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Insurance Rate Changing: A Fuzzy Logic Approach

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ABSTRACT

This article describes how fuzzy logic can be used to make insurance pricing decisions that consistently consider supplementary data, including vague or linguistic objectives of the insurer. The theory of fuzzy logic was developed in the 1970s to improve the accuracy and efficiency of expert systems, and through it, one can account for vague notions whose boundaries are not clearly defined. Using group health insurance data from an insurance company, I show how to build and fine-tune fuzzy logic models for changing rates to reflect supplementary data.

INTRODUCTION

Actuaries and other insurance decision-makers often consider conditions that are ancillary to statistical experience data. These conditions may be rules that reflect company philosophy, such as, "We require *stable* rates." They may involve competitive information or sales volume, and an actuary could create rules that represent how those data affect the rates. Rules may also include financial data, such as, "Raise the rates if we experience *high* loss ratios or *low* profit margins." Decision-makers might not explicitly state these rules and may apply them only on an intuitive level.

The theory of fuzzy sets provides a way to explicitly and consistently deal with linguistic constraints. Through fuzzy sets, one can account for vague notions whose boundaries are not clearly defined, such as "stable" rates. Fuzzy logic provides a uniform way to handle factors that influence pricing decisions. Fuzzy logic allows one to combine (possibly) conflicting goals and constraints. This article describes how to develop a fuzzy logic model with which actuaries and decision-makers can adjust insurance rates by considering constraints or information that are ancillary to statistical experience data. In related research (Young, 1996b), I show how an actuary/decision-maker can adjust rates by combining supplementary

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information with statistical data specific to the rate change itself, such as an actual-to-expected study.

Fuzzy sets have been applied only recently to insurance decision problems. DeWit (1982) describes how to use fuzzy sets in individual underwriting; Lemaire (1990) extends his work. Ostaszewski (1993) suggests several areas in actuarial science in which fuzzy sets may prove useful. Cummins and Derrig (1993) apply a form of fuzzy logic to calculate fuzzy trends in property-liability insurance. Young (1993) applies fuzzy sets in group health underwriting. In a discussion of that article, Erbach and Seah (1993) mention that Erbach, along with two colleagues, developed an underwriter for a Canadian insurer in 1987 by using a mixture of fuzzy set theory and other techniques. Derrig and Ostaszewski (1995) present interesting research on fuzzy clustering that provides an alternative way to view risk classification. Cummins and Derrig (1995) use fuzzy arithmetic to price property-liability insurance.

Chorafas (1994) describes several applications of fuzzy sets in finance. In particular, he remarks that the Financial Accounting Standards Board is currently trying to apply fuzzy set theory to contingent chains theory. He also mentions that a researcher at Goldman Sachs works with fuzzy logic in modeling customers' options. Siegel, de Korvin, and Omer (1995) describe many applications of fuzzy sets in accounting and finance.

The next section introduces fuzzy sets and defines operations that correspond to the linguistic connectors *and* and *or* and the modifier *not*. We use these operators to define the "if" clauses in our fuzzy rules which may be compound statements involving these connectors and modifier. I also describe the mechanics of fuzzy logic; in particular, I explain fuzzy *modus ponens*.[1]

The third section describes how to construct a competitive rate changing model using fuzzy logic. An illustrative model is built that uses fuzzy constraints exclusively to adjust insurance rates. I do not necessarily advocate using such a model without considering experience studies but present this simplified model to demonstrate more clearly how to represent linguistic rules. One possible use of such a model is in adjusting the rates when experience data are not available. For example, suppose an actuary does full-blown experience studies once every six months; then he or she could use this model to adjust the rates more frequently. Also, the model could be used simply to monitor the *adequacy* of the rates between experience studies, in which *adequacy* is given by the output of the model. See Young (1996b) for details about including statistical data in a fuzzy logic model and an example using workers' compensation insurance data. The end of that section presents one method for fine-tuning fuzzy logic rules to account for relevant learning data and provide references for other methods.

Finally, I build and fine-tune rate changing models using group health insurance data from an insurance company. Fuzzy models that contain actual-to-ex-

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[1] References for fuzzy sets include Dubois and Prade (1980), Kosko (1990), and Zadeh (1965). Some references for fuzzy *modus ponens* are Bellman and Zadeh (1970), Driankov, Hellendoorn, and Reinfrank (1993), Kandel and Langholz (1994), Klir and Folger (1988), Mamdani (1974), Zadeh (1975a, 1975b, 1976), and Zimmermann (1987).

pected data are compared to those that do not. The models are compared to least squares regression lines.

FUZZY SETS AND LOGIC

Fuzzy Sets

Fuzzy sets describe concepts that are vague (Zadeh, 1965). The fuzziness of a set arises from the lack of well-defined boundaries, which, in turn, is due to the imprecise nature of language; that is, objects can possess an attribute to various degrees. A fuzzy set corresponding to a given characteristic assigns a value to an object, the degree to which the object possesses the attribute. Vagueness (measured by fuzzy sets) is different from randomness (measured by probability). See Kosko (1990) and Young (1993) for further details about the difference between fuzzy sets and probability.

Examples of fuzzy sets encountered in insurance pricing are *stable* rates, *large* profits, and *small* amounts of business renewed or written. Indeed, rates can be *stable* to different degrees depending on the relative or absolute changes in a product's rate. Also, profits can be large to different degrees depending on the relative or absolute amount of profits.

Zadeh (1965) defines a fuzzy set as follows: A *fuzzy set*, A, in a universe of discourse, X, is a function of the form

$$f_A : X \to I = [0, 1].$$

The function f_A is called the *membership function* of A, and, for any x in X, $f_A(x)$ in [0, 1] represents the grade of membership of x in A.□

This definition generalizes the one for a nonfuzzy, or crisp, set, C. Such a set is given by a characteristic function:

$$\chi_C : X \to \{0, 1\},$$

in which $\chi_C(x) = 1$ if x is in C; otherwise, $\chi_C(x) = 0$. We see, therefore, that fuzzy sets essentially expand the notion of set to allow partial membership in a set.

Some basic operations on fuzzy sets include union, intersection, and complement. The *union*, $A \cup B$, of two fuzzy sets, A and B, is given by

$$f_{A \cup B}(x) \equiv \max[f_A(x), f_B(x)], \ x \in X, \tag{1}$$

and the *intersection*, $A \cap B$, is given by

$$f_{A \cap B}(x) \equiv \min[f_A(x), f_B(x)], \ x \in X. \tag{2}$$

The *complement*, $-A$, of fuzzy set A is given by

$$f_{-A}(x) \equiv 1 - f_A(x),\ x \in X. \tag{3}$$

The operation of union acts as an *or* operator, intersection acts as an *and* operator, and complement acts as a *not* operator. Thus, for example, $f_{A\cap B}(x)$ represents the degree to which x is a member of both A and B. The *max* and *min* operators satisfy the property

$$\max(a, b) + \min(a, b) = a + b,$$

or

$$f_{A\cup B}(x) + f_{A\cap B}(x) = f_A(x) + f_B(x). \tag{4}$$

The given definitions are not the only acceptable ones for these operations. Klir and Folger (1988) specify axioms that union, intersection, and complement operators satisfy. Also, Dubois and Prade (1980) discuss alternative operators. Another commonly used intersection is the *algebraic product*

$$f_{A\bullet B}(x) = f_A(x) \bullet f_B(x).$$

The union operator corresponding to the algebraic product is

$$f_{A\oplus B}(x) = f_A(x) + f_B(x) - f_A(x) \bullet f_B(x).$$

Note that $f_{A\oplus B}(x) + f_{A\bullet B}(x) = f_A(x) + f_B(x)$. Compare with equation (4).

