© The Journal of Risk and Insurance, 1995 Vol. 62, No. 3, 535-574

Examining Changes in Reserves Using Stochastic Interest Models

Siu-Wai Lai
Edward W. Frees

ABSTRACT

A fundamental problem in actuarial science is the determination of the reserves necessary to meet future obligations. Reserves are useful quantities because they summarize a vector of discounted cash flows. However, through this summarization, they mask the dynamic nature of interest rates. To study the effects on reserves of the dynamic nature of a stochastic interest environment, we look at a change in discounted reserves and study the potential short-term consequences of changes in the interest environment. Both the traditional linear ARIMA models and newer nonlinear autoregressive conditionally heteroskedastic (ARCH) processes are used to model the force of interest stochastically. We find that, in general, the next period reserve is a function of the previous interest rate. However, this is not true when the force of interest can be modeled as a white noise process. Explicit formulas compute changes in discounted reserves for linear interest rate processes. For nonlinear processes, we describe some approximations and exact simulation algorithms for these computations. An empirical example using a U.S. Treasury security series is presented and evidence of ARCH behavior is found.

Introduction

On balance sheets, assets must equal liabilities plus net worth of the firm. An important component of these liabilities for insurers is the reserve, the portion of a firms' assets set aside to meet future uncertain obligations arising from insurance contracts. Although the obligations of each contract are contingent upon uncertain future events, and thus may be modeled stochastically, the reserve set aside is a single number. There are limitations when using a single number to summarize a stochastic quantity. However, reserves play a promi-

---

Siu-Wai Lai is a Ph.D. student of statistics at the University of Wisconsin–Madison. Edward W. Frees is Professor of Business and Statistics at the University of Wisconsin–Madison. The first author gratefully acknowledges a Ph.D. grant support from the Society of Actuaries. The research of the second author was supported by a research grant from the Society of Actuaries, Actuarial Education and Research Fund, and a professorship funded by Time Insurance Company. The authors wish to thank the Editor, two anonymous reviewers, and James Hickman for their valuable comments.

nent role in financial statements and thus these quantities are important to managers of insurance organizations.

Several important problems in actuarial science rely heavily on the determination of a reserve. To illustrate, if a company or a block of business is to be traded on the open market, a value must be determined for the associated set of obligations. Thus, it is useful to think of a reserve as the "value" associated with a set of stochastic obligations. As another application, reserves traditionally have been used as a measure of an organization's financial strength. In this context, the reserve should be larger than the "value" of obligations, because a conservative approach should be taken for assessing potential future obligations.

Life insurance and annuity reserves are calculated by summarizing discounted cash flows, where the discounting is done with respect to investment earnings, as well as decrements due to mortality, disability, policy lapse, and so on, that may be applicable to a particular policy. For brevity, this articles considers only investment earnings and the mortality decrement. Extensions to the multidecrement case are straightforward.

In the traditional insurance literature, as in Jordan (1967), the deterministic assumption dominates the development of the theory of life contingencies. Namely, mortality occurs according to a known mortality table and the interest rate is assumed to have a deterministic value. One step further is to allow the age at death to be a random variable, although the interest rate is assumed to be deterministic—a "semi-stochastic" approach followed in Bowers et al. (1986). Insurance literature has generalized the traditional theory of life contingencies by introducing stochastic variation in interest rates (Boyle, 1976; Waters, 1978; Panjer and Bellhouse, 1980; Bellhouse and Panjer, 1981; Giaccotto, 1986; Dhaene, 1989; Frees, 1990; and Beekman and Fuelling, 1990, 1991). Additional stochastic interest models from financial economics are discussed below.

This article computes reserves as (conditional) expectations of sums of future cash flows. Motivation for this approach can be found in, for example, Bowers et al. (1986) for the semi-stochastic approach and Bühlmann (1992) for models using stochastic interest.

Here, we are primarily concerned with quantifying changes in reserves from one financial period to the next. Changes in reserves could be used to quantify the amount of profit released, as in Ramlau-Hansen (1988), which studied gains and losses emerging from margins built into mortality and other decrements and ignored those arising from investments. To complement that work, we focus on changes arising from stochastic interest rates and do not explicitly consider margins built into other decrement rates.

Changes in value of future obligations due to dynamic models of interest have been considered extensively in the financial economics literature, in particular as part of *immunization theory,* which deals with *instantaneous* changes in value. Here, we examine changes in value from one financial period to the next.

The new idea of examining changes in reserves can be illustrated by considering the following simple scenario. Let $\{y_s\}$ represent the random force of interest in the sth period. As argued in Frees (1990), y_s can be interpreted as a one-period spot rate. Consider an obligation that can be expressed as a T-year pure discount bond. At time zero, the random present value of one unit payable at time T is

$$\nu_T = \prod_{s=1}^{T} e^{-y_s} = \exp(-\sum_{s=1}^{T} y_s).$$

Without loss of generality, it is assumed that the time interval is one year. Suppose an insurer must pay one unit T years later with certainty, but under a stochastic interest rate environment. The reserve at time zero is denoted by

$$V_0^{(T)} = E(\nu_T) = E[\exp(-\sum_{s=1}^{T} y_s)],$$

where the expectation is taken at time zero. Because the interest rates prior to time zero are all known at time zero, $V_0^{(T)}$ is a deterministic value.

After one year, the maturity time of the payment shortens by one and the reserve (at time one) becomes

$$V_1^{(T-1)} = E[\exp(-\sum_{s=2}^{T} y_s)],$$

where the expectation is taken at time one. In general, because future spot rates may depend on the first-period spot rate, the quantity $V_1^{(T-1)}$ is a function of the random first-period spot rate y_1.

Although the obligations are random, their expected values define the (nonrandom) reserve. Following standard terminology, the reserve is the portion of liabilities that can be recognized in the insurer's balance sheet. Assets are held to meet the random obligations, where, at time zero, the known value of assets equals the reserve. In the absence of perfect matching, the random obligations and stochastic appreciaton of the assets diverge over time. This article is about quantifying one aspect of the asset's adequacy in meeting the obligations.

This article examines the change from the initial reserve V_0 to the time one reserve V_1. To examine this change, we discount the time one reserve $V_1^{(T-1)}$ back to time zero and then study its distribution. That is, we investigate the distribution and the statistical properties of the random reserve

$$e^{-y_1} V_1^{(T-1)},$$

from a time zero point of view. Although V_0 represents the current expected value of the obligation, V_1 represents the value at the subsequent time period. Thus, for budgeting and other purposes, V_1 and its discounted value $e^{-y_1} V_1$ are important quantities for risk and other financial managers.

At time zero, the discounted next-period reserve is stochastic because it is a function of the stochastic first-period interest rate y_1. However, it is a deterministic value at time one since y_1 becomes known at that time. In the case of a T-year pure discount bond and random force of interest $\{y_s\}$,

$$V_0^{(T)} = E(e^{-y_1} V_1^{(T-1)}), \tag{1}$$

where the expectation is taken at time zero. However, as shown in the Appendix, the relationship in equation (1) must be modified for a life insurance policy.

The next section of this article describes the model used in the analysis and discusses the continuous-time equilibrium term structure model and the discrete-time ARIMA/linear process. Then, the linear process for interest rates is investigated. The following section considers a nonlinear process for interest rates, the autoregressive conditionally heteroskedastic (ARCH) process, which is widely used in economics. An empirical example on the interest rates is presented before the conclusion. The proofs of all results are given in the Appendix.

The Basic Model

Insurance Model

We consider here the individual risk model for insurance contracts, using the notation of Bowers et al. (1986). Denote the valuation time to be h so that, at the initial valuation, $h = 0$. Assume that there are n policies in the block of business. For the ith policy, the age at issue is x_i, the duration is k_i when $h = 0$, the curtate random time of decrement is K_i, and the curtate-future-lifetime is J_i (i.e., $J_i = K_i - k_i - h$). Suppose that a death benefit b_{i,K_i+1} is payable at the end of the year of loss and that the annual premiums $a_{i,m}$ are payable at the beginning of each year up to and including the year of loss. Then, at time point $h + \tau + 1$, the random cash flow of the ith policy is

$$F_{i,\tau+1}^{(h)}(J_i) = \begin{cases} -a_{i,k_i+h+\tau+1} & \text{if } J_i > \tau \\ b_{i,k_i+h+\tau+1} & \text{if } J_i = \tau \\ 0 & \text{if } J_i < \tau, \end{cases} \tag{2}$$

where $F_{i,0}^{(h)} = -a_{i,k_i+h}$, and the probability function of J_i is: $\text{Prob}(J_i = \tau) = {}_{\tau|}q_{x_i+k_i+h}$, $\tau = 0, 1, ...$; and $\text{Prob}(J_i > \tau) = {}_{\tau+1}p_{x_i+k_i+h}$. Here, ${}_{\tau|}q_x$ and ${}_{\tau}p_x$ are the traditional deferred decrement probabilities and survival functions calculated from a mortality table.

Because the definition of cash flow is quite general, it can be used for general insurance as well as combinations of whole life insurance, term life insurance, deferred life insurance, annuities, and pure discount bonds.

Interest Rate Model

Let y_s denote the force of interest in the sth year (s = 1, 2,...). It is natural to assume that y_s has a parametric form,

$$y_s = f(\underline{\theta};\varepsilon_1,...,\varepsilon_s),$$

where $\underline{\theta}$ is a vector of parameters, $\{\varepsilon_s\}$ are independent and identically distributed (i.i.d.), and f is a known function. To illustrate, the next section considers the recursive linear process

$$y_t = a + \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t - \iota_1\varepsilon_{t-1} - ... - \iota_q\varepsilon_{t-q}.$$

The traditional ARMA/ARIMA model belongs to this class of model. In the fourth section, we consider an ARCH model, a nonlinear representation for the force of interest. Before proceeding, the following basic assumption and notations are made.

Assumption

y_t is a Borel measurable function of $(\varepsilon_1, \varepsilon_2,..., \varepsilon_t)$.

Notations

Let $M(t) = E(e^{t\varepsilon})$, the moment-generating function of ε.

$$\sum_{s=i}^{j} = 0 \text{ for } j<i,$$

and,

$$\prod_{s=i}^{j}=1 \text{ for } j<1,$$

$E(\cdot|X)$ means taking expectation conditional on the information generated by X.

The Reserves

To introduce reserves, we first define the loss at valuation time h for the ith policy,

$$L_i^{(h)} = \sum_{\tau=0}^{\infty} F_{i,\tau}^{(h)} \exp(-\sum_{s=h+1}^{h+\tau} y_s).$$

The sum of losses for the whole block of business at time h is

$$S_L^{(h)} = \sum_{i=1}^{n} \sum_{\tau=0}^{\infty} F_{i,\tau}^{(h)} \exp(-\sum_{s=h+1}^{h+\tau} y_s).$$

The reserve at time h is defined as $V_h = E_h(S_L^{(h)})$, where $E_h(\cdot) = E(\cdot|\varepsilon_1,..., \varepsilon_h)$, so that V_h is a function of $(\varepsilon_1,..., \varepsilon_h)$. The discussion below concentrates on V_0 and V_1, the reserves at the beginning of valuation and its following period, respectively. The results can be extended to consider further periods.

The following discussion assumes the random cash flows ($F_{i,\tau}^{(h)}$) are independent of the stochastic interest rates (y_s). The following result for the reserves is a basic one that we will use frequently.

Proposition 1. Under the basic assumption above, we have

$$V_0 = E_0(S_L^{(0)}) = \sum_{\tau=0}^{\infty} (\sum_{i=1}^{n} c_{\tau,k_i}^{(i)}) \cdot E[\exp(-\sum_{s=1}^{\tau+1} y_s)] - \sum_{i=1}^{n} a_{i,k_i}, \tag{3}$$

$$V_1 = E_1(S_L^{(1)}) = \sum_{\tau=0}^{\infty} (\sum_{i=1}^{n} c_{\tau,k_i+1}^{(i)}) \cdot E[\exp(-\sum_{s=2}^{\tau+2} y_s)|\varepsilon_1] - \sum_{i=1}^{n} a_{i,k_i+1}, \tag{4}$$

where

$$c_{\tau,k_i}^{(i)} = b_{i,k_i+\tau+1} \cdot {}_{\tau}|q_{x_i+k_i} - a_{i,k_i+\tau+1} \cdot {}_{\tau+1}p_{x_i+k_i}, \tag{5}$$

provided the expectations exist.■

In fact, $E(F_{i,\tau+1}^{(0)}) = c_{\tau,k_i}^{(i)}$, the expected cash flow of the ith policy at time h = 0, and $E(F_{i,\tau+1}^{(1)}) = c_{\tau,k_i+1}^{(i)}$, the expected cash flow of the ith policy at time h = 1. Although V_0 is a constant, V_1 is usually stochastic since it is a function of ε_1, the disturbance of force of interest generated in period one. When considering the discounted reserve, we simply multiply V_1 by e^{-y_1}. Actuarial science traditionally presents recursive calculations of reserves. The Appendix provides the recursive calculations relating V_1 to V_0, which generalize the T-year pure discount bond case described in the introduction.

Other Interest Rate Models

Several models of stochastic interest have been developed in the financial economics literature. These models have been developed from economic frameworks including equilibrium analysis and no arbitrage arguments. Thus, the emphasis differs from the purely statistical models introduced above. Here, we describe some of the similarities and differences of these competing models.

One disadvantage of the statistical models is that they allow only one-period investments, unlike the financial economic models, which consider the entire term structure of interest rates. It is well known (see, e.g., Ingersoll, 1987, chap. 13), in a world with deterministic interest, that precluding arbitrage opportunities means these two economic models are equivalent. However, this has also been shown to be true in certain stochastic interest environments, as follows.

