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The Valuation of Option Features in Retirement Benefits

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ABSTRACT

This article extends and applies a contingent claims valuation approach to option features in retirement fund benefits. The approach incorporates retirement, mortality, and other decrements and allows for differences between the benefit payments and standard option payoffs. A discrete lattice implementation commonly used for valuing financial options is found to be unsatisfactory for the valuation of these option features. Simulation is found to be more efficient. Simulation is then used to calculate benefit values for a range of ages for a simplified fund. The results of the valuations for a number of simulation scenarios demonstrate that a traditional deterministic actuarial valuation can understate the cost of these benefits by as much as 25 to 35 percent.

Introduction

In a number of countries, including the United States and Australia, retirement funds have been designed that offer "greater of" retirement, resignation, and death benefits that are the maximum of two alternative benefit amounts based on a multiple of service and salary or the accumulation of defined contributions. Interest is added to the defined contributions at a crediting rate based on the fund investment return. The origin and form of these option features in Australian retirement funds is discussed in more detail in Britt (1991).

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The benefits provided by a number of large state retirement funds in the United States are of this form.

Traditional actuarial valuation techniques are not designed to value these "greater of" benefits because they are based on deterministic assumptions for the salary growth rate and the crediting rate and because they do not provide a framework for incorporating the price of risk. Details of the traditional actuarial valuation approach are found in Bowers et al. (1986). The valuation of this benefit choice requires the use of a stochastic model for the salary growth rate and the crediting rate and a basis for adjusting for the price of risk. A method that is suited to this valuation problem is the contingent claims approach, which has been developed in the finance literature. This method provides a basis for deciding the market values of retirement fund liabilities either by assuming traded assets are available to price the relevant risks or by assuming a simple form for preferences and the market prices of risk. A market valuation of the liability differs from the actuarial valuation that is based on conservative assumptions and is primarily designed to establish contribution rates and solvency levels for these funds.

Option pricing techniques have been applied to the valuation of a range of actuarial liabilities. Initially, such techniques were applied to investment guarantees provided in maturity benefits of life insurance products, as discussed in Boyle and Schwartz (1976). Wilkie (1989) discussed the use of these techniques in the valuation of pension payments from U.K. pension schemes. The contingent claims approach based on arbitrage-free pricing has also been applied to the valuation of life insurance policy cash flows. For example, Manistre (1990) uses no-arbitrage interest rate models to value the effect of interest-sensitive withdrawals on the value of life insurance liabilities.

This article considers the valuation of "greater of" benefits in a multivariate contingent claims valuation framework. Conceptually, the "greater of" benefits can be treated as equivalent to an option on the maximum of two risky assets along the lines of Johnson (1987) and Stulz (1982). However, some additional issues to be considered when applying such techniques to these benefits are the incorporation of decrements of resignation, mortality, and retirement into the calculations; the path-dependent form of the benefit payments; the difficulties in applying standard numerical techniques, including simulation, discrete lattice models, and other numerical solutions to partial differential equations; and the absence of traded assets to determine a market price of risk directly.

Shimko (1989, 1992a) develops a contingent claims approach to the valuation of insurance claims. Valuation involves the derivation and solution of a partial differential equation for the benefit values including appropriate boundary conditions. The partial differential equation parameters are found from arbitrage-free or equilibrium pricing assumptions. This article extends and applies the contingent claims approach to the case of retirement benefits, including those with option features in the form of "greater of" benefits. The retirement fund decrement rates are treated as equivalent to the claim rate for insurance claims.

Although not as important as for traded options, the efficient numerical computation of these values is an important practical consideration. Techniques for solving the partial differential equation include finite difference or lattice approximations and simulation. Standard applications of these techniques are computationally intensive when the value of the benefit or the boundary conditions are path dependent. Simulation is computationally more efficient for such path dependent problems but does not capture the optimal dynamic aspects of the valuation. In a pension fund, the value of any optimal dynamic decisions by fund beneficiaries is small and is not allowed for here. Tilley (1993) develops a technique that allows the value of optimal dynamic decisions to be approximated for standard options using simulation. This technique could also be extended to the valuation of insurance and pension benefits containing option features.

This article examines the efficiency of lattice methods and simulation methods for valuing "greater of" retirement benefits. Simulation techniques are required for retirement benefits because the path dependence of the benefit values makes lattice approaches impractical for the long-term nature of the option features.

Simulation techniques are then implemented to value benefits in a simplified fund for a range of ages and for a number of simulation scenarios. The results demonstrate that a traditional deterministic actuarial valuation can understate the cost of these benefits by as much as 25 to 35 percent.