Another property that the *negative*, *max*, and *min* operators satisfy is

$$-(A\cup B) = (-A)\cap(-B), \tag{5}$$

known as *De Moivre's formula*. Indeed, if A and B are fuzzy sets in X, then

$$\begin{aligned} f_{-(A\cup B)}(x) &= 1 - f_{A\cup B}(x) \\ &= 1 - \max(f_A(x), f_B(x)) \\ &= \min(1 - f_A(x), 1 - f_B(x)) \\ &= f_{(-A)\cap(-B)}(x) \end{aligned}$$

for all x in X. Similarly,

$$\begin{aligned} f_{-(A\oplus B)}(x) &= 1 - f_{A\oplus B}(x) \\ &= 1 - (f_A(x) + f_B(x) - f_A(x) \bullet f_B(x)) \\ &= (1 - f_A(x)) \bullet (1 - f_B(x)) \\ &= f_{(-A)\bullet(-B)}(x), \end{aligned}$$

so we have that $-(A \oplus B) = (-A) \bullet (-B)$. Compare with equation (5).

For simplicity, we will use equations (1) through (3) for union, intersection, and complement—namely *max*, *min*, and *negative*—respectively. The algebraic product and its companion union operator are introduced to show that alternative operators exist. See Klir and Folger (1988) and Dubois and Prade (1980) for more complete collections of such operators. Also, Lemaire (1990) discusses the use of alternative operators in insurance applications.

Fuzzy Logic

Here, I present various versions of fuzzy *modus ponens* to show that there are many accepted fuzzy logic models. Then, I describe the simple model that is used throughout the article. The model is widely used in real-time, industrial applications (Kandel and Langholz, 1994).

Zadeh has been particularly interested in applying fuzzy sets to describe linguistic variables in artificial intelligence (Zadeh 1975a, 1975b, 1976). A few years after fuzzy sets were introduced, Bellman and Zadeh (1970) developed the first fuzzy logic model in which goals and constraints were defined as fuzzy sets, and their intersection was the fuzzy set of the decision. Lemaire (1990) shows how this technique can be used in underwriting individual life insurance. Young (1993) similarly applies the technique to group health underwriting.

Cummins and Derrig (1993) use Bellman and Zadeh's method to calculate a trend factor in property-liability insurance, and they calculate several possible trends using accepted statistical procedures. For each trend, they determine the degree to which the estimate is *good* by intersecting several fuzzy goals. They suggest that one may choose the trend that has the highest degree of *goodness*. Cummins and Derrig also propose that one may calculate a trend that accounts for all the trends by forming a weighted average using the membership degrees as weights. It is this latter method that more closely relates to the technique proposed below.

This article intends to show how actuaries may incorporate information that they usually do not explicitly include in their pricing models—for example, amount of business written or profit earned. I follow the method of Zadeh (1975a, 1975b, 1976) by applying fuzzy *modus ponens*. In particular, I use the fuzzy *modus ponens* proposed by Mamdani (1974).

Classical *modus ponens* can be represented by

Implication: If p, then q.
Premise: p. (6)
Conclusion: q,

in which p and q are crisp statements.

Example 1. At the beginning of the year, an actuary might say, "If the number of insurance policies decreases 5 percent during the year, then I will decrease the rates 5 percent." This rule is crisp and can serve as the implication in classical *modus ponens*. At the end of the year, the actuary learns that the number of policies decreased by 5 percent; in other words, the premise is satisfied. The conclusion follows, and the actuary will, therefore, decrease the rates 5 percent.

Fuzzy *modus ponens* arises with questions such as, "What will I do if the number of my company's policies decreases by a *moderate* amount?" One may be led to the fuzzy implication statement, "If the number of my company's policies decreases by a *moderate* amount, then I will decrease the rates *moderately*." Note that this fuzzy statement encompasses a broader concept than the crisp statement.□

Fuzzy *modus ponens* takes a form similar to expression (6), see Ostaszewski (1993, pp. 17–19 and pp. 63–66):

Implication:	If x is A, then y is B.	
Premise:	x is $\widetilde{A}$.	(7)
Conclusion:	y is $\widetilde{B}$,	

in which x is a variable in the universe of discourse X, and A and $\widetilde{A}$ are fuzzy sets in X. Similarly, y is a variable defined on Y, and B and $\widetilde{B}$ are fuzzy sets in Y. The fuzzy set $\widetilde{A}$ measures the degree to which input $x=\widetilde{x}$ satisfies the attribute given by the fuzzy set A. Based on that degree of satisfaction, one defines the fuzzy set $\widetilde{B}$.

Example 2. The last statement in Example 1 may be included in a fuzzy *modus ponens* as follows:

Implication:	If the number of my company's policies decreases by a *moderate* amount, then I will decrease the rates *moderately*.
Premise:	The number of my company's policies decreased *somewhat moderately*.
Conclusion:	I will decrease the rates *somewhat moderately*.

In this example, x = decrease in number of policies, y = rate decrease, A = *moderate* decrease in number of policies, and B = *moderate* rate decrease. In general, the actuary wants a specific rate change $\widetilde{y}$, given a specific policy number change $\widetilde{x}$, not just a fuzzy set representing the rate change. Suppose the value $\widetilde{x}$ of x is 4 percent, *moderate* to degree 0.8, or *somewhat moderate*. One possible value for the rate decrease $\widetilde{y}$ is the value of y that corresponds to a *moderate* decrease in rates to degree 0.8, or *somewhat moderate*.□

The fuzzy hypothesis A may be a compound statement, such as, "our company has been writing a *great deal* of business and earning a *small* amount of profit." In this case, we intersect the fuzzy sets corresponding to a *great deal* of business and a *small* amount of profit with the *min* operator, as in equation (2).

Also, if a compound hypothesis involves the connector *or* and modifier *not*, then use the *max* and *negative* operators, respectively, to combine the individual fuzzy sets.

The remainder of this subsection describes various methods for calculating the components in fuzzy *modus ponens*. In Mamdani's fuzzy *modus ponens*, one may define the fuzzy set "if A, then B" as follows:

$$m_{A \Rightarrow B}(x, y) = \min(m_A(x), m_B(y)).$$

One can define "if A, then B" otherwise by replacing the *min* operator with a different intersection operator, such as *algebraic product*, but we choose *min* for its simplicity. Also, see Ostaszewski (1993, p. 18) for two additional forms of implication.