Let P(t,s) denote the price at time t of a pure discount bond paying one at time s, $t \leq s$. If r(u) is the instantaneous risk-free interest rate at time u, $t \leq u \leq s$, then the local expectation hypothesis of Cox, Ingersoll, and Ross (1981) is written as

$$P(t,s) = E[\exp(-\int_t^s r(u)du) | r(t)].$$

They show that only this proposition (of three considered) can be sustained in a continuous-time rational expectations equilibrium. In their appendix, the logarithmic utility model also supports this form of the expectation hypothesis in discrete time:

$$P(t,s) = E_t[\exp(-\sum_{u=t}^{s-1} y_u)],$$

where E_t denotes the expectation taken at time t. Hence, the use of

$$E[\exp(-\sum_{s=1}^{T} y_s)]$$

as a price at time zero is justified.

Next, we try to find evidence to support the use of an ARIMA/linear process on the interest rates, as well as the conditional heteroskedasticity aspect for the error variance. A single-factor model for the instantaneous risk-free interest rate r usually is assumed to follow a diffusion process of the form

$$dr = (a+br)dt + \sigma r^{\beta} dB_t, \tag{6}$$

where B is a standard Brownian motion, and a, b, σ, and β are parameters with $\beta \geq 0$. This specification includes as special cases almost all the models that have appeared in the literature: $\beta = 0$ corresponds to the model proposed by Vasicek (1977); $\beta = 1/2$ was considered by Cox, Ingersoll, and Ross (1985); and $\beta = 1$ was used by Courtadon (1982) and also by Dothan (1978) with an additional restriction that $a = b = 0$; and others.

Here, we concentrate our discussion on the Vasicek and Cox, Ingersoll, and Ross models. The continuous-time Vasicek process is

$$dr = \kappa(\theta - r)dt + \sigma dB_t.$$

The change in the level of r between time t and t+1 may be written (see Vasicek, 1977, eqs. [25] and [26]) as

$$r_{t+1} - r_t = (1-e^{-\kappa})(\theta - r_t) + u_{t+1}$$

or

$$r_{t+1} = \theta(1-e^{-\kappa}) + e^{-\kappa} r_t + u_{t+1}, \tag{7}$$

where

$$\mathrm{Var}(u_{t+1}) = \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa}),$$

the drift (constant term) is $\theta(1-e^{-\kappa})$ and the distribution of u_t is normal. Obviously, the discrete-time econometric specification shown in equation (7) of the Vasicek model is a traditional AR(1) model.

The other well-known term structure model is the Cox, Ingersoll, and Ross model:

$$dr = \kappa(\theta\text{-}r)dt+\sigma\ \sqrt{r}\ \ dB_t.$$

Note that, although the models are arbitrage-free, they are approximations due to the lack of replication. The discrete-time version may be represented (see Cox, Ingersoll, and Ross, 1985, eq. [19]) as

$$r_{t+1}\text{-}r_t\ =\ (1\text{-}e^{-\kappa})(\theta\text{-}r_t)+u_{t+1}$$

or

$$r_{t+1}\ =\ \theta(1\text{-}e^{-\kappa})+e^{-\kappa}r_t+u_{t+1},$$

where

$$Var_t(u_{t+1})\ =\ \lambda_1+\lambda_2\ r_t,$$

$$\lambda_1\ =\ \frac{\theta\sigma^2}{2\kappa}\ (1-e^{-\kappa})^2,\quad \lambda_2\ =\ \frac{\sigma^2}{\kappa}\ (e^{-\kappa}-e^{-2\kappa}),$$

and the drift (constant term) is $\theta(1\text{-}e^{-\kappa})$. The conditional distribution of u_{t+1} is noncentral χ^2 and converges to normal distribution only when the time interval between two consecutive observations tends to zero. Over short periods of time, the deviation from normality is unlikely to be significant, and we treat u_{t+1} as normal (Chan et al., 1992, and Brown and Schaefer, 1994, also adopt this approach). Write the discrete-time version as the traditional AR(1) format:

$$r_t = a+\phi r_{t-1}+u_t,$$

where $a = \theta(1\text{-}e^{-\kappa})$, $\phi = e^{-\kappa}$. Then

$$r_t\ =\ \theta+\sum_{i=0}^{\infty}\ \phi^i u_{t-i}.$$

Substituting into the $Var_t(u_{t+1})$ yields

$$Var_t(u_{t+1})\ =\ \lambda_1+\lambda_2 r_t\ =\ (\lambda_1+\lambda_2\theta)+\lambda_2\sum_{i=0}^{\infty}\ \phi^i u_{t-i}.$$

In order to guarantee the positivity of $Var_t(u_{t+1})$, we use u_{t-i}^2 or $|u_{t-i}|$ on the right-hand side. The autoregressive conditional heteroskedasticity (ARCH) feature appears. Linear declining weights in the expression of $Var_t(u_t)$ were used in Engle (1982) and Bollerslev (1986) for weighting the past (squared) residuals. We found exponentially declining weights. Because the weights decay exponentially, only the first few past residuals are effectively useful in the dynamic evolvement of the variance of error term. (Of course, the degree of decay depends upon the magnitude of ϕ.)

The ARCH process has been applied to the term structure in finance literature. Longstaff and Schwartz (1992) present a two-factor general equilibrium model of the term structure.[1] The discrete-time approximation of the continuous-time specification is then formulated as a generalized autoregressive conditional heteroskedasticity (GARCH) framework introduced by Bollerslev (1986). Also, Bollerslev, Engle, and Wooldridge (1988) use a trivariate GARCH model to implement a capital asset pricing model with time-varying covariances, assuming the market consists of only bills, bonds, and stocks. In this article, we work in the univariate case; extension to a multivariate aspect will be investigated in future research.

Empirical studies suggest that model (6) does not match the observed data well (see Becker, 1991, and Chan et al., 1992). If the spot rate is assumed to follow a stochastic differential equation (6) in a continuous-time setting, then its instantaneous changes depend only on the current value and an independent random error process. The discrete-time econometric specification of the short-term riskless rate then usually results in the AR(1) formulation *only.* Extensions to higher-order ARIMA or linear processes can be used to model the short-term rates and may capture more information about the variation of the term structure.

To avoid the creation of arbitrage opportunities, the market is assumed to be efficient. However, it is well known that frictions in the real market place exist. That is, there are transaction costs, taxes, restrictions on short sales, asymmetries in information available to investors, etc. Perhaps due to these frictions, the spot rate curves derived by the theoretical equilibrium/arbitrage-free models do not conform well to observed data on bond yields and prices. In addition, these frictions exist for pricing assets in an efficient, highly active bond market. Further frictions arise in pricing liabilities due to the absence of an efficient secondary market for insurance reserves.

For the concept of changes in values over a period, since no particular model needs to be assumed for the term structure, we can use our idea of changes in reserves for equilibrium models as well as actuarial models. For solvency valuation purposes, we only need to value the *net* cash flows; that is, the excess of incomes over outgoing liabilities. In this case, the valuation method is less important than the case of valuing assets or liabilities. See Frees (1990) for solvency considerations of a life insurer.

Linear Interest Rate Process

Define a linear interest process so that the force of interest $\{y_t\}$ can be represented as

$$y_t = a+\phi_1 y_{t-1}+...+ \phi_p y_{t-p}+\varepsilon_t-\iota_1\varepsilon_{t-1}-...-\iota_q\varepsilon_{t-q}, \tag{8}$$

where $(a, \phi_1,..., \phi_p,\iota_1,..., \iota_q)$ are parameters, $\{\varepsilon_t\}$ are i.i.d., $E(\varepsilon_t) = 0$, $(y_0,y_{-1},..., y_{1-p})$, and $(\varepsilon_0,\varepsilon_{-1},..., \varepsilon_{1-q})$ are known. The traditional ARMA/ARIMA

[1] The two factors are the short-term interest rate and the volatility of the short-term interest rate.

model belongs to this class of process. The linear interest rate process also can be written in the form

$$y_t = \gamma_t + \sum_{i=0}^{t-1} \beta_i \varepsilon_{t-i}, t \geq 1, \tag{9}$$

where

$$\gamma_t = \sum_{i=1}^{p} y_{i-p} \sum_{j=\max(0,i-t)}^{i-1} \phi_{p-j} \alpha_{j-i+t} - \sum_{i=1}^{q} \varepsilon_{i-q} \sum_{j=\max(0,i-t)}^{i-1} \iota_{q-j} \alpha_{j-i+t} + a \sum_{i=0}^{t-1} \alpha_i, \tag{10}$$

$$\alpha_0 = 1,\ \beta_0 = 1;\ \alpha_i = \sum_{j=1}^{\min(i,p)} \phi_j \alpha_{i-j},\ i \geq 1;\ \beta_i = \alpha_i - \sum_{j=1}^{\min(i,q)} \iota_j \alpha_{i-j},\ i \geq 1 \tag{11}$$

(see, e.g., Box and Jenkins, 1976, or Dhaene, 1989). In other words, y_t is affine (linear plus constant) in $(\varepsilon_1,\dots,\varepsilon_t)$.

If the force of interest follows a linear process, then the two discount factors in Proposition 1 can be expressed explicitly by using the next theorem.

Theorem 1. For a one-unit T-year default-free pure discount bond, if $\{y_t\}$ follows a linear interest process, then

$$V_0^{(T)} = E[\exp(-\sum_{s=1}^{T} y_s)] = \prod_{l=0}^{T-1} M(-\sum_{i=0}^{l} \beta_i) \cdot \exp(-\sum_{s=1}^{T} \gamma_s),$$

and

$$V_1^{(T-1)} = E[\exp(-\sum_{s=2}^{T} y_s) | \varepsilon_1] = \prod_{l=0}^{T-2} M(-\sum_{i=0}^{l} \beta_i) \cdot \exp(-\sum_{s=2}^{T} \gamma_s) \cdot \exp(-\varepsilon_1 \sum_{i=1}^{T-1} \beta_i),$$

where $\{\beta_i\}$ and $\{\gamma_s\}$ are defined in equations (10) and (11).■

Therefore, supposing a linear process for the force of interest, we can combine Proposition 1 and Theorem 1 to write the reserves V_0 and V_1. To illustrate, it is of interest to write the special case of independent interest.

Corollary 1 (Independent Interest Case). If $\phi_1 = \dots = \phi_p = \iota_1 = \dots = \iota_q = 0$ so that $\{y_t\}$ are i.i.d., then

$$V_0 = \sum_{\tau=0}^{\infty} (\sum_{i=1}^{n} c_{\tau,k_i}^{(i)}) \cdot [e^{-a} M(-1)]^{\tau+1} - \sum_{i=1}^{n} a_{i,k_i},$$

and

$$V_1 = \sum_{\tau=0}^{\infty} (\sum_{i=1}^{n} c_{\tau,k_i+1}^{(i)}) \cdot [e^{-a} M(-1)]^{\tau+1} - \sum_{i=1}^{n} a_{i,k_i+1}. ■$$

From equation (4), it is evident that, in general, V_1 is a function of ε_1, and is thus a random variable. However, in the i.i.d. case, V_1 becomes a constant, which means that it is independent of ε_1. The intuition is that in this i.i.d. case, $\{y_t\}$ are purely random, and thus ε_1 does not provide any sequential information about $(y_2,y_3,\dots)$ and hence the next-period reserve V_1. Therefore, by employing the expectation approach for pricing and assuming that the force of interest is generated by a white noise series, the measure V_1 (also for V_h, $h \geq 1$) is a deterministic value, which may not capture the real events. Examining an autocorrelated interest environment may model practical situations more appropriately than the independent interest benchmark case.

Default-Free Pure Discount Bond Example

The simplest special case for the cash flows is a pure discount bond. Suppose that an insurer must pay one unit T years later with certainty. Then, using Theorem 1 and a linear process for the interest rates as in equation (8), we have

$$e^{-y_1} \cdot V_1^{(T-1)} = \prod_{l=0}^{T-2} M(-\sum_{i=0}^{l} ß_i) \cdot \exp(-\sum_{s=1}^{T} \gamma_s) \cdot \exp(-\varepsilon_1 \sum_{i=0}^{T-1} ß_i), \quad T \geq 2.$$

Making the normality assumption for ε's yields

$$e^{-y_1} \cdot V_1^{(T-1)} \overset{d}{=} e^{U},$$

where

$$U \sim N(-\sum_{s=1}^{T} \gamma_s + \frac{1}{2}\sigma^2 \sum_{l=0}^{T-2} (\sum_{i=0}^{l} ß_i)^2, \sigma^2(\sum_{i=0}^{T-1} ß_i)^2), \quad T \geq 2,$$

$$\sigma^2 = \mathrm{Var}(\varepsilon_t).$$

Thus, the discounted reserve $e^{-y_1} \cdot V_1^{(T-1)}$ has a lognormal distribution in this special case. If σ^2 increases, so does the mean and variance of $e^{-y_1} \cdot V_1^{(T-1)}$.

For a single payment with fixed payment date as above, the exact distribution of $e^{-y_1}V_1$ can be obtained. However, for a general insurance contract such as life insurance or deferred annuity, analytically the exact distribution is extremely difficult to find. In this case, a moment-matching method can be used to approximate the distribution. A simulation example for the distribution under a general insurance contract is provided below.

A Simulation Example for Linear Process

To illustrate the distributions of V_1 and $e^{-y_1}V_1$, we consider a block of whole life business. For simplicity, policies are categorized into three groups of ten so the total size is 30. Assume that, for each category, ages at issue are x = 30, 30, 40, and durations are k = 5, 10, 5, respectively. All death benefits are one dollar. The mortality decrements are those in the 1979–1981 U.S. Life Table that appear in Bowers et al. (1986). Here, assume that we have the following stationary AR(2) model for the interest rate:

$$y_t = 0.08 + 0.6(y_{t-1} - 0.08) - 0.3(y_{t-2} - 0.08) + \varepsilon_t, \ t \geq 1,$$

where $\varepsilon_t \sim$ i.i.d. $N(0, \sigma = 0.025)$, $y_0 = 0.06$, and $y_{-1} = 0.07$, as in Giaccotto (1986) and Dhaene (1989). To compute the level premiums for each of three categories, first use constant force of interest 0.08 ($=E(y_t)$) to get the net level annual premiums, and then add 20 percent as the relative security loading to obtain the final level premiums.

