GARCH process in specifying equity index and exchange rate processes, and a jump process valuation of currency options outperforms the Black-Scholes model, and Kim, Oh, and Brooks (1994) find that the Poisson-distributed jumps in the movement of the major market index and its component stocks constitute nondiversiable risk, and thus should be priced in option valuation.

enough time value that outweighs the value to lock in the profit early. In conclusion, the results suggest that the Black formula should be used with caution when the option is out-of-the money and the possibility of an unexpected arrival of a significant catastrophe is high.

Table 1
The Black (B) and Information-Time (IT) European Capped Call Values

Price/Strike		Maturity (Years)				
X/k	Model	0.025	0.05	0.1	0.25	0.5
0.5	B/IT∞	0	0	0	0.029	0.247
	IT30	0	0	0.002	0.045	0.263
	IT15	0	0.001	0.005	0.058	0.279
	IT2	0.013	0.028	0.060	0.179	0.433
1	B/IT∞	0.756	1.067	1.504	2.355	3.277
	IT30	0.536	0.909	1.416	2.312	3.248
	IT15	0.416	0.757	1.281	2.254	3.216
	IT2	0.165	0.325	0.631	1.442	2.577
2	B/IT∞	9.988	9.975	9.950	9.891	9.877
	IT30	9.993	9.981	9.954	9.898	9.884
	IT15	9.996	9.987	9.964	9.908	9.893
	IT2	10.006	10.011	10.021	10.041	10.059

Note: All option values are calculated assuming annual volatility $\sigma = 60$ percent, annual risk-free interest rate $r = 5$ percent, and futures price $X = 20$. IT30, IT15, and IT2 denote information-time values with annual jump arrival intensities $j = 30$, 15, and 2, respectively. B denotes the Black value and is identical to IT∞, the information-time value when the jump arrival intensity is infinity. The option is capped at 200.

Table 2 reports the corresponding American values, and Table 3 reports the early exercise values. As predicted, the longer the maturity, the larger the values. For the Black formula, the deeper in-the-money, the larger the values; but this is not necessarily the case for the information-time formula because of the value to wait for jump surprises. Except for the in-the-money case, the Black early exercise values are smaller than the information-time values because of the differential distribution assumptions.

Table 4 shows the 20/40 and 40/60 call spread values. For the 40/60 case, the Black formula underprices the spread as one may expect. For the 20/40 case, however, the Black formula overprices the spread when the maturity is short. Recall that the Black formula is a limiting case of the information-time formula, so the largest mispricing occurs when $j = 2$.

Table 2
The Black (B) and Information-Time (IT) American Capped Call Values

Price/Strike X/k	Model	Maturity (Years) 0.025	0.05	0.1	0.25	0.5
0.5	B/IT∞	0	0	0	0.03	0.250
	IT30	0	0	0.004	0.061	0.336
	IT15	0	0.001	0.007	0.070	0.337
	IT2	0.014	0.028	0.062	0.186	0.452
1	B/IT∞	0.756	1.067	1.505	2.361	3.298
	IT30	0.549	0.943	1.507	2.644	4.402
	IT15	0.424	0.775	1.329	2.431	3.685
	IT2	0.168	0.331	0.642	1.474	2.592
2	B/IT∞	10.000	10.000	10.000	10.000	10.030
	IT30	10.000	10.000	10.000	10.000	10.000
	IT15	10.000	10.000	10.000	10.000	10.000
	IT2	10.024	10.047	10.091	10.210	10.387

Note: All option values are calculated assuming annual volatility σ = 60 percent, annual risk-free interest rate r = 5 percent, and futures price X = 20. IT30, IT15, and IT2 denote information-time values with annual jump arrival intensities j = 30, 15, and 2, respectively. B denotes the Black value and is identical to IT∞, the information-time value when the jump arrival intensity is infinity. The option is capped at 200.

Table 3
The Black (B) and Information-Time (IT) Early Exercise Values

Price/Strike X/k	Model	Maturity (Years) 0.025	0.05	0.1	0.25	0.5
0.5	B/IT∞	0	0	0	0.001	0.003
	IT30	0	0	0.002	0.006	0.073
	IT15	0	0.001	0.002	0.012	0.058
	IT2	0.001	0.000	0.002	0.007	0.019
1	B/IT∞	0	0	0.001	0.006	0.021
	IT30	0.013	0.034	0.091	0.332	0.794
	IT15	0.008	0.018	0.048	0.185	0.469
	IT2	0.003	0.006	0.011	0.032	0.015
2	B/IT∞	0.012	0.025	0.050	0.109	0.123
	IT30	0.007	0.019	0.046	0.102	0.116
	IT15	0.004	0.013	0.036	0.092	0.107
	IT2	0.018	0.036	0.070	0.169	0.328

Note: All option values are calculated assuming annual volatility σ = 60 percent, annual risk-free interest rate r = 5 percent, and futures price X = 20. IT30, IT15, and IT2 denote information-time values with annual jump arrival intensities j = 30, 15, and 2, respectively. B denotes the Black value and is identical to IT∞, the information-time value when the jump arrival intensity is infinity. The option is capped at 200.

Table 4
The Black (B) and Information-Time (IT) American Call Spread Values

Strike/Strike		Maturity (Years)				
k/k	Model	0.025	0.05	0.1	0.25	0.5
	B/IT∞	0.756	1.067	1.505	2.331	3.048
	IT30	0.549	0.943	1.503	2.583	3.706
	IT15	0.424	0.774	1.322	2.369	3.348
	IT2	0.154	0.303	0.580	1.288	2.140
40/60	B/IT∞	0	0	0	0.030	0.228
	IT30	0	0	0	0.056	0.288
	IT16	0	0	0	0.064	0.281
	IT2	0.013	0.024	0.053	0.146	0.321

Note: All option values are calculated assuming annual volatility $\sigma = 60$ percent, annual risk-free interest rate $r = 5$ percent, and futures price $X = 20$. IT30, IT15, and IT2 denote information-time values with annual jump arrival intensities $j = 30$, 15, and 2, respectively. B denotes the Black value and is identical to IT∞, the information-time value when the jump arrival intensity is infinity. The option is capped at 200.

CONCLUSION

Based on the assumption that CAT futures returns follow a subordinated process in calendar-time, we transform the problem of option pricing with nonstationary futures returns in calendar-time to an isomorphic option pricing problem with stationary futures returns but random maturity in information-time. Using Dynkin's (1965) Feynman-Kac formula that permits a random terminal date, we solve the option pricing partial differential equation. The resulting information-time option pricing formula is risk neutral and requires one to estimate only two parameters: the information arrival intensity and the information-time futures volatility. In this context, the time change has served to reduce the computational complexity of the option pricing problem. This exercise demonstrates that the concept of "randomized operational time" may also be valuable in insurance applications.

Since the Black formula (1976) is a limiting case of the information-time formula when the jump arrival intensity approaches infinity and the jump size simultaneously approaches zero, the use of the formula produces larger pricing errors the lower the catastrophe arrival intensity and the larger the jump size volatility. Our numerical analysis ensures that the formula produces the largest undervaluation error for out-of-money short-maturity options when a small number of significant catastrophes may strike during the option's maturity, since, in this case, a large catastrophic loss may bring the option back into the money.

Overall, our preliminary simulation results suggest the information-time formula to be a viable alternative to the calendar-time norm. Nevertheless, the extent of the usefulness of the formula can be determined only by a careful empirical evaluation using market data. Resolution of this issue must wait for future research.

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