The premise $\widetilde{A}$ is the fuzzy set that measures the degree to which the hypothesis A in the implication is satisfied by a specific input value $\widetilde{x}$. We define $\widetilde{A}$ by

$$m_{\widetilde{A}}(x) = \min(m_A(x), m_A(\widetilde{x})).$$

That is, $\widetilde{A}$ is A truncated at $m_A(\widetilde{x})$. The fuzzy conclusion $\widetilde{B}$ is defined by

$$m_{\widetilde{B}}(y) = \sup_x \min(m_{\widetilde{A}}(x), m_{A \Rightarrow B}(x, y));$$

see, for example, Ostaszewski (1993, p. 19). This expression simplifies to

$$\begin{aligned} m_{\widetilde{B}}(y) &= \sup_x \min[\min(m_A(x), m_A(\widetilde{x}), \min(m_A(x), m_B(y))] \\ &= \min[m_A(\widetilde{x}), m_B(y)], \end{aligned}$$

so $\widetilde{B}$ is B truncated at $m_A(\widetilde{x})$.

To get a specific output value, $\widetilde{y}$, one defuzzifies $\widetilde{B}$. There are several ways to calculate $\widetilde{y}$: (1) use the abscissa of the center of gravity of $\widetilde{B}$, (2) use the average value of the ys at which $\widetilde{B}$ is maximum, (3) use the smallest value of y at which $\widetilde{B}$ is a maximum, or (4) use the value of y at which B is a maximum (see, for example, Kandel and Langholz, 1994). The first three methods are useful when only one fuzzy rule is at work. The models presented in this article combine more than one rule and, therefore, the fourth method suffices.

Several fuzzy rules can be combined by blending the output fuzzy sets. If $\widetilde{B}_i$ is the fuzzy set that results from the ith rule, i = 1, ..., n, then set $\widetilde{B}$ equal to the union of the $\{\widetilde{B}_i\}$ and, finally, defuzzify $\widetilde{B}$ to calculate $\widetilde{y}$. Alternatively, one can blend the individual output values, $\tilde{y}_i$, of each set $\tilde{B}_i$ by forming a weighted average:

$$\tilde{y} = \frac{\sum_{i=1}^{n} \tilde{y}_i w_i}{\sum_{i=1}^{n} w_i},$$

in which w_i is the weight given to $\tilde{y}_i$. The value w_i may be set equal to the area of $\tilde{B}_i$ or to $m_{A_i}(\tilde{x})$, the height at which B_i is truncated to define $\tilde{B}_i$. The latter value is preferred in engineering control because it is easier to compute in real-time applications (Driankov, Hellendoorn, and Reinfrank, 1993).

This article follows this alternative way of calculating $\tilde{y}$—as a weighted average of the $\{\tilde{y}_i\}$. We set $w_i = m_{A_i}(\tilde{x})$ because it is easier to interpret, and, in a certain model, the output is a piecewise linear function of the input. Recall that $\tilde{y}_i$ is set equal to the value at which B_i attains its maximum; denote that value by y_i. In fact, for all practical purposes, the fuzzy set B_i may be replaced with the crisp value y_i.

To summarize, the fuzzy system here is a collection of n fuzzy rules:

If x is A_1, then y is y_1.
If x is A_2, then y is y_2.
...
If x is A_n, then y is y_n.

If we are given input data $\tilde{x}$ (possibly multidimensional if the A_i are compound hypotheses), then measure the degree to which $\tilde{x}$ satisfies the hypothesis A_i in rule i, i = 1, ..., n, namely, $m_{A_i}(\tilde{x})$. To calculate the output $\tilde{y}$, form the weighted average

$$\tilde{y} = \frac{\sum_{i=1}^{n} y_i\, m_{A_i}(\tilde{x})}{\sum_{i=1}^{n} m_{A_i}(\tilde{x})}.$$

FUZZY RATE CHANGING

The previous section describes how to obtain a crisp output $\tilde{y}$ given a fuzzy logic model and crisp input $\tilde{x}$. This section explains how to construct and fine-tune a fuzzy logic model. To make the exercise more concrete, we consider a hypothetical example. The example shows how an actuary might apply fuzzy constraints to adjust rates. We follow these steps:

(1) Verbally state linguistic rules: These rules may reflect current or desired company philosophy. They may arise from the business sense of actuaries. They may result from the combined input of several functions in the insurance company, such as underwriting and marketing. In the last case, the statements may even conflict with each other, and the fuzzy logic system will blend the conflicting goals.

(2) Create the fuzzy sets corresponding to the hypotheses: The input may be multidimensional; for example, one might consider both amount of business and profit earned. We assume that the linguistic variables are naturally ordered.

(a) To create the fuzzy sets for the jth dimension of the input, partition the input space $X_j = [x_{j,1}, x_{j,n(j)}]$ into $n(j) - 1$ disjoint subintervals, one fewer than the number, $n(j)$, of fuzzy sets defined on X_j. Write the boundary points of the subintervals

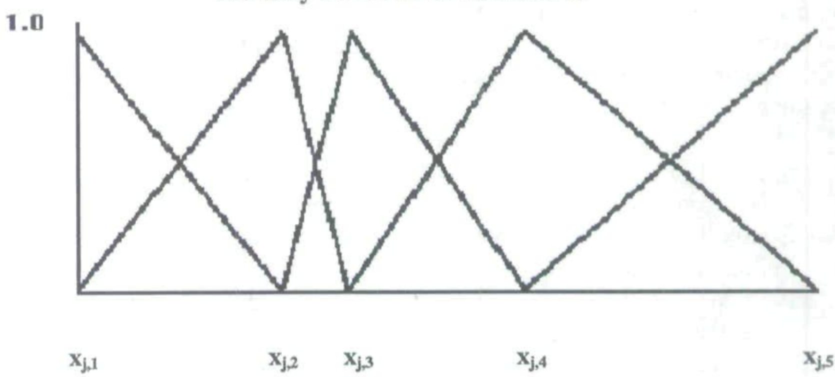
$$x_{j,1} \leq x_{j,2} \leq \ldots \leq x_{j,n(j)}.$$

Even though the input space X_j may be infinitely long, in the example below, we describe how to determine $x_{j,1}$ and $x_{j,n(j)}$ so that we can effectively limit X_j to the finite interval $[x_{j,1}, x_{j,n(j)}]$.

(b) The graph of the leftmost fuzzy set $A_{j,1}$ is defined to be the line segment joining the points $(x_{j,1}, 1)$ and $(x_{j,2}, 0)$ and 0 elsewhere. The graph of each of the middle $n(j) - 2$ sets $A_{j,k(j)}$ is the triangular fuzzy set (piece-wise linear) that connects the points $(x_{j,k(j)-1}, 0)$, $(x_{j,k(j)}, 1)$, and $(x_{j,k(j)+1}, 0)$ and 0 elsewhere, $k(j) = 2, \ldots, n(j) - 1$. Finally, the n(j)th fuzzy set $A_{j,n(j)}$ is the line segment joining the points $(x_{j,n(j)-1}, 0)$ and $(x_{j,n(j)}, 1)$ and 0 elsewhere. Note that, for any input value of x_j, the sum (over k(j)) of its membership values in the sets $A_{j,k(j)}$ is 1; thus, the $A_{j,k(j)}$ are said to *partition* X_j.

Figure 1 illustrates a partition with $n(j) = 5$. The equations for the fuzzy sets $A_{j,k(j)}$, $k(j) = 1, \ldots, 5$, follow the figure.