Combining Proposition 1 and Theorem 1 yields analytic expressions for V_1 and its discounted version, $e^{-y_1}V_1$. To approximate these expressions, we performed 500 simulations. The resulting frequency histograms are shown in Figures 1a and 1b. Normally distributed curves are superimposed for comparison purposes. The sample mean and standard deviation of the simulated distribution of reserve V_1 are 0.7875 and 0.009748, respectively. For the discounted reserve $e^{-y_1}V_1$, the corresponding measures are 0.7349 and 0.02799. Interestingly, the dispersion for the discounted reserve $e^{-y_1}V_1$ is almost three times that for reserve V_1 (0.02799/0.009748 = 2.9) in this example. The difference originates from the additional random portion—the discount factor e^{-y_1}.

Because the mean of the force of interest $E(y_t) = 0.08$, suppose that we had used $\delta = 0.08$ as constant force of interest to calculate the next-period reserve. Then, $V_1 = 0.765$, which underestimates the "true" reserve, as shown in Figure 1. For comparison, choosing another $\delta = 0.079$ as constant force of interest gives $V_1 = 0.808$, which overestimates the "true" distribution. This example indicates that ignoring the stochastic interest rate environment would easily miss the target distribution.

Since V_0 and discounted reserve $e^{-y_1}V_1$ are both valued at time zero under the AR(2) interest rate environment, we can compare $V_0 = 0.602$ and the distribution of $e^{-y_1}V_1$ to see the changes in reserves.[2] To illustrate, we may wish to know how much to add to the current reserve so that the next-period funds can meet future obligations measured by V_1 with 95 percent probability. That is, we want to find A such that $P[(V_0 + A)\, e^{y_1} > V_1] = 0.95$. Hence A = (95th percentile of $e^{-y_1}V_1$) - $V_0 \approx 0.782 - 0.602 = 0.18$. The value 0.782 is the empirical distribution value. In fact, it can be approximated by using a normal variate.[3]

---

[2] Proposition 1 and Theorem 1 gives

$$V_0 = \sum_{\tau=0}^{\infty} \left(\sum_{i=1}^{n} c_{\tau,k_i}^{(i)}\right) \cdot \prod_{l=0}^{\tau} M\left(-\sum_{i=0}^{l} \beta_i\right) \cdot \exp\left(-\sum_{s=1}^{\tau+1} \gamma_s\right) - \sum_{i=1}^{n} a_{i,k_i},$$

where $M(u) = \exp(\sigma^2 u^2/2)$, the moment-generating function of ε_t.

[3] The sample values of skewness and kurtosis for V_1 and $e^{-y_1}V_1$ in this simulation example are 0.0643 and 3.19, and 0.155 and 3.21, respectively. The kurtosis defined here is equal to three for a normal variate. Thus, a normal distribution could be used to approximate the reserve V_1 even though slightly positive skewness and slightly thicker tails feature are exhibited from the sample quantities. The positivity of skewness of $e^{-y_1}V_1$ is larger than that of V_1, but a normal approximation to the distribution of discounted reserve $e^{-y_1}V_1$ seems adequate.

Figure 1a
Frequency Distribution for Random Reserve V_1

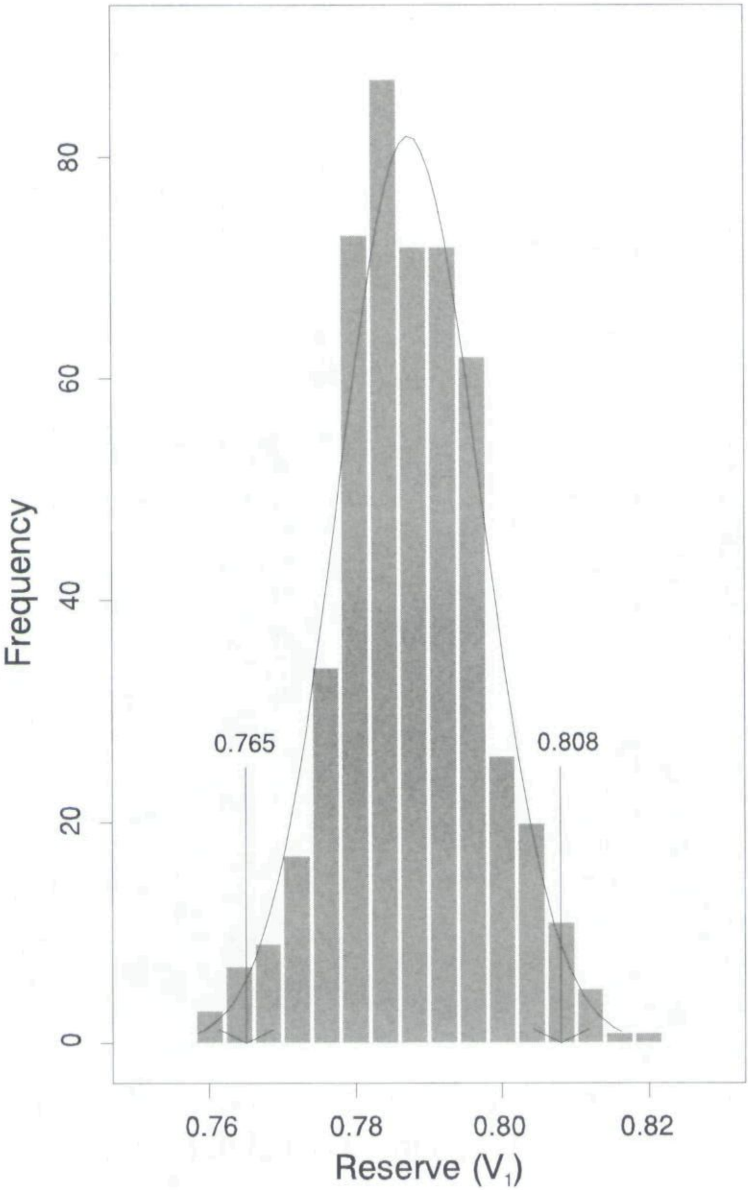
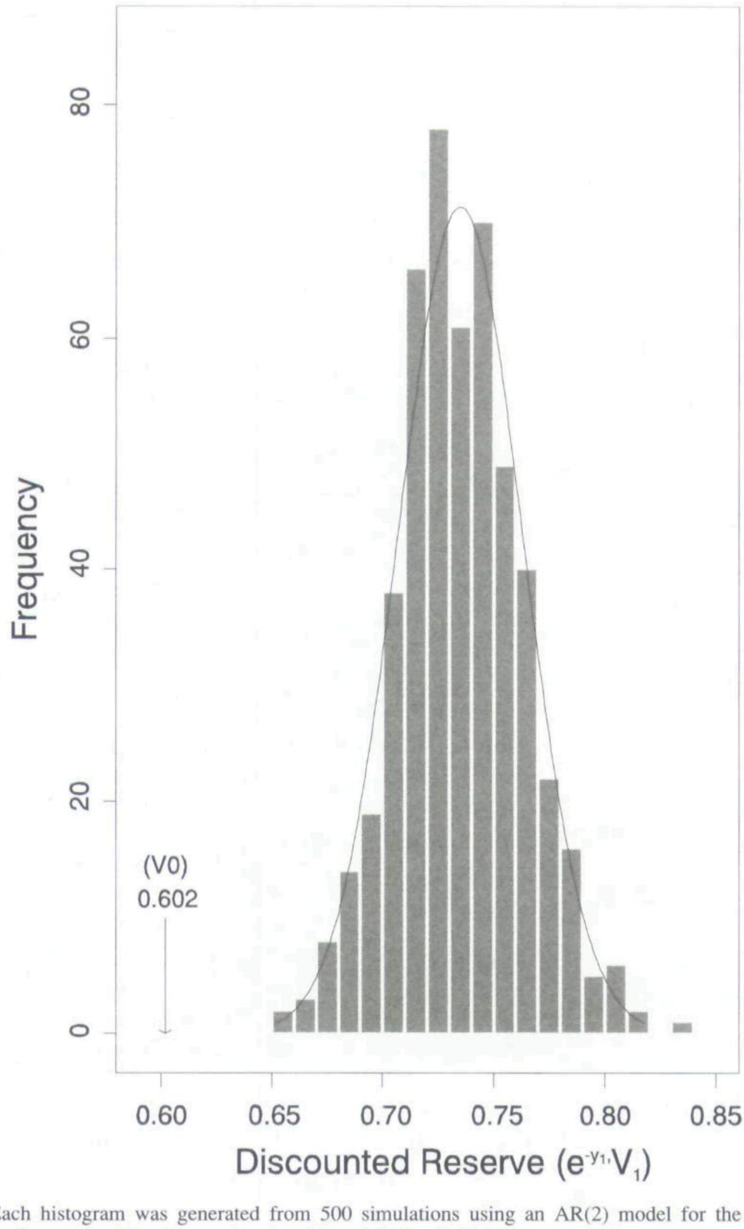


Figure 1b

Frequency Distribution for Discounted Reserve $e^{-y_1}V_1$



Note: Each histogram was generated from 500 simulations using an AR(2) model for the force of interest. For comparison, Figure 1a plots $V_1 = 0.765$ and 0.808, corresponding to constant force of interest 0.08 and 0.079, respectively. Figure 1b plots the initial reserve, $V_0 = 0.602$.

Nonlinear Interest Rate Process

Introduction to ARCH Process

Under the traditional linear time series setting, the conditional variance of one-step-ahead prediction is time invariant. Applied researchers have recognized the importance of explicitly modeling time-varying second- and higher-order moments. One of the most prominent tools that has emerged for describing such changing variances is the ARCH model of Engle (1982) and its various extensions. Bollerslev et al. (1992) contains an overview of some of the developments in the formulation of ARCH models and a survey of the numerous empirical applications using financial data. In particular, this survey includes a discussion of the modeling of interest rates.

The simplest nontrivial ARCH model is the first-order linear model given by

$$\varepsilon_t|\psi_{t-1} \sim N(0,h_t),$$
$$h_t = \delta_0 + \delta_1\varepsilon_{t-1}^2, \tag{12}$$

where ψ_{t-1} is the information set (σ-field) available at time t-1, and $\delta_0 > 0$, $\delta_1 \geq 0$, unknown parameters. The quantity ε_t could be the observation or the disturbance term from the relevant model. The nonlinear ARCH process is serially uncorrelated with nonconstant variances conditional on the past, but constant unconditional variances. This model captures the tendency for volatility clustering, that is, for large changes to be followed by other large changes, and small changes to be followed by other small changes, but of unpredictable sign. The nonlinearity stems from the variance of the disturbance term ε_t. If δ_1 were equal to zero, then the model would be the conventional Gaussian white noise. However, for $\delta_1 \neq 0$, the effect of equation (12) is to make the variance of the disturbance terms at time t dependent on the realized value of the disturbance term in the previous period.

We have just seen a simple example of an ARCH process for the innovations. To a certain extent, reserves V_0 and V_1, as obtained from equations (3) and (4), involve the moment-generating function of ε_t's. Because of the nonexistence of higher moments under the ARCH model of equation (12) and hence its moment-generating function (see Engle, 1982, and Bollerslev, 1986), we consider a model similar to expression (12) but in absolute value form:

$$\varepsilon_t|\psi_{t-1} \sim N(0,h_t),$$
$$h_t = \delta_0 + \delta_1|\varepsilon_{t-1}|, \tag{13}$$

where $\delta_0 > 0$, $\delta_1 \geq 0$. This model was mentioned in Engle's (1982) seminal article but is seldom used in the ARCH literature, perhaps due to the mathematical tractability difficulty in absolute values. Engle and Bollerslev (1986), Schwert (1990), and Higgins and Bera (1992) discuss the absolute version of the ARCH model. The absolute value form of the model in expression (13) shares the features described above for model (12); and its moment-generating function exists, a desirable characteristic. We note that δ_1 in model (13) depends on the measurement unit.

The discussion below uses the following model for the force of interest:

$$y_t = a + \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t - \iota_1 \varepsilon_{t-1} - ... - \iota_q \varepsilon_{t-q}, \tag{14a}$$

$$\varepsilon_t | \psi_{t-1} \sim N(0, h_t), \tag{14b}$$

$$h_t = \delta_0 + \delta_1 |\varepsilon_{t-1}|, \ t \geq 1. \tag{14c}$$

That is, we allow for conditional (prediction) variance to depend on the absolute value of immediately previous innovation.

Approximate Results for Reserves Under ARCH Processes

We examine the effect of δ_1 on the reserves V_0 and V_1 under the model described by expressions (14a), (14b), and (14c), supposing δ_1 to be small. Before stating the result for reserves, a lemma for the finiteness of the moment-generating function of innovations is given first. According to Engle (1982), "The absolute value form...can be shown to have finite variance for any parameter values." In fact, we can prove a stronger, new result that the moment-generating function is finite, hence so are all moments of ε_t's.

Lemma 1. If ε_t's follows model (14b) and (14c), then the moment-generating function of $(\varepsilon_1, ..., \varepsilon_T)$ is finite for all $\delta_0 > 0$, $\delta_1 \geq 0$.■

With the above lemma, we are ready to state the basic main result for a pure discount bond under the ARCH process.