Figure 1
A Fuzzy Partition of an Interval



$$m_{A_{j,1}}(x)=\max\left[0,\min\left(1,\frac{x_{j,2}-x}{x_{j,2}-x_{j,1}}\right)\right],$$
$$m_{A_{j,k(j)}}(x)=\max\left[0,\min\left(\frac{x_{j,k(j)+1}-x}{x_{j,k(j)+1}-x_{j,k(j)}},\frac{x-x_{j,k(j)-1}}{x_{j,k(j)}-x_{j,k(j)-1}}\right)\right],\ k(j)=2,3,4,$$
$$m_{A_{j,5}}(x)=\max\left[0,\min\left(\frac{x-x_{j,4}}{x_{j,5}-x_{j,4}},1\right)\right].$$

(c) Combine the fuzzy sets that comprise each hypothesis into one fuzzy set using the operators *min*, *max*, and *negative*, corresponding to the linguistic connectors *and* and *or* and modifier *not*.

(3) Determine the output values $\{y_i\}$ for the conclusions: Set the output value y_i to the desired output if the hypothesis of rule i, i = 1, ..., n, is met to degree 1.0. Underlying Examples 1 and 2 is that the hypothesis is satisfied to degree 1.0 when the number of policies decreases by exactly 5 percent. In that case, the desired effect is to decrease the rates by 5 percent. Hence, the output value for that rule is –5 percent.

(4) Fine-tune the fuzzy rules, if applicable: If learning data are available—either historical data that are still relevant or hypothetical data from experts—then use that data to modify the values $x_{j,k(j)}$ and the values y_i. Those values may be modified to optimize any one of a number of objectives. One objective function is discussed at the end of this section, and it is used with a numerical example in the next section.

The fuzzy rules in the following example are strictly hypothetical and not necessarily appropriate for business decision-making. Also, for each example, if none of the hypotheses is satisfied, then the rate change is zero percent by default.

Example 3. Consider the following three fuzzy rules: (a) If we are writing a *large* amount of business and earning a *small* profit, then *raise* the rates. (b) If we are writing a *moderate* amount of business and earning a *moderate* profit, then do *not change* the rates. (c) If we are writing a *small* amount of business and earning a *large* profit, then *lower* the rates.

The rationale behind these linguistic rules is that, if the company is writing a large amount of business but not earning much money on it, the rates are probably too low (that is, they are very competitive). Similarly, if the company is writing a small amount of business and earning a large profit on it, then the rates are probably too high.

Ways to measure amount of business include total premiums, number of accounts, retention ratio, and premiums for new business. Ways to measure profit include loss ratio, expense ratio, return on surplus, profit as a percentage of premium, and profit as a percentage of claims. Here, the amount of business is measured by total premium volume, and profit is measured by the sum of the loss and expense ratios, sometimes referred to as the combined ratio or underwriting ratio.

Other variables to consider are recent changes in amount of business written, profit, and rates. Suppose the company wants to serve its customers by keeping the rates stable. One way to reflect this philosophy is to augment the three illustrative fuzzy rules as follows: (a') If we are writing a *large* amount of business, earning a *small* profit, and have *not changed* the rates recently, then *raise* the rates. (b') If we are writing a *moderate* amount of business, earning a *moderate* profit, and have *changed* the rates recently, then do *not change* the rates. (c') If we are writing a *small* amount of business, earning a *large* profit, and have *not changed* the rates recently, then *lower* the rates.

In this example, each hypothesis is a compound statement that combines two fuzzy sets with the connector *and.* Denote the space of premium volume (measuring amount of business) by X_1 and the space of underwriting ratios (measuring profit) by X_2. On each of these spaces, we define three fuzzy sets—one for each fuzzy rule.

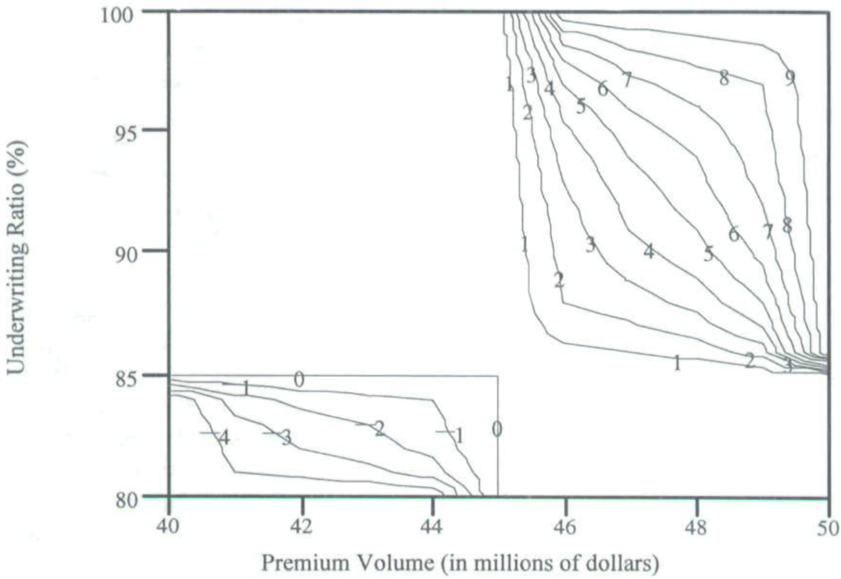
To find the endpoints of each of these spaces and the intermediate boundary points, work backward as follows: Determine the maximum and minimum rate changes. For example, let us say that the maximum change in rates that one is willing to apply is +10 percent, and the minimum is –5 percent. Then, determine the premium volumes and underwriting ratios that would lead to those extreme changes. Suppose that we would raise the rates 10 percent if the premium volume were greater than or equal to 50 (in millions of dollars) and if the underwriting ratio were greater than or equal to 100 percent. Also, suppose that we would lower the rates 5 percent if the premium volume were less than or equal to 40 and if the underwriting ratio were less than or equal to 80 percent.

Then, the space of premium volume is effectively $X_1 = [40, 50]$, and the space of underwriting ratios is $X_2 = [80, 100]$. In this representation, all premium volumes less than 40 are identified with 40 because the rate change resulting from any premium volume less than 40 is the same as the rate change if the premium volume were identically 40—similarly for the other endpoints. If one wants to allow a smaller than 40 premium volume to be offset by a smaller than 80 percent underwriting ratio, then rewrite the fuzzy rules to account for this.

To find the intermediate points at which no rate change occurs, decide what values of premium volume and underwriting ratios would lead to no change. Suppose that these values are 45 and 85, respectively. Figure 2 plots the contours of the rate changes against premium volume and the underwriting ratio. The underwriting ratio is along the vertical axis, and premium volume is along the horizontal axis. Each contour line shows the combination of underwriting ratios and premium volumes that imply a percentage change in rates (for example, one percent) equal to the numerical label on the contour line. Note that by using the *min* operator to join the clauses in our compound hypotheses, the region for "no change" is relatively large.

Such a large region of "no change" might be appropriate if this model is used to change the rates between experience studies; we may want to change the rates only when *all* the variables indicate that we do so. On the other hand, if we want the rates to respond to either of the variables, premium volume or underwriting ratio, then consider replacing the *and*s by *or*s. That is, replace the *min* operators in the three rules with *max*. Figure 3 plots such contours.

Figure 2
Contours of Rate Change Against Underwriting Ratio and Premium Volume (Using *And* in All Three Rules)



Note: Interpret an area between two contour curves as representing the region of the corresponding surface that varies continuously from one contour height to the next. For example, in the area between 0 and +1, the rate change varies from 0 percent to 1 percent continuously.

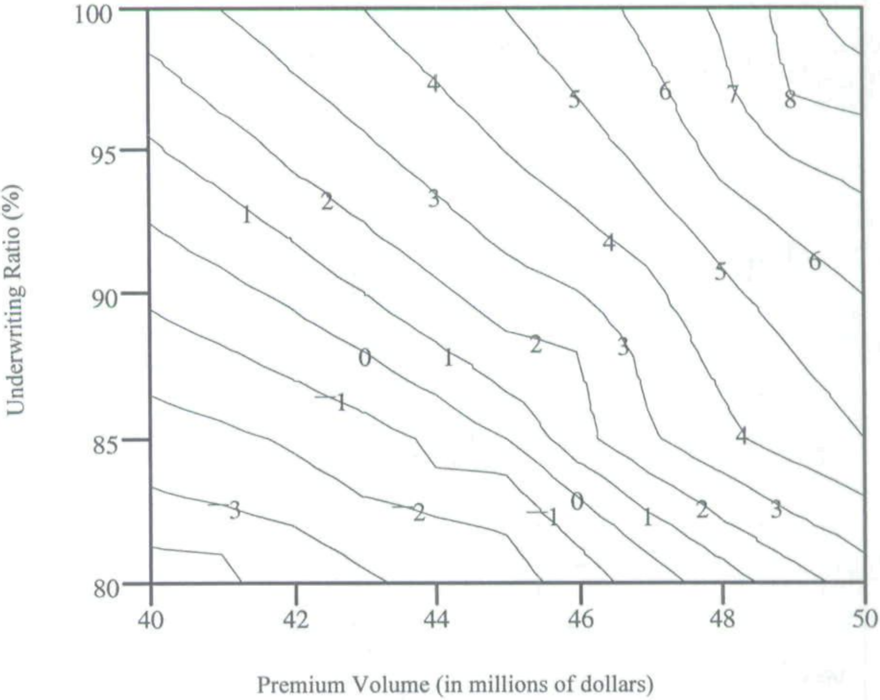
For a third scenario, suppose we want to increase the rates if *either* the premium volume *or* the underwriting ratio is large, but we want to decrease the rates only if *both* the premium volume *and* the underwriting ratio are small. In that case, replace only the *min* operator in the first rule with *max*, and leave *min* in the other two rules. The resulting contours are plotted in Figure 4.

Clearly, there are many ways to formulate the fuzzy rules, but an actuary should at least check contour plots to see which formulation coincides with the philosophy or practices of his or her company.□

After setting up a fuzzy logic model based on the expert opinion of the actuary and others, as described above, the next step is to fine-tune the model to reflect more accurately the past actions of the company or its desired actions. Also, the actuary may wish to update the model as more information becomes available. Given learning data of the form $\{(\tilde{x}_\ell, \tilde{y}_\ell) : \ell = 1,..., L\}$ pairs of input and output values, one may fine-tune the model using the following simple method: Perturb the parameters $\{x_{j,k(j)}\}$ and $\{y_i\}$ to minimize the sum of squared errors

$$\sum_{\ell=1}^{L}(\tilde{y}_\ell - \tilde{y}(\tilde{x}_\ell))^2,$$

Figure 3
Contours of Rate Change Against Underwriting Ratio and Premium Volume (Using *Or* in All Three Rules)



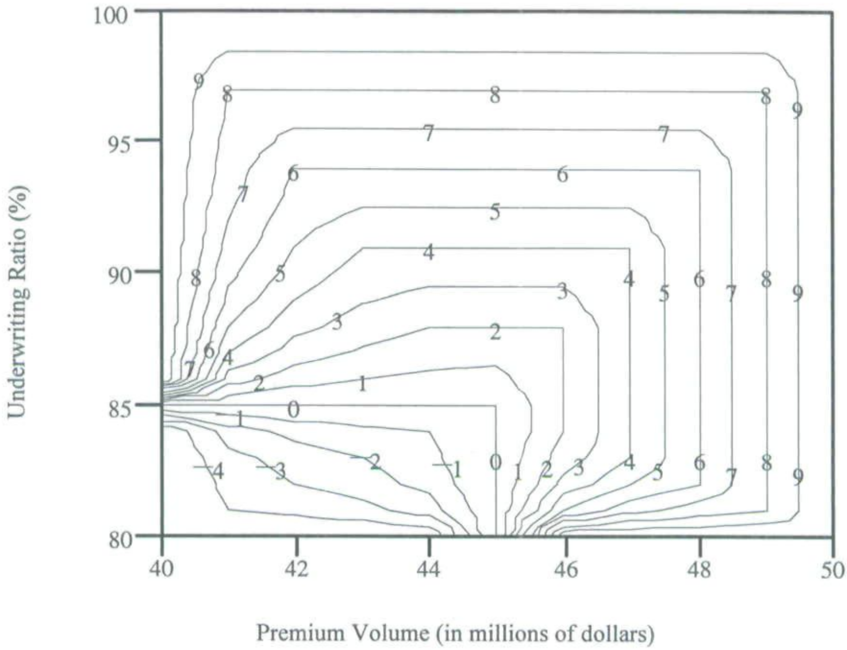
Note: Interpret an area between two contour curves as representing the region of the corresponding surface that varies continuously from one contour height to the next. For example, in the area between 0 and +1, the rate change varies from 0 percent to 1 percent continuously.

in which $\tilde{y}(\tilde{x}_\ell)$ is the output of the fuzzy logic model, given the input $\tilde{x}_\ell$. One may also weight these errors to reflect the relative importance of each ordered pair. In the next section, we minimize such a weighted sum of squared errors:

$$\sum_{\ell=1}^{L} w_\ell (\tilde{y}_\ell - \tilde{y}(\tilde{x}_\ell))^2, \tag{8}$$

in which w_ℓ, $\ell = 1, \ldots, L$, is the weight for the pair $(\tilde{x}_\ell, \tilde{y}_\ell)$.

Figure 4
Contours of Rate Change Against Underwriting Ratio and Premium Volume (Using *Or* in the First Rule and *And* in the Second and Third Rules)



Note: Interpret an area between two contour curves as representing the region of the corresponding surface that varies continuously from one contour height to the next. For example, in the area between 0 and +1, the rate change varies from 0 percent to 1 percent continuously.

To explore other methods for fine-tuning a fuzzy logic model, see Glorennec (1994), Katayama, Kajitani, and Nishida (1994), and Driankov, Hellendoorn, and Reinfrank (1993), who describe several methods to adjust the parameters to fit learning data. Also, Young (1996a) proposes using a measure of implication derived from fuzzy subsets to fine-tune fuzzy logic models. This measure of implication measures the degree to which the input implies the output. To fine-tune a given model, therefore, perturb the parameters of the model to maximize this measure of implication.

In sum, to build a fuzzy logic system, (1) Verbally state linguistic "if-then" rules. (2) Create the fuzzy sets corresponding to the hypotheses of the rules. (3) Determine the output values for the conclusions. (4) Fine-tune the fuzzy rules, if applicable.