Theorem 2. For a one-unit T-year default-free pure discount bond, under the model described by expressions (14a), (14b), and (14c) for the force of interest, we have that, as $\delta_1 \to 0$,

$$(1) \qquad V_0^{(T)} = E[\exp(-\sum_{s=1}^{T} y_s)] = \chi_0^{(T)} + O(\delta_1^3)$$

$$(2) \qquad V_1^{(T-1)} = E[\exp(-\sum_{s=2}^{T} y_s)|\varepsilon_1] = \chi_1^{(T)} + O(\delta_1^3),$$

where

$$\chi_0^{(T)} = \exp(-\sum_{s=1}^{T} \gamma_s + \frac{\delta_0}{2} \sum_{l=0}^{T-1} \zeta_l^2) \cdot [1 + \frac{\delta_1}{2} C_1^{(T)}(\varepsilon_0) + \frac{\delta_1^2}{2} C_2^{(T)}(\varepsilon_0)],$$

$$\chi_1^{(T)} = \exp(-\sum_{s=2}^{T} \gamma_s - \varepsilon_1 \sum_{i=1}^{T-1} \beta_i + \frac{\delta_0}{2} \sum_{l=0}^{T-2} \zeta_l^2) \cdot [1 + \frac{\delta_1}{2} C_1^{(T-1)}(\varepsilon_1) + \frac{\delta_1^2}{2} C_2^{(T-1)}(\varepsilon_1)],$$

$$C_1^{(T)}(x) = |x| \zeta_{T-1}^2 + \sqrt{\delta_0} \sum_{l=0}^{T-2} \zeta_l^2 q(\sqrt{\delta_0}\, \zeta_{l+1}),$$

$$C_2^{(T)}(x) = \frac{|x|}{\sqrt{\delta_0}} \zeta_{T-2}^2 [q(\sqrt{\delta_0}\, \zeta_{T-1}) - \frac{1}{\sqrt{2\pi}} e^{-\delta_0 \zeta_{T-1}^2/2}]$$

$$+ \frac{1}{4}\delta_0 \sum_{l=0}^{T-2} \zeta_l^4 [1 + \delta_0 \zeta_{l+1}^2 - q^2(\sqrt{\delta_0}\, \zeta_{l+1})]$$

$$+ \sum_{l=0}^{T-3} \zeta_l^2 q(\sqrt{\delta_0}\, \zeta_{l+2})[q(\sqrt{\delta_0}\, \zeta_{l+1}) - \frac{1}{\sqrt{2\pi}} e^{-\delta_0 \zeta_{l+1}^2/2}] + \frac{1}{4}[C_1^{(T)}(x)]^2,$$

$$q(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} + x[2\Phi(x) - 1],$$

$\Phi(x)$ = distribution function of a standard normal variable, and

$$\zeta_l = \sum_{i=0}^{l} \beta_i. \blacksquare$$

We can see that the terms outside the square brackets of $\chi_0^{(T)}$ and $\chi_1^{(T)}$ correspond to the results of the linear process case ($\delta_1 = 0$) under normality. It can be shown that both coefficients of the linear- and quadratic-order terms (with respect to δ_1)—C_1 and C_2—are nonnegative. Thus for a pure discount bond, when δ_1 is small, the reserves V_0 and V_1 are both larger in the absolute-value form ARCH process case than in the linear process case. Moreover, we expect V_0 and V_1 to increase if δ_1 increases.

Once we have Theorem 2 for a pure discount bond, we can have corresponding results for V_0 and V_1 in the general block of business case. That is, by retrieving equations (3) and (4) in Proposition 1 and substituting $\chi_0^{(\tau+1)}$ and $\chi_1^{(\tau+2)}$ for those two discount factors, respectively, the desired results are obtained.

The above analysis studies the case where δ_1 is assumed to be small. For larger values of δ_1, we resort to simulation techniques, as described below.

A Simulation Example for an ARCH Process

Unlike the ARMA model, the ARCH model takes into account the conditional heteroscedasticity. The model that we use for an ARCH process resembles that for a linear process. The insurance model is the same as the linear process case. However, the model for interest rate variations becomes:

$$y_t = 0.08 + 0.6(y_{t-1} - 0.08) - 0.3(y_{t-2} - 0.08) + \varepsilon_t,$$

$$\varepsilon_t | \psi_{t-1} \sim N(0, h_t),\ h_t = (0.025)^2 + \delta_1 |\varepsilon_{t-1}|,\ t \geq 1,$$

with $\delta_1 = 0, 0.001, 0.003, \ldots, 0.08$, where $y_0 = 0.06$, $y_{-1} = 0.07$, and $\varepsilon_0 = -0.01$. We use this ARCH model as a nonlinear time series example to simulate the next period reserve. The largest value of δ_1 chosen to investigate is 0.08 because, in Appendix B, we show that $sd(\varepsilon_t) \leq \sqrt{\delta_0 + \delta_1(\delta_1 + \sqrt{\delta_1^2 + 2\pi\delta_0})/\pi}$. With

$\delta_0 = (0.025)^2$ and $\delta_1 = 0.08$, we have $sd(\varepsilon_t) \le 0.072$, which is about 2.9 times the standard deviation of ε_t in the linear process case ($0.072/0.025 = 2.88$).

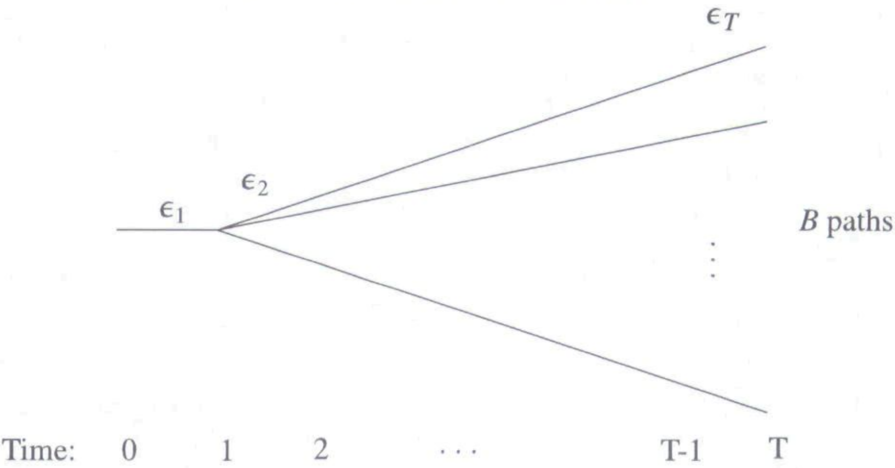
The valuation algorithm. Part of our simulation ideas comes from Tilley (1993), who presents a simulation algorithm for valuing American-style options. For nonlinear time series models, like ARCH processes, it is difficult to compute analytically $E[\exp(-\sum_{s=1}^{T} y_s)]$ or $E[\exp(-\sum_{s=2}^{T} y_s)|\varepsilon_1]$, and hence the distribution of the reserve, since the error term ε's are no longer independent. However, simulations can provide some insight into the distribution of V_1 and $e^{-y_1}V_1$. For simplicity, consider first a one-unit T-year pure discount bond. The reserve at time one is

$$E[\exp(-\sum_{s=2}^{T} y_s)|\varepsilon_1].$$

Because it involves an expectation that could not be expressed simply, we need to approximate it by simulation. Figure 2 illustrates how the simulation works.

Figure 2

Simulation Paths for the ARCH Process



Given a value of ε_1 (its 98th percentile, for example), we simulate $(\varepsilon_2,\ldots, \varepsilon_T)$ on each path from time 1 to T. Suppose we have B (= 1,200, say) such repetitions. Then the expectation $E[\exp(-\sum_{s=2}^{T} y_s)|\varepsilon_1]$ can be approximated by

$$B^{-1} \sum_{k=1}^{B} \exp(-\sum_{s=2}^{T} y_s^{(k)}),$$

where $y_s^{(k)}$ is the simulated sth period force of interest on path k (k = 1,..., B) for a certain percentile of ε_1.

Now extend a pure discount bond to the case of a whole block of business with n insureds. Let ω be the limiting age. From equation (4), we have

$$V_1 = \sum_{\tau=0}^{M-2} f_\tau E[\exp(-\sum_{s=2}^{\tau+2} y_s)|\varepsilon_1] - \sum_{i=1}^{n} a_{i,k_i+1},$$

where $M = \omega - \min_{1 \le i \le n}(x_i + k_i)$ and $f_\tau = \sum_{i=1}^{n} c^{(i)}_{\tau,k_i+1}$, since $f_\tau = 0$ for $\tau > M - 2$. In this case, we use the same ideas as above, replacing $T = \tau + 2$, $0 \le \tau \le M - 2$, then we obtain a value of V_1 that corresponds to a given percentile of ε_1.

Sensitivity of the reserve to the variance autoregression parameter. Our numerical example uses different values of δ_1 in the simulation. Thus, we can study the sensitivity of V_1 and $e^{-y_1}V_1$ to a change in the value of the variance autoregression parameter δ_1. For example, we might like to see how large this parameter can be before it matters. Therefore, we include the case $\delta_1 = 0$ for comparison purposes. For each percentile of ε_1, we implement the simulation method described above to obtain the values of $V_1(\mu_p,\delta_1)$, where μ_p denotes the 100pth percentile of the distribution of ε_1. By examining many different percentiles of ε_1, especially the tails, we can present the whole picture of sensitivity of the variance autoregression parameter on the next-period reserve.

The simulated values of V_1, and associated simulation standard errors, are summarized in Table 1. Plotting those values of V_1 together in Figure 3a presents a clearer picture. We see that, for each fixed value of ε_1, as δ_1 increases, the reserve V_1 also increases. This aspect is consistent with the discussion following Theorem 2 for a pure discount bond. Fixing δ_1, a larger value of ε_1 reduces the reserve V_1, as shown in Table 1 and Figure 3a. However, this is not necessarily always true. That is, recall from Theorem 1 that there need not be a monotonic relationship between the value of ε_1 and reserve V_1. From Figure 3a, it seems that V_1 increases quadratically for $\delta_1 \in [0, 0.08]$ so the approximation up to a second-order term appears adequate. When $\delta_1 < 0.02$, the linear-order term already gives a good approximation to the reserve V_1. The plot also indicates that the dispersion of V_1 increases as δ_1 increases.

A corresponding graph for the discounted reserve $e^{-y_1}V_1$ is shown in Figure 3b in which similar features appear. In this example, the discounted reserve is less influenced by δ_1 than the reserve, as indicated by the growing rate of curves in the figure. As noted in the linear process case, the dispersion of discounted reserve $e^{-y_1}V_1$ is larger than that of reserve V_1, for various values of δ_1. In addition, the discounted reserve seems to grow linearly for δ_1 up to about 0.03.

Next, we compare the values of reserve V_1 obtained from two different methods: by simulation and by the approximation result in Theorem 2. Table 2 shows the values of V_1 obtained from approximation approach. The corre-

Figure 3a

Plot of Reserve V_1 Against δ_1 for
Given Percentile of ε_1 Using Simulation Method

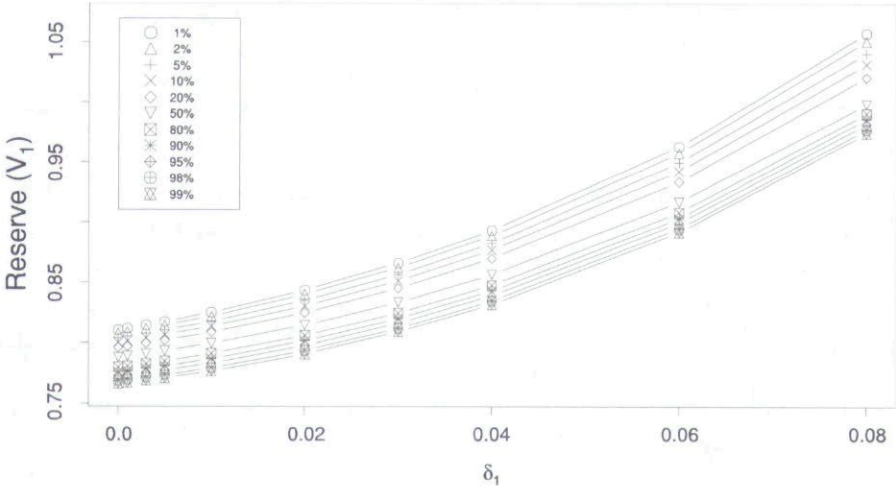
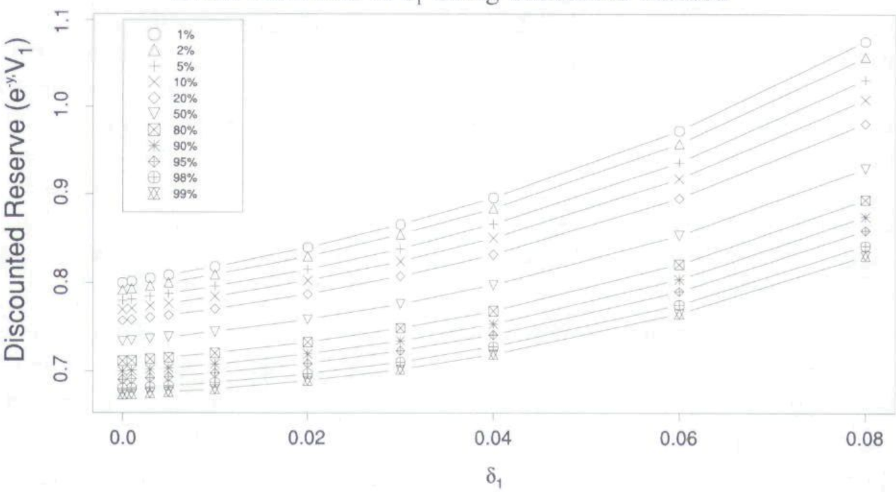


Figure 3b

Plot of Discounted Reserve $e^{-y_1}V_1$ Against δ_1 for
Given Percentile of ε_1 Using Simulation Method



Note: The 1 percent denotes the first percentile of ε_1, and so forth.