GROUP HEALTH INSURANCE EXAMPLE

This section presents an example of building and fine-tuning fuzzy logic models, using group health insurance data from an anonymous cooperating insurance company, here referred to as Group Health Insurer (GHI). To protect the interests of this insurer, the data have been masked by linear transformation and by relabeling the geographic regions and dates involved. Data for the eight consecutive rating periods in the early 1990s are available from the author upon request.

A comparison of fuzzy logic models that include an actual-to-expected claims ratio to those that do not is conducted to see how much accuracy one loses (or gains) when relying only on ancillary data. Accuracy is measured by a weighted sum of squared errors, as in expression (8). Linear functions fit via least squares regression are also included to serve as benchmarks for the accuracy of the fuzzy models.

Data from eight consecutive rating periods are used for this study. GHI performs an actual-to-expected (a/e) study every three months to monitor the rates in 17 geographical regions of the United States and to provide a basis for adjusting those rates, if so indicated. GHI compares the actual paid claims of group health policyholders with the corresponding expected manually-rated claims. These manually-rated claims are adjusted for demographic factors of each group (age/sex and geographical area), for plan design, and for trend.

In addition to the a/e ratio, GHI considers financial and marketing data in choosing a rate change: Paid and incurred 12-month loss ratios, initial premiums, and canceled premiums. This study considers only incurred loss ratios, except for one rating period for which it was not available. In that period, the 12-month paid loss ratio is used. Also, the earliest loss ratio is only a 10-month loss ratio. Some of the early amounts for initial premiums and cancellations that are not based on 12 months of data are annualized.

To help me develop linguistic rules, the pricing actuaries at GHI explained how they use the data for changing rates. As a rule of thumb, if the a/e ratio is high, then they raise the rates, and vice versa. However, they temper this rate change by examining whether the incurred loss ratio is high or low. I, therefore, substitute loss ratio for a/e ratio in building fuzzy logic models that rely only on ancillary data. A typical fuzzy rule might be,

If the loss ratio is high, then raise the rates.

The actuaries also consider the amount of claims rated in their a/e study in each region: If the a/e ratio is based on little data, then they do not weight it very heavily in deciding on a rate change for that region. For this reason, the data are weighted for each period according to the manually-rated claims in each region, after normalizing the weights so that they add to 1.00. Then, each rating period is weighted equally. That is, a weighted sum of squared errors for each rating period is calculated; then those eight numbers are added to get a total sum of squared errors. In practice, one might also wish to weight data according to the number of months on which it is based.

The actuaries at GHI agree that large initial premiums or cancellations of business may indicate that they should raise or lower the rates, respectively. They mentioned, however, that they do not always respond to that data. Of the two actions, they are more likely to lower the rates if the cancellations are large.

In addition to information provided by the pricing actuaries, I calculate weighted correlations between variables. I use the same weights that are used in fine-tuning the fuzzy logic models and in calculating the linear regressions. The correlations between the loss ratio and the rate change are negative in two of the eight periods, 1 and 7. Because the loss ratio is substituted for the a/e ratio, only the six periods, 2 through 6 and 8, are used to fine-tune the models. That is, the data from periods 1 and 7 are omitted because they are not qualitatively relevant to the linguistic rate changing models.

Weighted correlations are also calculated between the rate change and initials (I), cancellations (C), the ratio of initial premiums to cancellations (I/C), and the difference between initial premiums and cancellations (I–C). In most periods, these correlations were small. However, the correlations between rate change and cancellations are negative in four periods, so that one might want to include a rules such as,

If the loss ratio is low and the cancellations are large, then lower the rates.

This rule also coincides with the statements of GHI's pricing actuaries.

The actuaries also said that GHI targets stable rates. Therefore, the weighted correlations between the absolute value of the rate change and the absolute value of the previous rate change are calculated. (Either or both of these numbers may be zero.) If stability is a goal, then one would expect those correlations to be negative, and such is not the case. On the other hand, one might also achieve stability by splitting a rate action across two rating periods. Suppose the evidence calls for a 7 percent decrease in rates. An insurance company might propose a 4 percent decrease in one rating period and a 3 percent decrease in the following period if the data continues to support such a decrease. One may, therefore, consider a fuzzy model based on the following rules: (a) If the loss ratio is very low, then lower the rates. (b) If the loss ratio is somewhat low and the absolute value of the previous rate change is low, then lower the rates. (c) If the loss ratio is somewhat high and the absolute value of the previous rate change is low, then raise the rates. (d) If the loss ratio is very high, then raise the rates. In such a model, the default is no rate change if all of the hypotheses are satisfied to degree zero. In this case, none of the rules applies.

In a fine-tuned fuzzy logic system for the one above, some of the intermediate partitioning points collapse to the same point. This degeneracy is interpreted as indicating that fewer rules may fit better. In general, whenever models are fine-tuned with three or more rules, a similar collapsing occurs. For this reason, I present only the results of fine-tuning models with two rules.

Tables 1 through 4 present the results of estimating four types of models. To fine-tune the fuzzy logic models, Microsoft Excel's Solver is used to minimize the weighted sum of squared errors from rating periods 2 through 6 and 8, given the initial values. These initial values are the endpoints of the input spaces and rate changes that correspond to the conclusions. Each model involves only two fuzzy rules, so we only specify the boundary points of the input spaces and two extreme rate changes. These values were obtained from the pricing actuaries at GHI. In the optimization, the left-hand endpoint is constrained to be less than or equal to the right-hand endpoints. In general, the solution obtained by Microsoft Excel's Solver yields a local minimum. Although the global minimum may not have been reached, the solution found here may be desirable because in some sense it is close to the experts' initial system.

Table 1 presents data from the model of rate changes as a function of rules that depend on the actual-to-expected ratio and the absolute value of the previous rate change, similar to the rules mentioned above. The table also presents results obtained by substituting the loss ratio for the a/e ratio and fine-tuning that model. In optimizing both fuzzy models, the left-hand endpoint of the absolute value of the previous rate change is fixed at zero. For comparison with standard models, we include the weighted least squares regression lines that use the same independent variables—namely, a/e ratio or loss ratio and absolute value of the previous rate change. The fuzzy models fit better, as measured by the sum of squared errors, than do the linear regressions. The models that use the a/e ratio fit better than those that substitute loss ratio for a/e ratio by roughly 0.5 percent in average error. Put another way, in depending on ancillary data we lose only about 0.5 percent in average accuracy, compared with the accuracy of the models that include the a/e ratio.

The form of the presentation and the results are similar in the following three models. The data presented in Table 2 are based on a model that considers whether the business is growing or declining, as measured by the ratio of initial premiums to cancellations. In optimizing these fuzzy models, the left-hand endpoint of that ratio is fixed at zero. In Table 3, the ratio of initial premiums to cancellations is replaced with the amount of cancellations, because of information from GHI's actuaries. The left-hand endpoint of the amount of cancellations is again fixed at zero. Finally, the data presented in Table 4 reflect models that depend on the a/e ratio only or on the loss ratio only.

Table 1
Rate Change as a Function of the Absolute Previous Rate Change and of the Loss Ratio or the Actual-to-Expected Ratio

Panel A: Fuzzy Model that Includes the Actual-to-Expected Ratio

(a) If a/e is low and if absolute previous rate change is low, then lower the rates.
(b) If a/e is high and if absolute previous rate change is low, then raise the rates.
(c) If neither rule (a) nor rule (b) applies, then do not change the rates.

	Input to Excel Solver	Output from Solver
Actual to Expected Ratio	[0.85, 1.10]	[0.89, 1.09]
Absolute Previous Rate Change	[0.00, 5.00]	[0.00, 7.42]
Rate Change	{−5.00, 5.00}	{−3.52, 2.83}
Sum of Squared Errors	14.48	
Average Error	$\sqrt{14.48/6}=1.554$	

Panel B: Fuzzy Model that Substitutes Loss Ratio for Actual-to-Expected Ratio

(a) If loss ratio is low and if absolute previous rate change is low, then lower the rates.
(b) If loss ratio is high and if absolute previous rate change is low, then raise the rates.
(c) If neither rule (a) nor rule (b) applies, then do not change the rates.

	Input to Excel Solver	Output from Solver
Loss Ratio	[70.00, 90.00]	[75.41, 86.60]
Absolute Previous Rate Change	[0.00, 5.00]	[0.00, 5.00]
Rate Change	{−5.00, 5.00}	{−2.56, 0.57}
Sum of Squared Errors	23.61	
Average Error	$\sqrt{23.61/6}=1.984$	

Panel C: Linear Regression that Includes the Actual-to-Expected Ratio

Rate Change	-24.477 + 24.602 * (a/e ratio) – 0.237 * (absolute previous rate change)
Sum of Squared Errors	14.96
Average Error	$\sqrt{14.96/6}=1.579$

Panel D: Linear Regression that Substitutes Loss Ratio for Actual-to-Expected Ratio

Rate Change	−9.718 + 0.116 * (loss ratio) – 0.407 * (absolute previous rate change)
Sum of Squared Errors	24.46
Average Error	$\sqrt{24.46/6}=2.019$

Table 2
Rate Change as a Function of the Change of Business and of the Loss Ratio or the Actual-to-Expected Ratio

Panel A: Fuzzy Model that Includes the Actual-to-Expected Ratio

(a) If a/e is low and if amount of business is declining, then lower the rates.
(b) If a/e is high and if amount of business is growing, then raise the rates.
(c) If neither rule (a) nor rule (b) applies, then do not change the rates.

	Input to Excel Solver	Output from Solver
Actual-to-Expected Ratio	[0.85, 1.10]	[0.93, 0.98]
Initial/Cancel	[0.00, 2.00]	[0.00, 2.42]
Rate Change	{−5.00, 5.00}	{−2.88, 0.89}
Sum of Squared Errors	15.46	
Average Error	$\sqrt{15.46/6}=1.605$	

Panel B: Fuzzy Model that Substitutes Loss Ratio for Actual-to-Expected Ratio

(a) If loss ratio is low and if amount of business is declining, then lower the rates.
(b) If loss ratio is high and if amount of business is growing, then raise the rates.
(c) If neither rule (a) nor rule (b) applies, then do not change the rates.

	Input to Excel Solver	Output from Solver
Loss Ratio	[70.00, 90.00]	[76.20, 86.50]
Initial/Cancel	[0.00, 2.00]	[0.00, 3.00]
Rate Change	{−5.00, 5.00}	{−2.42, 0.96}
Sum of Squared Errors	22.54	
Average Error	$\sqrt{22.54/6}=1.938$	

Panel C: Linear Regression that Includes the Actual-to-Expected Ratio

Rate Change	−26.695 + 26.712 * (a/e ratio) – 0.119 * (initial/cancellation)
Sum of Squared Errors	15.76
Average Error	$\sqrt{15.76/6}=1.621$

Panel D: Linear Regression that Substitutes Loss Ratio for Actual-to-Expected Ratio

Rate Change	−11.902 + 0.135 * (loss ratio) + 0.124 * (initial/cancellation)
Sum of Squared Errors	27.04
Average Error	$\sqrt{27.04/6}=2.123$

Table 3
Rate Change as a Function of the Amount of Cancellations and of the Loss Ratio or the Actual-to-Expected Ratio

Panel A: Fuzzy Model that Includes the Actual-to-Expected Ratio

(a) If a/e is low and if amount of cancellations is high, then lower the rates.
(b) If a/e is high and if amount of cancellations is low, then raise the rates.
(c) If neither rule (a) nor rule (b) applies, then do not change the rates.

	Input to Excel Solver	Output from Solver
Actual-to-Expected Ratio	[0.85, 1.10]	[0.93, 1.15]
Cancellation	[0.00, 4.00]	[0.00, 5.87]
Rate Change	{−5.00, 5.00}	{−2.57, 3.62}
Sum of Squared Errors	20.14	
Average Error	$\sqrt{20.14/6}=1.832$	

Panel B: Fuzzy Model that Substitutes Loss Ratio for Actual-to-Expected Ratio

(a) If loss ratio is low and if amount of cancellations is high, then lower the rates.
(b) If loss ratio is high and if amount of cancellations is low, then raise the rates.
(c) If neither rule (a) nor rule (b) applies, then do not change the rates.

	Input to Excel Solver	Output from Solver
Loss Ratio	[70.00, 90.00]	[73.62, 85.40]
Cancellation	[0.00, 4.00]	[0.00, 7.54]
Rate Change	{−5.00, 5.00}	{−2.38, 0.48}
Sum of Squared Errors	24.84	
Average Error	$\sqrt{24.84/6}=2.035$	

Panel C: Linear Regression that Includes the Actual-to-Expected Ratio

Rate Change	−27.607 + 27.232 * (a/e ratio) + 0.091 * (cancellation)
Sum of Squared Errors	15.49
Average Error	$\sqrt{15.49/6}=1.607$

Panel D: Linear Regression that Substitutes Loss Ratio for Actual-to-Expected Ratio

Rate Change	−11.994 + 0.142 * (loss ratio) − 0.100 * (cancellation)
Sum of Squared Errors	26.71
Average Error	$\sqrt{26.71/6}=2.110$

Table 4
Rate Change as a Function of the Loss Ratio or the Actual-to-Expected Ratio

Panel A: Fuzzy Model that Includes the Actual-to-Expected Ratio

(a) If a/e is low, then lower the rates.
(b) If a/e is high, then raise the rates.

	Input to Excel Solver	Output from Solver
Actual-to-Expected Ratio	[0.85, 1.10]	[0.87, 1.16]
Rate Change	{−5.00, 5.00}	{−3.78, 4.53}
Sum of Squared Errors	14.67	
Average Error	$\sqrt{14.67/6}=1.564$	

Panel B: Fuzzy Model that Substitutes Loss Ratio for Actual-to-Expected Ratio

(a) If loss ratio is low, then lower the rates.
(b) If loss ratio is high, then raise the rates.