Table 1
Simulated Values of Reserve V_1 for Different Combinations of Percentiles of ε_1 and Values of δ_1

δ_1	Percentile of ε_1										
	1%	*2%*	*5%*	*10%*	*20%*	*50%*	*80%*	*90%*	*95%*	*98%*	*99%*
0	0.8107	0.8080	0.8041	0.8006	0.7963	0.7883	0.7804	0.7762	0.7728	0.7690	0.7665
	(0.0075)	(0.0075)	(0.0075)	(0.0074)	(0.0074)	(0.0073)	(0.0073)	(0.0072)	(0.0072)	(0.0072)	(0.0072)
0.001	0.8120	0.8093	0.8053	0.8017	0.7974	0.7893	0.7813	0.7772	0.7737	0.7700	0.7674
	(0.0077)	(0.0076)	(0.0076)	(0.0076)	(0.0075)	(0.0075)	(0.0074)	(0.0074)	(0.0073)	(0.0073)	(0.0073)
0.003	0.8147	0.8119	0.8078	0.8042	0.7998	0.7914	0.7834	0.7782	0.7757	0.7719	0.7693
	(0.0080)	(0.0079)	(0.0079)	(0.0078)	(0.0078)	(0.0077)	(0.0077)	(0.0076)	(0.0076)	(0.0076)	(0.0076)
0.005	0.8176	0.8147	0.8105	0.8068	0.8022	0.7936	0.7855	0.7813	0.7778	0.7740	0.7713
	(0.0082)	(0.0082)	(0.0082)	(0.0081)	(0.0081)	(0.0080)	(0.0080)	(0.0079)	(0.0079)	(0.0079)	(0.0078)
0.010	0.8253	0.8223	0.8178	0.8138	0.8090	0.7998	0.7915	0.7872	0.7837	0.7797	0.7771
	(0.0090)	(0.0090)	(0.0089)	(0.0089)	(0.0088)	(0.0087)	(0.0087)	(0.0086)	(0.0086)	(0.0086)	(0.0086)
0.020	0.8436	0.8402	0.8351	0.8306	0.8252	0.8148	0.8063	0.8018	0.7981	0.7940	0.7912
	(0.0108)	(0.0108)	(0.0107)	(0.0106)	(0.0105)	(0.0103)	(0.0103)	(0.0103)	(0.0103)	(0.0102)	(0.0102)
0.030	0.8660	0.8622	0.8565	0.8514	0.8453	0.8336	0.8248	0.8202	0.8164	0.8122	0.8093
	(0.0129)	(0.0128)	(0.0127)	(0.0126)	(0.0124)	(0.0122)	(0.0122)	(0.0122)	(0.0121)	(0.0121)	(0.0121)
0.040	0.8930	0.8887	0.8824	0.8767	0.8698	0.8565	0.8477	0.8430	0.8391	0.8347	0.8317
	(0.0152)	(0.0151)	(0.0149)	(0.0148)	(0.0146)	(0.0143)	(0.0143)	(0.0143)	(0.0143)	(0.0142)	(0.0142)
0.060	0.9628	0.9574	0.9495	0.9423	0.9336	0.9165	0.9079	0.9031	0.8990	0.8945	0.8914
	(0.0208)	(0.0206)	(0.0204)	(0.0201)	(0.0199)	(0.0193)	(0.0194)	(0.0194)	(0.0194)	(0.0194)	(0.0194)
0.080	1.0576	1.0508	1.0408	1.0318	1.0207	0.9983	0.9907	0.9859	0.9818	0.9771	0.9739
	(0.0281)	(0.0279)	(0.0275)	(0.0271)	(0.0267)	(0.0257)	(0.0260)	(0.0260)	(0.0261)	(0.0261)	(0.0261)

Note: The values in parentheses beneath the values of V_1 correspond to simulation standard errors. The number of simulations is 1,200.

sponding plot for Table 2 would look like Figure 3a. To compare these two methods let us consider the following ratio:

$$\text{Standardized Error} = \frac{\text{Approximation Value} - \text{Simulated Value}}{\text{Simulation Standard Error}}.$$

The standardized error values (reported in Table 3) are all less than 0.34 in absolute value. Theorem 2 therefore provides good approximations to the reserve V_1 under the absolute-value version ARCH model for the force of interest. All of the ratio values are negative, from which we deduce that the approximation method would slightly underestimate the "true" reserve V_1.

Empirical Example

To illustrate the importance of conditionally heteroskedastic models, we use data from the Citibank Interest Rate data base. Treasury interest rates are fundamental and usually used as a reference to other rates (for example, corporate rates). We consider the three-month Treasury bill yield as the short-term spot rate. This series consists of $N = 124$ monthly Treasury yields from the first month of 1982 through the fourth month of 1992. The monthly values are averages of daily figures that were originally obtained from the Board of Governors of the Federal Reserve System.

Figure 4 plots the time series and its first differences for three-month U.S. Treasure bills. The stationarity in mean is obtained after first differencing, but the variance appears to be changing over time. The summary statistics of the original and first differenced series are shown in Table 4.

For a simple illustration, we consider only an AR(1) model for the differenced series, which corresponds to the Vasicek and Cox, Ingersoll, and Ross models described earlier. It was shown above that the moment-generating function exists for the absolute-value form ARCH process described in equation (14c). Although the moment-generating function of the regular ARCH model described in equation (12) does not exist (except $\delta_1 = 0$), it is widely used in the literature. Hence, we also consider it in the data analysis for comparison. In sum, we fit an ARIMA(1,1,0) model for the three-month Treasury bill series (y_t), with the non-, absolute-, and regular-ARCH processes for the conditional variance of error term. Maximum likelihood estimation was used to fit each model. Estimation procedures for non-ARCH and regular ARCH models are well known. Estimation procedures for the absolute ARCH model and the consideration of the other competitive models for this yield rate series can be found in Lai (1995). The fitted models for ARIMA(1,1,0) are presented below. Asymptotic standard errors are in parentheses. Symbols *, †, and ‡ indicate statistical significance at 10 percent, 5 percent, and 1 percent, respectively. B denotes the backward shift operator.

Table 2

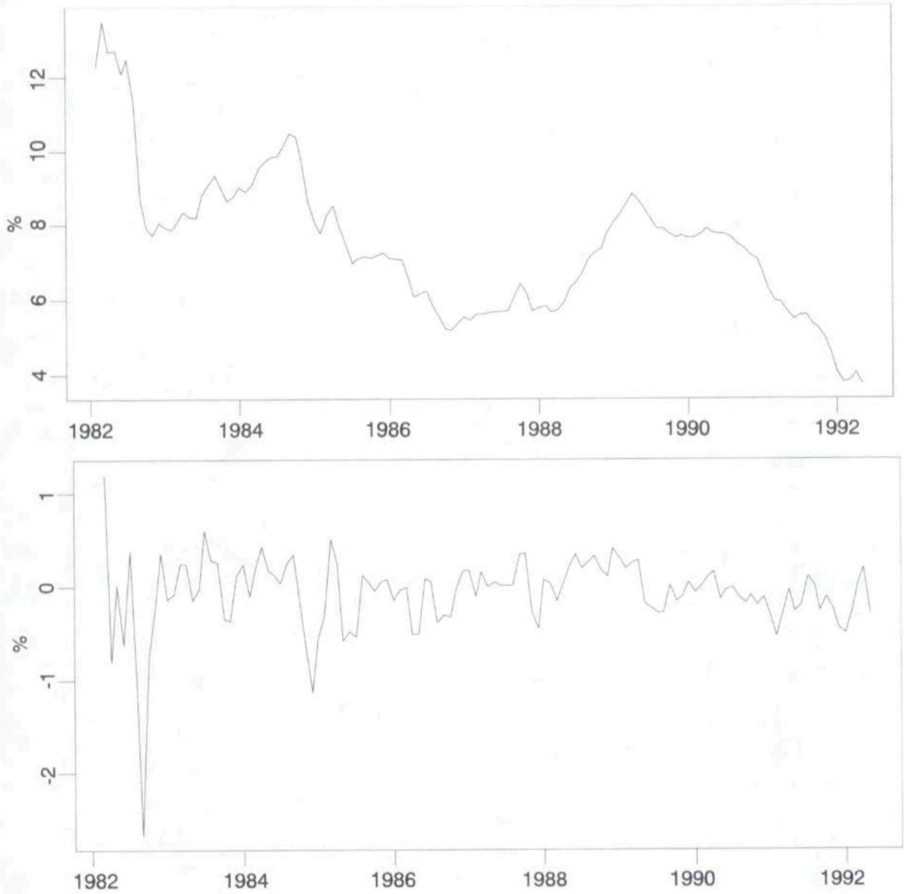
Values of V_1 Obtained from the Approximation Result of Theorem 2

	Percentile of ε_l										
δ_l	*1%*	*2%*	*5%*	*10%*	*20%*	*50%*	*80%*	*90%*	*95%*	*98%*	*99%*
0	0.8088	0.8061	0.8022	0.7987	0.7945	0.7865	0.7786	0.7744	0.7710	0.7672	0.7647
0.001	0.8099	0.8072	0.8033	0.7998	0.7955	0.7874	0.7794	0.7752	0.7718	0.7680	0.7654
0.003	0.8124	0.8096	0.8055	0.8019	0.7975	0.7893	0.7812	0.7770	0.7735	0.7697	0.7670
0.005	0.8150	0.8121	0.8080	0.8043	0.7998	0.7913	0.7832	0.7789	0.7754	0.7715	0.7689
0.010	0.8223	0.8193	0.8149	0.8110	0.8062	0.7972	0.7889	0.7845	0.7809	0.7769	0.7742
0.020	0.8406	0.8372	0.8323	0.8278	0.8225	0.8123	0.8036	0.7990	0.7953	0.7912	0.7883
0.030	0.8637	0.8599	0.8543	0.8493	0.8432	0.8318	0.8228	0.8181	0.8143	0.8100	0.8071
0.040	0.8918	0.8874	0.8810	0.8754	0.8685	0.8555	0.8465	0.8417	0.8378	0.8335	0.8305
0.060	0.9628	0.9572	0.9489	0.9416	0.9328	0.9162	0.9071	0.9024	0.8985	0.8942	0.8913
0.080	1.0541	1.0468	1.0363	1.0269	1.0155	0.9942	0.9856	0.9811	0.9775	0.9734	0.9706

Table 3
Values of Standardized Error Obtained from
Simulated Values of Table 1 and Approximate Values of Table 2

	Percentile of ε_1										
δ_1	1%	2%	5%	10%	20%	50%	80%	90%	95%	98%	99%
0	-0.247	-0.248	-0.248	-0.247	-0.247	-0.248	-0.248	-0.247	-0.248	-0.248	-0.247
0.001	-0.267	-0.266	-0.264	-0.264	-0.261	-0.258	-0.261	-0.263	-0.264	-0.267	-0.267
0.003	-0.297	-0.294	-0.290	-0.288	-0.284	-0.276	-0.284	-0.288	-0.292	-0.294	-0.298
0.005	-0.316	-0.315	-0.309	-0.305	-0.299	-0.287	-0.298	-0.305	-0.309	-0.314	-0.318
0.010	-0.334	-0.330	-0.323	-0.317	-0.309	-0.291	-0.309	-0.318	-0.324	-0.331	-0.335
0.020	-0.279	-0.276	-0.271	-0.265	-0.256	-0.235	-0.257	-0.265	-0.271	-0.277	-0.280
0.030	-0.178	-0.178	-0.175	-0.173	-0.167	-0.147	-0.167	-0.173	-0.176	-0.178	-0.179
0.040	-0.083	-0.085	-0.089	-0.089	-0.088	-0.068	-0.088	-0.090	-0.089	-0.086	-0.084
0.060	-0.002	-0.013	-0.026	-0.034	-0.041	-0.019	-0.041	-0.035	-0.026	-0.013	-0.004
0.080	-0.127	-0.143	-0.165	-0.181	-0.194	-0.161	-0.194	-0.181	-0.165	-0.144	-0.128

Figure 4
Three-Month U.S. Treasury Bill Yield Rate
Time Series and Its First Difference



Note: The period is from January 1982 through April 1992, for a total of 124 monthly observations.

ARIMA(1,1,0) model for U.S. three-month Treasury-bill yield rates.

Non-ARCH

$(1 - \underset{(0.084)}{0.451^{\ddagger}}B)\,(1\text{-}B)y_t = \underset{(0.0335)}{-0.0386} + \varepsilon_t,\; h_t = \underset{(0.0155)}{0.117^{\ddagger}},\; AIC_3 = 0.7894.$

Absolute-ARCH

$(1 - \underset{(0.078)}{0.365^{\ddagger}}B)\,(1\text{-}B)y_t = \underset{(0.017)}{-0.0281^{*}} + \varepsilon_t,\; h_t = \underset{(0.010)}{0.0346^{\ddagger}} + \underset{(0.058)}{0.265^{\ddagger}}|\varepsilon_{t-1}|,\; AIC_4 = 0.4043.$

Regular-ARCH

$(1 - \underset{(0.089)}{0.393^{\ddagger}}B)\,(1\text{-}B)y_t = \underset{(0.023)}{-0.0166} + \varepsilon_t,\; h_t = \underset{(0.009)}{0.0484^{\ddagger}} + \underset{(0.145)}{0.467^{\ddagger}}\,\varepsilon^2_{t-1},\; AIC_4 = 0.3150.$

Table 4
Summary Statistics for a Three-Month U.S. Treasury Bill Rate Time Series and Its First Differences

	Sample Size	Sample Mean	Standard Deviation	Minimum	Maximum
Original Series	124	7.44	1.92	3.75	13.48
First Difference Series	123	-0.07	0.41	-2.67	1.20

	Autocorrelations									
Lag	*1*	*2*	*3*	*4*	*5*	*6*	*9*	*12*	*15*	*18*
Original Series	0.937	0.845	0.756	0.667	0.582	0.491	0.353	0.283	0.215	0.142
First Difference Series	0.367	0.066	-0.046	0.091	0.091	-0.139	0.015	-0.003	0.027	0.003

Here, the Akaike information model selection criterion (AIC) is given by

$$AIC_r = [-2 \log(\text{maximized likelihood}) + 2r]/N^*,$$

where r denotes the number of parameters estimated by maximum likelihood, and $N^* = N - (p+1)$ is the "effective" sample size. The significance of coefficient δ_1 indicates the usefulness of incorporating the ARCH process in the model.

Conclusion

This article examines the discrete-time short-term consequences on reserves due to changes in the interest rate environment. Generally, when viewed at initial time, the next period reserve is a random variable, which is a function of the disturbance generated in the first period. However, in the special case of a white noise process for the force of interest, the next period reserve is a deterministic value. Linear and nonlinear ARCH process models for the force of interest are considered. Under linear interest rate processes, explicit expressions are given for the changes in reserves. In particular, for a pure discount bond, the next period reserve and its discounted value have a lognormal distribution.