	Input to Excel Solver	Output from Solver
Loss Ratio	[70.00, 90.00]	[75.50, 99.91]
Rate Change	{−5.00, 5.00}	{−2.26, 2.98}
Sum of Squared Errors	25.38	
Average Error	$\sqrt{25.38/6}=2.057$	

Panel C: Linear Regression that Includes the Actual-to-Expected Ratio

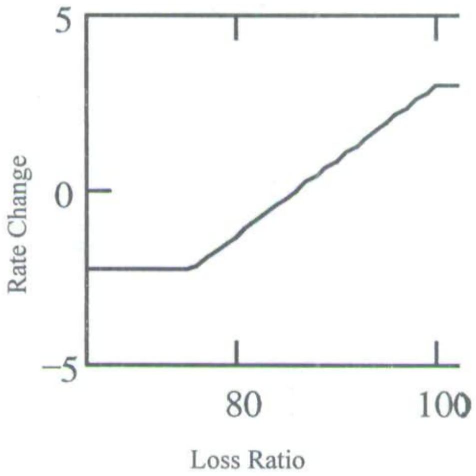
Rate Change	−26.545 + 26.435 * (a/e ratio)
Sum of Squared Errors	15.82
Average Error	$\sqrt{15.82/6}=1.624$

Panel D: Linear Regression that Substitutes Loss Ratio for Actual-to-Expected Ratio

Rate Change	−11.859 + 0.136 * (loss ratio)
Sum of Squared Errors	27.12
Average Error	$\sqrt{27.12/6}=2.126$

Of the fuzzy models that substitute the loss ratio for the a/e ratio, the one in which the rate change depends on the loss ratio and the ratio of initials to cancellations, seems to fit well (see Table 2, Panel B). The fit is reasonable, with an average error less than 2 percent. One drawback is that the maximum rate increase is only about 1 percent; however, this maximum increase is less for the corresponding models shown in Tables 1 and 3. Also, the fuzzy model that bases the rate change solely on the loss ratio (see Table 4, Panel B), comes close to the fit of the more complicated model in Table 2, Panel B. The former model results in a piecewise linear function, as shown in Figure 5.

Figure 5
Rate Change as a Function of Loss Ratio (Fuzzy Model in Table 4, Panel B)



CONCLUSION

This article describes and demonstrates how to build and fine-tune a fuzzy logic system from linguistic rules to finished model. This method might be applied to any line of business, not just group health insurance, as discussed above. For example, Young (1996b), applies fuzzy logic to workers' compensation insurance.

The emphasis here is on models that depend only on data that supplement statistical data specifically related to claim experience. The models are compared to ones that include actual-to-expected data to judge their performance. Future research will explore more fully models that include statistical data, as well as ancillary data.

Even though a given fuzzy logic model may fit only slightly better than a standard linear regression model, the main advantage in fuzzy logic is that one can begin with verbal rules and systematically create a mathematical model that reflects those rules. A decision-maker can, thus, consistently act in accordance with those rules through use of fuzzy logic. By fine-tuning a model using relevant historic data, the decision-maker can also judge whether his or her company has followed those rules. Also, through building and fine-tuning a model, a decision-maker can develop valuable intuition, because the process described here forces demonstration of the decision process.

REFERENCES

Bellman, R. and L. A. Zadeh, 1970, Decision-Making in a Fuzzy Environment, *Management Science*, 17: 141–164.

Chorafas, D. N., 1994, *Chaos Theory in the Financial Markets* (Chicago: Probus).

Cummins, J. D. and R. A. Derrig, 1993, Fuzzy Trends in Property-Liability Insurance Claim Costs, *Journal of Risk and Insurance*, 60: 429–465.

Cummins, J. D. and R. A. Derrig, 1995, Fuzzy Financial Pricing of Property-Liability Insurance, Working Paper, Center for Research on Risk and Insurance, Wharton School of the University of Pennsylvania, Philadelphia.

Derrig, R. A. and K. Ostaszewski, 1995, Fuzzy Techniques of Pattern Recognition in Risk and Claim Classification, *Journal of Risk and Insurance*, 62: 447–482.

DeWit, G. W., 1982, Underwriting and Uncertainty, *Insurance: Mathematics and Economics*, 1: 277–285.

Driankov, D., H. Hellendoorn, and M. Reinfrank, 1993, *An Introduction to Fuzzy Control* (Berlin: Springer-Verlag).

Dubois, D. and H. Prade, 1980, *Fuzzy Sets and Systems: Theory and Applications* (San Diego: Academic Press).

Erbach, D. W. and E. Seah, 1993, *Discussion of* The Application of Fuzzy Sets to Group Health Underwriting by V. R. Young, *Transactions of the Society of Actuaries*, 45: 585–587.

Glorennec, P. Y., 1994, Learning Algorithms for Neuro-Fuzzy Networks, in: A. Kandel and G. Langholz, eds., *Fuzzy Control Systems* (Boca Raton, Fla.: CRC Press).

Kandel, A. and G. Langholz, eds., 1994, *Fuzzy Control Systems* (Boca Raton, Fla.: CRC Press).

Katayama, R., Y. Kajitani, and Y. Nishida, 1994, A Self Generating and Tuning Method for Fuzzy Modeling Using Interior Penalty Method and Its Application to Knowledge Acquisition of Fuzzy Controller, in: A. Kandel and G. Langholz, eds., *Fuzzy Control Systems* (Boca Raton, Fla.: CRC Press).

Klir, G. J. and T. A. Folger, 1988, *Fuzzy Sets, Uncertainty, and Information* (Englewood Cliffs, N.J.: Prentice-Hall).

Kosko, B., 1990, Fuzziness vs. Probability, *International Journal of General Systems*, 17: 211–240.

Lemaire, J., 1990, Fuzzy Insurance, *ASTIN Bulletin*, 20(1): 33–55.

Mamdani, E. H., 1974, Application of Fuzzy Algorithms for Control of Simple Dynamic Plant, *Proceedings of the Institute of Electrical Engineers*, 121: 1585–1588.

Mizumoto, M., 1994, Fuzzy Controls under Product-Sum-Gravity Methods and New Fuzzy Control Methods, in: A. Kandel and G. Langholz, eds., *Fuzzy Control Systems* (Boca Raton, Fla.: CRC Press).

Ostaszewski, K., 1993, *Fuzzy Set Methods in Actuarial Science* (Schaumburg, Ill.: Society of Actuaries).

Siegel, P. H., A. de Korvin, and K. Omer, eds., 1995, *Applications of Fuzzy Sets and the Theory of Evidence to Accounting* (Greenwich, Conn.: JAI Press).

Young, V., 1993, The Application of Fuzzy Sets to Group Health Underwriting, *Transactions of the Society of Actuaries*, 45: 551–590.

Young, V., 1996a, Fuzzy Subsethood, *Fuzzy Sets and Systems*, 77: 371–384.

Young, V., 1996b, Adjusting Indicated Insurance Rates: Fuzzy Rules that Consider Statistical Data, unpublished manuscript, University of Wisconsin–Madison.

Zadeh, L. A., 1965, Fuzzy Sets, *Information and Control*, 8: 338–353.

Zadeh, L. A., 1975a, The Concept of Linguistic Variable and Its Application to Approximate Reasoning (Part 1), *Information Sciences*, 8: 199–249.

Zadeh, L. A., 1975b, The Concept of Linguistic Variable and Its Application to Approximate Reasoning (Part 2), *Information Sciences*, 8: 301–357.

Zadeh, L. A. 1976, The Concept of Linguistic Variable and Its Application to Approximate Reasoning (Part 3), *Information Sciences*, 9: 43–80.

Zimmermann, H. J., 1987, *Fuzzy Sets, Decision Making, and Expert Systems* (Boston: Kluwer Academic Publishers).