As an extension from linear processes to nonlinear processes, approximation formulas and simulation algorithms are presented. Compared with simulation results, we find that the approximation formulas perform well in the sense that the discrepancies between the approximations and simulations are small relative to the simulation standard error. Although we used only one type of insurance contract for our comparisons, the results are generalizable as long as cash flows are independent of interest rates. This is because, under this assumption, those two factors can be separated, as shown in Proposition 1. Theorems 1 and 2 refer to linear rate and nonlinear rate processes, respectively. A numerical example using U.S. Treasury securities indicates strong evidence of the ARCH process.

We consider only the initial and next period reserves, whose values will evolve with time. To handle this one-period-ahead problem sequentially, stochastic control theory may be used to access not only the dynamic nature of the time development, but also the mobility of an insurer's cash flows (claims, premiums, and expenses). Martin-Löf (1983) and Vandebroek and Dhaene (1990) apply the control theory in an insurance context, where they assume fixed interest rates.

In this article, the cash flows are assumed to be independent of interest rate variations. For many insurance contracts, this assumption appears to be unnecessarily restrictive. We hope to investigate this issue in the future.

Appendix A
Expected Value of Discounted Reserve

For a single policy, we have the following result, which can serve as a stepping stone to compute the expected value of discounted reserve for an entire block of business.

Proposition 2

Suppose that the basic assumption that y_t is a Borel measurable function of $(\varepsilon_1, \varepsilon_2,..., \varepsilon_t)$ holds. For single policy, we have

$$V_{h-1} + a_{k+h-1} = E(p_{x+k+h-1} \cdot e^{-y_h} \cdot V_h + q_{x+k+h-1} \cdot e^{-y_h} \cdot b_{k+h} | \varepsilon_1,..., \varepsilon_{h-1}),\ h \geq 1.$$

In particular, for h = 1,

$$V_0 + a_k = p_{x+k}E(e^{-y_1} \cdot V_1) + q_{x+k} \cdot b_{k+1}E(e^{-y_1}) \blacksquare \tag{15}$$

(compare Bowers et al., 1986, 7.8.2). The resources required at the beginning of policy year h equal the actuarial present value of year-end requirements (in expected value sense). Furthermore, we can employ equation (15) to calculate the expected discounted reserve $E(e^{-y_1} \cdot V_1)$ for each policy and then add them to obtain the entire block of business expected discounted reserve.

A recursive expression for $E[\exp(-\sum_{s=1}^{h} y_s) \cdot V_h]$, the expected value of discounted period h reserve, is obtained from Proposition 2:

$$E[\exp(-\sum_{s=1}^{h-1} y_s) \cdot V_{h-1}] + a_{k+h-1}E[\exp(-\sum_{s=1}^{h-1} y_s)]$$
$$= p_{x+k+h-1}E[\exp(-\sum_{s=1}^{h} y_s) \cdot V_h] + q_{x+k+h-1} \cdot b_{k+h}E[\exp(-\sum_{s=1}^{h} y_s)],\ h \geq 1.$$

Note that $V_0^{(h)} = E[\exp(-\sum_{s=1}^{h} y_s)]$; its value can therefore be computed by using Theorems 1 and 2 under different models for $\{y_t\}$.

Proof

For a single policy, let the loss at time h be

$$L_h = \sum_{\tau=0}^{\infty} F_\tau^{(h)} \exp(-\sum_{s=h+1}^{h+\tau} y_s),$$

where $F_\tau^{(h)}$ is defined similarly to equation (2). By definition,

$$V_h = E_h(L_h) = \sum_{\tau=0}^{\infty} E(F_\tau^{(h)}) \cdot E[\exp(-\sum_{s=h+1}^{h+\tau} y_s)|\varepsilon_1,\ldots,\varepsilon_h]$$
$$= -a_{k+h} + \sum_{\tau=0}^{\infty} E(F_{\tau+1}^{(h)}) \cdot E[\exp(-\sum_{s=h+1}^{h+\tau+1} y_s)|\varepsilon_1,\ldots,\varepsilon_h]$$
$$= -a_{k+h} + \sum_{\tau=0}^{\infty} (b_{k+h+\tau+1} \cdot {}_{\tau}|q_{x+k+h} - a_{k+h+\tau+1} \cdot {}_{\tau+1}p_{x+k+h})$$
$$\cdot E[\exp(-\sum_{s=h+1}^{h+\tau+1} y_s)|\varepsilon_1,\ldots,\varepsilon_h]. \tag{16}$$

So

$$e^{-y_h} \cdot V_h = -e^{y_h} \cdot a_{k+h} + \sum_{\tau=0}^{\infty} (b_{k+h+\tau+1} \cdot {}_{\tau}|q_{x+k+h} - a_{k+h+\tau+1} \cdot {}_{\tau+1}p_{x+k+h})$$
$$\cdot E[\exp(-\sum_{s=h}^{h+\tau+1} y_s)|\varepsilon_1,\ldots,\varepsilon_h],$$

which implies that

$$E(p_{x+k+h-1} \cdot e^{-y_h} \cdot V_h|\varepsilon_1,\ldots,\varepsilon_{h-1}) = -a_{k+h} \cdot p_{x+k+h-1} \cdot E(e^{-y_h}|\varepsilon_1,\ldots,\varepsilon_{h-1})$$
$$+ \sum_{\tau=0}^{\infty} (b_{k+h+\tau+1} \cdot {}_{\tau+1}|q_{\chi+k+h-1} - a_{k+h+\tau+1} \cdot {}_{\tau+2}p_{\chi+k+h-1})$$
$$\cdot E[\exp(-\sum_{s=h}^{h+\tau+1} y_s)|\varepsilon_1,\ldots,\varepsilon_{h-1}]. \tag{17}$$

From equation (16), we obtain

$$V_{h-1} + a_{k+h-1}$$
$$= \sum_{\tau=0}^{\infty} (b_{k+h+\tau} \cdot {}_{\tau}|q_{x+k+h-1} - a_{k+h+\tau} \cdot {}_{\tau+1}p_{x+k+h-1}) \cdot E[\exp(-\sum_{s=h}^{h+\tau} y_s)|\varepsilon_1,\ldots,\varepsilon_{h-1}]$$
$$= (b_{k+h} \cdot q_{x+k+h-1} - a_{k+h} \cdot p_{x+k+h-1} \cdot E(e^{-y_h}|\varepsilon_1,\ldots,\varepsilon_{h-1})$$
$$+ \sum_{\tau=0}^{\infty} (b_{k+h+\tau+1} \cdot {}_{\tau+1}|q_{x+k+h-1} - a_{k+h+\tau+1} \cdot {}_{\tau+2}p_{x+k+h-1}) \cdot E[\exp(-\sum_{s=h}^{h+\tau+1} y_s)|\varepsilon_1,\ldots,\varepsilon_{h-1}]$$
$$= b_{k+h} \cdot q_{x+k+h-1} \cdot E(e^{-y_h}|\varepsilon,\ldots,\varepsilon_{h-1}) + E(p_{x+k+h-1} \cdot e^{-y_h} \cdot V_h|\varepsilon_1,\ldots,\varepsilon_{h-1})$$

by noting equation (17). The proof is completed.■

Appendix B
An Upper Bound for the Variance of Innovation Under the ARCH Model

First, $E(\varepsilon_t) = EE(\varepsilon_t|\psi_{t-1}) = 0$, by equation (14b). Thus, $Var(\varepsilon_t) = E(\varepsilon_t^2) = EE(\varepsilon_t^2|\psi_{t-1}) = \delta_0 + \delta_1 E|\varepsilon_{t-1}|$, by equation (14c). Since for $X \sim N(0,\sigma^2)$, $E|X| = \frac{2}{\sqrt{2\pi}}\sigma$, we have

$$E|\varepsilon_{t-1}| = EE(|\varepsilon_{t-1}||\psi_{t-2}) = E(\frac{2}{\sqrt{2\pi}}h_{t-1}^{1/2}) = \frac{2}{\sqrt{2\pi}}E\sqrt{\delta_0+\delta_1|\varepsilon_{t-2}|}$$

$$\leq \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\delta_1 E|\varepsilon_{t-2}|}\text{, by the Jensen's inequality}$$

$$\leq \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\delta_1\frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\delta_1 E|\varepsilon_{t-3}|}}\text{, by the same argument}$$

$$\leq \ldots \leq \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\delta_1\frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\ldots+\delta_1\frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\delta_1 E|\varepsilon_{t-k}|}}}\text{, in general.}$$

Assuming that the series starts indefinitely far in the past with a finite first absolute moment, the limit of the upper bound on the right-hand side as k goes to infinity tends to L, where

$$L = \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\delta_1\frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\ldots}}.$$

Hence $L = \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0+\delta_1 L}$, which implies that $L = (\delta_1+\sqrt{\delta_1^2+2\pi\delta_0})/\pi$. Thus,

$$Var(\varepsilon_t) \leq \delta_0+\delta_1(\delta_1+\sqrt{\delta_1^2+2\pi\delta_0})/\pi.$$

Appendix C
Proofs

Proof of Proposition 1

By definition,

$$S_L^{(0)} = \sum_{i=1}^{n} \sum_{\tau=0}^{\infty} F_{i,\tau}^{(0)} \exp(-\sum_{s=1}^{\tau} y_s) = \sum_{\tau=0}^{\infty} (\sum_{i=1}^{n} F_{i,\tau+1}^{(0)}) \exp(-\sum_{s=1}^{\tau+1} y_s) - \sum_{i=1}^{n} a_{i,k_i},$$

and

$$S_L^{(1)} = \sum_{i=1}^{n} \sum_{\tau=0}^{\infty} F_{i,\tau}^{(1)} \exp(-\sum_{s=2}^{\tau+1} y_s) = \sum_{\tau=0}^{\infty} (\sum_{i=1}^{n} F_{i,\tau+1}^{(1)}) \exp(-\sum_{s=2}^{\tau+2} y_s) - \sum_{i=1}^{n} a_{i,k_i+1}.$$

Then we have the desired results for V_0 and V_1 by utilizing the independence between cash flows and interest rates, the definition of cash flow $F_{i,\tau+1}^{(h)}$ in equation (2) and the probability function of J_i.■

Proof of Theorem 1

The y_t expression from equation (9) implies that

$$\sum_{s=1}^{T} y_s = \sum_{s=1}^{T} \gamma_s + \sum_{s=1}^{T} \sum_{i=0}^{s-1} \beta_i \varepsilon_{s-i} = \sum_{s=1}^{T} \gamma_s + \sum_{s=1}^{T} \varepsilon_s (\sum_{i=0}^{T-s} \beta_i), \tag{18}$$

and

$$\sum_{s=2}^{T} y_s = \sum_{s=2}^{T} \gamma_s + \sum_{s=2}^{T} \sum_{i=0}^{s-1} \beta_i \varepsilon_{s-i} = \sum_{s=2}^{T} \gamma_s + \varepsilon_1 \sum_{i=1}^{T-1} \beta_i + \sum_{s=2}^{T} \varepsilon_s (\sum_{i=0}^{T-s} \beta_i). \tag{19}$$

Therefore,

$$E[\exp(-\sum_{s=1}^{T} y_s)] = \exp(-\sum_{s=1}^{T} \gamma_s) \cdot \prod_{l=0}^{T-1} M(-\sum_{i=0}^{l} \beta_i),$$

and

$$E[\exp(-\sum_{s=2}^{T} y_s) | \varepsilon_1] = \exp(-\sum_{s=2}^{T} \gamma_s) \cdot \exp(-\varepsilon_1 \sum_{i=1}^{T-1} \beta_i) \cdot \prod_{l=0}^{T-2} M(-\sum_{i=0}^{l} \beta_i),$$

by using the i.i.d. property of ε's and reindexing.■

Proof of Lemma 1

Let $\underline{t} = (t_1,\ldots, t_T)'$. Then the moment-generating function of $\underline{\varepsilon} = (\varepsilon_1,\ldots, \varepsilon_T)'$ is

$$M_{\underline{\varepsilon}}(\underline{t}) = E[\exp(\sum_{i=1}^{T} t_i\varepsilon_i)] = E[\exp(\sum_{i=1}^{T-1} t_i\varepsilon_i) E(e^{t_T\varepsilon_T}|\psi_{T-1})]$$

$$= e^{\delta_0 t_T^2/2} E[\exp(\sum_{i=1}^{T-1} t_i\varepsilon_i) \cdot \exp(\frac{\delta_1}{2} t_T^2|\varepsilon_{T-1}|)] \text{ (by equations [14b] and [14c])}$$

$$\leq e^{\delta_0 t_T^2/2}\{E[\exp(2\sum_{i=1}^{T-1} t_i\varepsilon_i)]\}^{1/2}\{E[\exp(\delta_1 t_T^2|\varepsilon_{T-1}|)]\}^{1/2}$$

(by the Hölder inequality).

Since for all $b > 0$ and $X \sim N(0,\sigma^2)$,
$E(e^{b|X|}) = E[e^{-bX}I(X < 0)] + E[e^{bX}I(X \geq 0)]$ (where I is an indicator function)

$$\leq E(e^{-bX}) + E(e^{bX}) = 2e^{b^2\sigma^2/2}.$$

Thus,

$$E[\exp(\delta_1 t_T^2|\varepsilon_{T-1}|)] = EE[\exp(\delta_1 t_T^2|\varepsilon_{T-1}|)|\psi_{T-2}] \leq E[2\exp\{\tfrac{1}{2}\delta_1^2 t_T^4(\delta_0 + \delta_1|\varepsilon_{T-2}|)\}]$$

$$= 2e^{\delta_0\delta_1^2 t_T^4/2} E[\exp(\tfrac{1}{2}\delta_1^3 t_T^4|\varepsilon_{T-2}|)] \leq C_1 < \infty,$$

by a recursive argument, where

$$C_1 = 2^{T-1}\exp[\frac{2\delta_0}{\delta_1^2}\sum_{j=2}^{T} \frac{(\delta_1 t_T)^{(2^j)}}{2^{(2^j-1)}} + \frac{2|\varepsilon_0|}{\delta_1} \cdot \frac{(\delta_1 t_T)^{(2^T)}}{2^{(2^{T-1})}}].$$

Hence,

$$M_{\underline{\varepsilon}}(\underline{t}) = E[\exp(\sum_{i=1}^{T} t_i\varepsilon_i)] \leq C_1^{1/2} e^{\delta_0 t_T^2/2}\{E[\exp(2\sum_{i=1}^{T-1} t_i\varepsilon_i)]\}^{1/2}$$

$$= C_2\{E[\exp(2\sum_{i=1}^{T-1} t_i\varepsilon_i)]\}^{1/2}, \text{ say}$$

$$\leq C_3 < \infty, \text{ for } t_i \in \Re,\ i = 1,\ldots T,$$

by a recursive argument, where

$$C_3 = C_2^{2(1-2^{-(T-1)})} \cdot \exp[2^{T-2}t_1^2(\delta_0 + \delta_1\varepsilon_0^2)].\blacksquare$$

Proof of Theorem 2 requires the following lemma.

Lemma 2

If the ARCH model of equations (14b) and (14c) governs the innovation process $\{\varepsilon_t\}$, then for a fixed positive integer T and as $\delta_1 \to 0$, we have that

(1) the joint density of $\underline{\varepsilon} = (\varepsilon_1,..., \varepsilon_T)'$ is

$$f(\varepsilon_1,..., \varepsilon_T) = (2\pi\delta_0)^{-T/2}\exp(-\frac{1}{2\delta_0}\sum_{i=1}^{T}\varepsilon_i^2) \cdot [1 + \frac{\delta_1}{2}\psi_{1,0}(\underline{\varepsilon}) + \frac{\delta_1^2}{2}\psi_{2,0}(\underline{\varepsilon})] + O(\delta_1^3), \text{ and}$$

(2) the joint density of $(\varepsilon_2,..., \varepsilon_T)$ given ε_1 is

$$f(\varepsilon_2,..., \varepsilon_T|\varepsilon_1) = (2\pi\delta_0)^{-(T-1)/2}\exp(-\frac{1}{2\delta_0}\sum_{i=2}^{T}\varepsilon_i^2) \cdot [1 + \frac{\delta_1}{2}\psi_{1,1}(\underline{\varepsilon}) + \frac{\delta_1^2}{2}\psi_{2,1}(\underline{\varepsilon})] + O(\delta_1^3),$$

where

$$\psi_{1,k}(\underline{\varepsilon}) = \frac{1}{\delta_0^2}\sum_{i=k}^{T-1}|\varepsilon_i|\varepsilon_{i+1}^2 - \frac{1}{\delta_0}\sum_{i=k}^{T-1}|\varepsilon_i|, \text{ and}$$

$$\psi_{2,k}(\underline{\varepsilon}) = \tfrac{1}{4}[\psi_{1,k}(\underline{\varepsilon})]^2 - \frac{1}{\delta_0^3}\sum_{i=k}^{T-1}\varepsilon_i^2\varepsilon_{i+1}^2 + \frac{1}{2\delta_0^2}\sum_{i=k}^{T-1}\varepsilon_i^2, \; k = 0,1.\ddagger$$

Proof of Lemma 2

Taylor's series expansion is used to derive the results. From the conditional densities of equations (14b) and (14c), we have

$$f(\varepsilon_1,..., \varepsilon_T) = (2\pi)^{-T/2}[\prod_{i=0}^{T-1}(\delta_0 + \delta_1|\varepsilon_i|)]^{-1/2} \cdot \exp(-\tfrac{1}{2}\sum_{i=1}^{T}\frac{\varepsilon_i^2}{\delta_0 + \delta_1|\varepsilon_{i-1}|}). \qquad (20)$$

When $\delta_1 = 0$, then $f(\varepsilon_1,..., \varepsilon_T) = \phi(\underline{\varepsilon})$, where $\phi(\underline{\varepsilon}) = (2\pi\delta_0)^{-T/2}\exp[-\sum_{i=1}^{T}\varepsilon_i^2/(2\delta_0)]$.

From equation (20),

$$\ln f = -\frac{T}{2}\ln(2\pi) - \tfrac{1}{2}\sum_{i=0}^{T-1}\ln(\delta_0 + \delta_1|\varepsilon_i|) - \tfrac{1}{2}\sum_{i=1}^{T}\frac{\varepsilon_i^2}{\delta_0 + \delta_1|\varepsilon_{i-1}|}.$$

Differentiating with respect to δ_1 gives

$$\frac{1}{f}\frac{df}{d\delta_1} = -\tfrac{1}{2}\sum_{i=0}^{T-1}\frac{|\varepsilon_i|}{\delta_0 + \delta_1|\varepsilon_i|} + \tfrac{1}{2}\sum_{i=1}^{T}\frac{\varepsilon_i^2|\varepsilon_{i-1}|}{(\delta_0 + \delta_1|\varepsilon_{i-1}|)^2}. \qquad (21)$$

Differentiating again on equation (21) gives

$$-\frac{1}{f^2}(\frac{df}{d\delta_1})^2 + \frac{1}{f}(\frac{d^2f}{d\delta_1^2}) = \tfrac{1}{2}\sum_{i=0}^{T-1}\frac{\varepsilon_i^2}{(\delta_0 + \delta_1|\varepsilon_i|)^2} - \sum_{i=1}^{T}\frac{\varepsilon_i^2\varepsilon_{i-1}^2}{(\delta_0 + \delta_1|\varepsilon_{i-1}|)^3}. \qquad (22)$$

Thus,

$$\frac{df}{d\delta_1} = \frac{f}{2}\left[\sum_{i=1}^{T} \frac{\varepsilon_i^2|\varepsilon_{i-1}|}{(\delta_0+\delta_1|\varepsilon_{i-1}|)^2} - \sum_{i=0}^{T-1} \frac{|\varepsilon_i|}{\delta_0+\delta_1|\varepsilon_i|}\right], \tag{23}$$

so,

$$\left.\frac{df}{d\delta_1}\right|_{\delta_1=0} = \frac{\phi(\underline{\varepsilon})}{2}\left(\frac{1}{\delta_0^2}\sum_{i=0}^{T-1} |\varepsilon_i|\varepsilon_{i+1}^2 - \frac{1}{\delta_0}\sum_{i=0}^{T-1} |\varepsilon_i|\right) = \frac{\phi(\underline{\varepsilon})}{2}\psi_{1,0}(\underline{\varepsilon}).$$

Also,

$$\frac{d^2f}{d\delta_1^2} = \frac{1}{f}\left(\frac{df}{d\delta_1}\right)^2 + f\left[\tfrac{1}{2}\sum_{i=0}^{T-1} \frac{\varepsilon_i^2}{(\delta_0+\delta_1|\varepsilon_i|)^2} - \sum_{i=1}^{T} \frac{\varepsilon_i^2\varepsilon_{i-1}^2}{(\delta_0+\delta_1|\varepsilon_{i-1}|)^3}\right], \tag{24}$$

and

$$\left.\frac{d^2f}{d\delta_1^2}\right|_{\delta_1=0} = \frac{\phi(\underline{\varepsilon})}{4}[\psi_{1,0}(\underline{\varepsilon})]^2 + \phi(\underline{\varepsilon})\left(\frac{1}{2\delta_0^2}\sum_{i=0}^{T-1} \varepsilon_i^2 - \frac{1}{\delta_0^3}\sum_{i=0}^{T-1} \varepsilon_i^2\varepsilon_{i+1}^2\right) = \phi(\underline{\varepsilon})\psi_{2,0}(\underline{\varepsilon}).$$

From Taylor's formula, as δ_1 approaches zero,

$$f(\varepsilon_1,\ldots,\varepsilon_T) = \phi(\underline{\varepsilon}) + \frac{\phi(\underline{\varepsilon})}{2}\psi_{1,0}(\underline{\varepsilon})\cdot\delta_1 + \frac{\phi(\underline{\varepsilon})\psi_{2,0}(\underline{\varepsilon})}{2!}\cdot\delta_1^2 + O(\delta_1^3),$$

so result (1) is proven.

For $f(\varepsilon_2,\ldots,\varepsilon_T|\varepsilon_1)$ in (2), the derivation is essentially the same as $f(\varepsilon_1,\ldots,\varepsilon_T)$, replacing $\prod_{i=0}^{T-1}$ and $\sum_{i=1}^{T}$ by $\prod_{i=1}^{T-1}$ and $\sum_{i=2}^{T}$, respectively. The proof of Lemma 2 is completed.■

Proof of Theorem 2

The proof of Theorem 2 is organized as follows. First, we write Taylor's expansion for $f(\underline{\varepsilon})$ with an explicit form for the remainder. Second, we derive the linear- and quadratic-order expressions (with respect to δ_1) for $V_0^{(T)}$. Third, the boundedness of the remainder term is established.

For the first step, we differentiate equation (22) to give

$$\frac{2}{f^3}\left(\frac{df}{d\delta_1}\right)^3 - \frac{3}{f^2}\frac{df}{d\delta_1}\frac{d^2f}{d\delta_1^2} + \frac{1}{f}\frac{d^3f}{d\delta_1^3} = -\sum_{i=0}^{T-1} \frac{|\varepsilon_i|^3}{(\delta_0+\delta_1|\varepsilon_i|)^3} + 3\sum_{i=1}^{T} \frac{\varepsilon_i^2|\varepsilon_{i-1}|^3}{(\delta_0+\delta_1|\varepsilon_{i-1}|)^4}.$$

So, by substituting $df/d\delta_1$ and $d^2f/d\delta_1^2$ from equations (23) and (24), we have

$$\frac{d^3f}{d\delta_1^3} = \frac{3}{f}\frac{df}{d\delta_1}\frac{d^2f}{d\delta_1^2} - \frac{2}{f^2}\left(\frac{df}{d\delta_1}\right)^3 + fT_3 = f_{\delta_1}(\underline{\varepsilon})\cdot\left(\frac{1}{8}T_1^3 + \frac{3}{2}T_1T_2 + T_3\right),$$

where

$$T_1 = \sum_{i=0}^{T-1} \frac{|\varepsilon_i|\varepsilon_{i+1}^2}{(\delta_0 + \delta_1|\varepsilon_i|)^2} - \sum_{i=0}^{T-1} \frac{|\varepsilon_i|}{\delta_0 + \delta_1|\varepsilon_i|},$$

$$T_2 = \tfrac{1}{2}\sum_{i=0}^{T-1} \frac{\varepsilon_i^2}{(\delta_0 + \delta_1|\varepsilon_i|)^2} - \sum_{i=0}^{T-1} \frac{\varepsilon_i^2\varepsilon_{i+1}^2}{(\delta_0 + \delta_1|\varepsilon_i|)^3}, \text{ and}$$

$$T_3 = 3\sum_{i=0}^{T-1} \frac{|\varepsilon_i|^3\varepsilon_{i+1}^2}{(\delta_0 + \delta_1|\varepsilon_i|)^4} - \sum_{i=0}^{T-1} \frac{|\varepsilon_i|^3}{(\delta_0 + \delta_1|\varepsilon_i|)^3}.$$

By Taylor's formula with Lagrange form of the remainder,

$$f(\underline{\varepsilon}) = \phi(\underline{\varepsilon})[1 + \frac{\delta_1}{2}\psi_{1,0}(\underline{\varepsilon}) + \frac{\delta_1^2}{2}\psi_{2,0}(\underline{\varepsilon})] + R_3(\delta_1,\underline{\varepsilon}),$$

where

$$R_3(\delta_1,\underline{\varepsilon}) = \frac{\delta_1^3}{3!} \cdot \left.\frac{d^3 f}{d\delta_1^3}\right|_{\delta_1=\xi},\ 0 < \xi < \delta_1.$$

For the second step, by using equation (18),

$$V_0^{(T)} = E[\exp(-\sum_{s=1}^{T} y_s)] = \exp(-\sum_{s=1}^{T} \gamma_s)E[\exp(\sum_{s=1}^{T} \pi_s\varepsilon_s)],$$

where $\pi_s = -\sum_{i=0}^{T-s} ß_i$, $1 \le s \le T$. Let $e_i = \varepsilon_i/\sqrt{\delta_0}$, $0 \le i \le T$; $\pi_s^* = \sqrt{\delta_0}\,\pi_s$, $1 \le s \le T$; $\phi(\underline{\varepsilon}) = (2\pi\delta_0)^{-T/2} \exp[-\sum_{i=1}^{T} \varepsilon_i^2/(2\delta_0)]$, $\phi_0(\underline{e}) = (2\pi)^{-T/2} \exp(-\sum_{i=1}^{T} e_i^2/2)$, and Φ is the distribution function of a standard normal variate. Hence,

$$E[\exp(\sum_{s=1}^{T} \pi_s\varepsilon_s)]$$

$$= \int \exp(\sum_{s=1}^{T} \pi_s\varepsilon_s)[1 + \frac{\delta_1}{2}\psi_{1,0}(\underline{\varepsilon}) + \frac{\delta_1^2}{2}\psi_{2,0}(\underline{\varepsilon})]\phi(\underline{\varepsilon}) \prod_{i=1}^{T} d\varepsilon_i$$

$$+ \int \exp(\sum_{s=1}^{T} \pi_s\varepsilon_s)R_3(\delta_1,\underline{\varepsilon}) \prod_{i=1}^{T} d\varepsilon_i$$

$$= \int \exp(\sum_{s=1}^{T} \pi_s^* e_s)\phi_0(\underline{e}) \prod_{i=1}^{T} de_i + \frac{\delta_1}{2} \int \exp(\sum_{s=1}^{T} \pi_s^* e_s)Y_{1,0}(\underline{e})\phi_0(\underline{e}) \prod_{i=1}^{T} de_i$$

$$+ \frac{\delta_1^2}{2} \int \exp(\sum_{s=1}^{T} \pi_s^* e_s)Y_{2,0}(\underline{e})\phi_0(\underline{e}) \prod_{i=1}^{T} de_i$$

$$+ \frac{\delta_1^3}{6} \int \exp(\sum_{s=1}^{T} \pi_s\varepsilon_s) \cdot \left.\frac{d^3 f}{d\delta_1^3}\right|_{\delta_1=\xi} \prod_{i=1}^{T} d\varepsilon_i, \tag{25}$$

where

$$Y_{1,0}(\underline{e}) = \frac{1}{\sqrt{\delta_0}}(\sum_{i=0}^{T-1} |e_i|e_{i+1}^2 - \sum_{i=0}^{T-1} |e_i|), \text{ and}$$

$$Y_{2,0}(\underline{e}) = \frac{1}{\delta_0}[¼(\sum_{i=0}^{T-1} |e_i|e_{i+1}^2)^2 - ½\sum_{i=0}^{T-1} |e_i| \cdot \sum_{i=0}^{T-1} |e_i|e_{i+1}^2 + ¼(\sum_{i=0}^{T-1} |e_i|)^2 - \sum_{i=0}^{T-1} e_i^2 e_{i+1}^2 + ½\sum_{i=0}^{T-1} e_i^2].$$

By viewing e_i ~ i.i.d. N(0,1), the first integral of equation (25) is $\prod_{s=1}^{T} M_\Phi(\pi_s^*) = \exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2) = \exp(\frac{\delta_0}{2}\sum_{l=0}^{T-1} \zeta_e^2)$, by noting $\zeta_e = \sum_{i=0}^{l} ß_i$, that is, $\pi_s = -\zeta_{T-s}$, and after reindexing. The second integral (linear-order term with respect to δ_1) is

$$\frac{1}{\sqrt{\delta_0}}\{\sum_{i=0}^{T-1} E[\exp(\sum_{s=1}^{T} \pi_s^* e_s) \cdot |e_i|e_{i+1}^2] - \sum_{i=0}^{T-1} E[\exp(\sum_{s=1}^{T} \pi_s^* e_s) \cdot |e_i|]\}.$$

For i = 0,

$$E[\exp(\sum_{s=1}^{T} \pi_s^* e_s) \cdot |e_i|] = |e_0|\exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2) = \frac{|\varepsilon_0|}{\sqrt{\delta_0}}\exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2).$$

For i = 1,..., T,

$$E[\exp(\sum_{s=1}^{T} \pi_s^* e_s) \cdot |e_i|] = E(|e_i| \cdot e^{\pi_i^* e_i}) \cdot E[\exp(\sum_{\substack{s=1 \\ s\neq i}}^{T} \pi_s^* e_s)]$$

$$= e^{(\pi_i^*)^2/2} q(\pi_i^*) \cdot \exp(\frac{\delta_0}{2}\sum_{\substack{s=1 \\ s\neq i}}^{T} \pi_s^2) = \exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2) \cdot q(\sqrt{\delta_0}\pi_i),$$

since for X ~ N(0,1), $E(|X|e^{aX}) = e^{a^2/2}q(a)$, where $q(a) = \frac{2}{\sqrt{2\pi}}e^{-a^2/2} + a[2\Phi(a)-1]$.

Define π_0 such that $q(\sqrt{\delta_0 \pi_0}) = |\varepsilon_0|/\sqrt{\delta_0}$. Then

$$E[\exp(\sum_{s=1}^{T} \pi_s^* e_s) \cdot |e_i|] = \exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2) \cdot q(\sqrt{\delta_0}\,\pi_i),\ i = 0, 1,..., T.$$

A similar technique is used to find that

$$E[\exp(\sum_{s=1}^{T} \pi_s^* e_s) \cdot |e_i|e_{i+1}^2] = \exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2) \cdot q(\sqrt{\delta_0}\, \pi_i) \cdot (1+\delta_0\pi_{i+1}^2),\ i = 0, 1,..., T-1,$$

by using, for $X \sim N(0,1)$, $E(X^2 e^{aX}) = e^{a^2/2}(1+a^2)$. Hence, the second integral becomes

$$\frac{1}{\sqrt{\delta_0}}\exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2)\sum_{i=0}^{T-1} [q(\sqrt{\delta_0}\, \pi_i) \cdot (1+\delta_0\pi_{i+1}^2) - q(\sqrt{\delta_0}\, \pi_i)]$$

$$= \sqrt{\delta_0}\exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2)\sum_{i=0}^{T-1} q(\sqrt{\delta_0}\, \pi_i) \cdot \pi_{i+1}^2$$

$$= \sqrt{\delta_0}\exp(\frac{\delta_0}{2}\sum_{s=1}^{T} \pi_s^2)[\frac{|\varepsilon_0|}{\sqrt{\delta_0}}\pi_1^2 + \sum_{i=1}^{T-1} q(\sqrt{\delta_0}\, \pi_i) \cdot \pi_{i+1}^2]$$

$$= \exp(\frac{\delta_0}{2}\sum_{l=0}^{T-1} \zeta_l^2) \cdot [|\varepsilon_0|\zeta_{T-1}^2 + \sqrt{\delta_0}\sum_{l=0}^{T-2} \zeta_l^2 q(\sqrt{\delta_0}\, \zeta_{l+1})]$$

$$= \exp(\frac{\delta_0}{2}\sum_{l=0}^{T-1} \zeta_l^2) \cdot C_1^{(T)}(\varepsilon_0),$$

by noting $\pi_s = -\zeta_{T-s}$, $q(a) = q(-a)$ and reindexing. The linear-order term for $V_0^{(T)}$ is proven. For the quadratic-order term (the third integral of equation [25]), the derivations are more tedious but the ideas are the same as for the linear-order term. That proof is omitted. The third integral can be proven to be equal to $\exp(\frac{\delta_0}{2}\sum_{l=0}^{T-1} \zeta_l^2) \cdot C_2^{(T)}(\varepsilon_0)$.

For the last (remainder) term in equation (25), there exist constants a_1, a_2 such that

$$\frac{|\varepsilon_i|}{(\delta_0+\xi|\varepsilon_i|)^2} \le a_1, \text{ and } \frac{|\varepsilon_i|}{(\delta_0+\xi|\varepsilon_i|)^2} \le a_2, \text{ for } \varepsilon_i \in \Re,\ 0 \le a_1, a_2 < \infty.$$

Therefore,

$$\left|\frac{1}{8}T_1^3 + \frac{3}{2}T_1T_2 + T_3\right| \le \frac{1}{8}(a_1\sum_{i=1}^{T} \varepsilon_i^2 + Ta_2)^3 + \frac{3}{2}(a_1\sum_{i=1}^{T} \varepsilon_i^2 + Ta_2)(\frac{1}{2}Ta_2^2 + a_1a_2\sum_{i=1}^{T} \varepsilon_i^2)$$

$$+ (3a_1a_2^2\sum_{i=1}^{T} \varepsilon_i^2 + Ta_2^3). \qquad (26)$$

We claim that

$$R := \int \exp(\sum_{s=1}^{T} \pi_s \varepsilon_s) \cdot \left.\frac{d^3 f}{d\delta_1^3}\right|_{\delta_1=\xi} \prod_{i=1}^{T} d\varepsilon_i = O(1).$$

Because

$$|R| \leq \int \exp(\sum_{s=1}^{T} \pi_s \varepsilon_s) f_{\zeta(\underline{\varepsilon})} \cdot \left|\frac{1}{8}T_1^3 + \frac{3}{2}T_1 T_2 + T_3\right| \prod_{i=1}^{T} d\varepsilon_i,$$

and noting equation (26), it suffices to prove that for i = 1,..., T and all r > 0,

$$\int |\varepsilon_i|^r \cdot \exp(\sum_{s=1}^{T} \pi_s \varepsilon_s) f_\zeta(\underline{\varepsilon}) \prod_{i=1}^{T} d\varepsilon_i = E[|\varepsilon_i|^r \cdot \exp(\sum_{s=1}^{T} \pi_s \varepsilon_s)] < \infty.$$

It is obviously true since

$$E[|\varepsilon_i|^r \cdot \exp(\sum_{s=1}^{T} \pi_s \varepsilon_s)] \leq \{E[|\varepsilon_i|^{2r}]\}^{1/2}\{E[\exp(2\sum_{s=1}^{T} \pi_s \varepsilon_s)]\}^{1/2} \text{ (by the Hölder inequality)}$$

$$< \infty \text{ (by Lemma 1).}$$

Result (1) follows.

The proof for $V_1^{(T-1)}$ follows the same steps as for $V_0^{(T)}$, by noting that, from equation (19),

$$V_1^{(T-1)} = E[\exp(-\sum_{s=2}^{T} y_s)|\varepsilon_1] = \exp(-\sum_{s=2}^{T} \gamma_s - \varepsilon_1 \sum_{i=1}^{T-1} \beta_i) \cdot E[\exp(\sum_{s=2}^{T} \pi_s \varepsilon_s)|\varepsilon_1]$$

and using Lemma 2(2).■

References

Becker, D. N., 1991, Statistical Tests of the Lognormal Distribution as a Basis for Interest Rate Changes, *Transactions of the Society of Actuaries,* 43: 7-57, Discusion 59-72.

Beekman, J. A. and C. P. Fuelling, 1990, Interest and Mortality Randomness in Some Annuities, *Insurance: Mathematics and Economics,* 9: 185-196.

Beekman, J. A. and C. P. Fuelling, 1991, Extra Randomness in Certain Annuity Models, *Insurance: Mathematics and Economics,* 10: 275-287.

Bellhouse, D. R. and H. H. Panjer, 1981, Stochastic Modelling of Interest Rates with Applications to Life Contingencies—Part II, *Journal of Risk and Insurance,* 48: 628-637.

Bollerslev, T., 1986, Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics,* 31: 307-327.

Bollerslev, T., R. Y. Chou, and K. F. Kroner, 1992, ARCH Modeling in Finance: A Review of the Theory and Empirical Evidence, *Journal of Econometrics,* 52: 5-59.

Bollerslev, T., R. F. Engle, and J. M. Wooldridge, 1988, A Capital Asset Pricing Model with Time Varying Covariances, *Journal of Political Economy,* 96: 116-131.

Bowers, N. L., H. U. Gerber, J. C. Hickman, D. A. Jones, and C. J. Nesbitt, 1986, *Actuarial Mathematics* (Schaumburg, Ill.: Society of Actuaries).

Box, G. and G. Jenkins, 1976, *Time Series Analysis, Forecasting, and Control, Second Edition* (San Francisco: Holden-Day).

Boyle, P. P., 1976, Rates of Return as Random Variables, *Journal of Risk and Insurance,* 43: 693-713.

Brown, R. H. and S. M. Schaefer, 1994, The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model, *Journal of Financial Economics,* 35: 3-42.

Bühlmann, H., 1992, Stochastic Discounting, *Insurance: Mathematics and Economics,* 11: 113-127.

Chan, K. C., G. A. Karolyi, F. A. Longstaff, and A. B. Sanders, 1992, An Empirical Comparison of Alternative Models of the Short-Term Interest Rate, *Journal of Finance,* 47: 617-630.

Courtadon, G., 1982, The Pricing of Options on Default-Free Bonds, *Journal of Financial and Quantitative Analysis,* 17: 75-100.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1981, A Reexamination of Traditional Hypotheses About the Term Structure of Interest Rates, *Journal of Finance,* 36: 769-799.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica,* 53: 385-407.

Dhaene, J., 1989, Stochastic Interest Rates and Autoregressive Integrated Moving Average Processes, *ASTIN Bulletin,* 19: 131-138.

Dothan, L. U., 1978, On the Term Structure of Interest Rates, *Journal of Financial Economics,* 6: 59-69.

Engle, R. F., 1982, Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation, *Econometrica,* 50: 987-1008.

Engle, R. F. and T. Bollerslev, 1986, Modelling the Persistence of Conditional Variances, *Econometric Reviews,* 5: 1-50.

Frees, E. W., 1990, Stochastic Life Contingencies with Solvency Considerations, *Transactions of the Society of Actuaries,* 42: 91-129.

Giacotto, C., 1986, Stochastic Modelling of Interest Rates: Actuarial vs Equilibrium Approach, *Journal of Risk and Insurance*, 53: 435-453.

Higgins, M. L. and A. K. Bera, 1992, A Class of Nonlinear ARCH Models, *International Economic Review,* 33: 137-158.

Ingersoll, J. E., 1987, Theory of Financial Decision Making (Rowman & Littlefield).

Jordan, C. W., 1967, *Life Contingencies* (Chicago: Society of Actuaries).

Lai, S. W., 1995, Stochastic Models of Interest Rates in Actuarial Science, Ph.D. Thesis, University of Wisconsin–Madison.

Longstaff, F. A. and E. S. Schwartz, 1992, Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model, *Journal of Finance,* 47: 1259-1282.

Martin-Löf, A., 1983, Premium Control in an Insurance System, An Approach Using Linear Control Theory, *Scandinavian Actuarial Journal,* 1-27.

Panjer, H. H. and D. R. Bellhouse, 1980, Stochastic Modelling of Interest Rates with Applications to Life Contingencies, *Journal of Risk and Insurance,* 47: 91-110.

Ramlau-Hansen, H., 1988, The Emergence of Profit in Life Insurance, *Insurance: Mathematics and Economics,* 7: 225-236.

Schwert, W., 1990, Stock Volatility and the Crash of '87, *Review of Financial Studies,* 3: 77-102.

Tilley, J. A., 1993, Valuing American Options in a Path Simulation Model, *Transactions of the Society of Actuaries,* 45.

Vandebroek, M. and J. Dhaene, 1990, Optimal Premium Control in a Non-Life Insurance Business, *Scandinavian Actuarial Journal,* 3-13.

Vasicek, O., 1977, An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics,* 5: 177-188.

Waters, H. R., 1978, The Moments and Distributions of Actuarial Functions, *Journal of the Institute of Actuaries,* 105: 61-75.