Previous Valuation Techniques

Britt (1991) suggests applying option pricing theory to the valuation of "greater of" benefits and, following Wilkie (1989), uses an adaptation of the Garman-Kohlhagen (1983) formula for currency options to determine the value of these benefits at retirement. The value of the "greater of" benefit at retirement was determined using a traditional actuarial technique as the value of the defined benefit at retirement plus the value of the option to exchange the accumulation of defined contributions for the defined benefit. This option to exchange was valued as $Qe^{-rt}N(d_1) - Be^{-\delta t}N(d_2)$, where Q is the expected resignation benefit at retirement date, B is the expected defined benefit at retirement, t is the time to retirement, r and δ are the discount rates for Q and B, respectively, $d_1 = \log(Qe^{-rt}/Be^{-\delta t})/\sigma\sqrt{t} + \frac{1}{2}\sigma\sqrt{t}$, and $d_2 = d_1 - \sigma\sqrt{t}$. Without any formal justification, Britt used $r = \delta = \ln(1.1)$ and $\sigma = 0.75\sigma_i + 0.25\sigma_f$, where σ_i and σ_f are the standard deviations of the fund earning rate and salary growth rate, respectively.

Britt's formula approach fits well with the standard actuarial approach to the valuation of pension benefits, but the basis for determining the parameters is ad-hoc. The approach only applies to valuing "greater of" benefits on retirement and does not allow for other decrements. There is also an inconsistency since the defined benefit is valued using deterministic assumptions and the option to exchange the benefits is valued using stochastic assumptions.

Bell and Sherris (1991) use a simple example to illustrate how a standard numerical technique often used in option pricing could be used to value "greater of" benefits using the binomial equivalent of Margrabe's (1978) approach for options to exchange one asset for another. They do not attempt to implement this approach nor do they allow for decrements such as withdrawals and deaths.

The approaches of Britt and Bell and Sherris determine relative values for these benefits by using the value of one of the benefit payments as the numeraire along with simplifying assumptions about the form of the benefits as a function of the salary growth rates and crediting rates. This allows the approximate calculation of the additional cost of these benefits in terms of the cost of one or the other of the benefits.

Britt (1991) also values the benefits using a simulation approach, which has the advantage of being capable of allowing for decrements such as resignation and death. However, no formal basis for determining the parameters for the valuation is offered, and no consideration is given to incorporating the price of risk. In contrast, a contingent claims valuation approach provides a basis for incorporating the price of risk.

Theoretical Value of "Greater Of" Benefits

"Greater of" benefits are assumed to be a function of two state variables: the crediting rate, which is the interest rate credited to the accumulated contributions, and the growth rate of salary. In this article, the crediting rate is the earning rate on a diversified portfolio that represents the asset portfolio of a typical retirement fund.[1]

A stochastic term structure of risk-free interest rates is not incorporated for ease of exposition. Instead, it is assumed that marketable bonds are traded for all maturities so that interest rate risk is priced in the present value factors used to determine the value of risk-adjusted expected benefit payments. Thus, a market-based spot interest rate for a bond with the same maturity as the benefit payment is assumed to be available to value each risk-adjusted expected benefit payment.

The rate of salary growth, denoted by s, is assumed to follow a stochastic differential equation of the form

$$ds = \mu(s,t)dt + \sigma(s,t)dZ_s, \tag{1}$$

and the crediting rate, denoted by f, is assumed to follow a stochastic differential equation of the form

[1] The returns on a number of asset classes such as equities and fixed interest could have been taken as the state variables. The fund earning rate then could be modeled as the return on a mix of these asset classes, allowing alternative investment strategies to be modeled; a model for the term structure of interest rates could also be included. This alternative method of determining the crediting rate by modeling separate asset classes adds unnecessary complexity to the current work and is left for future research.

$$df = \mu(f,t)dt + \sigma(f,t)dZ_f, \tag{2}$$

with $dZ_s dZ_f = \rho_{fs}dt$, where ρ_{fs} denotes the instantaneous correlation coefficient between the standardized Wiener increments dZ_s and dZ_f.

It is possible to write the stochastic differential equation for f as

$$df = \mu(f,t)dt + \sigma(f,t)(\rho_{fs}dZ_s + \sqrt{(1-(\rho_{fs})^2)}dZ^*), \tag{3}$$

where dZ_s and dZ^* are independent Wiener increments (Shimko, 1992b). This result is used for the numerical evaluation of the "greater of" benefit and for simulation of the processes. It is important to recognize that the state variables s and f are not traded assets.

The "greater of" benefit for a member who joined a fund at time t at age x years, on retirement from the fund at time T after a membership of T-t years, when the life is aged x+T-t, will be the greater of D(x,t,T) = a fixed multiple of final salary for each year of service or part thereof (defined benefit), and A(x,t,T) = the accumulation of contributions as a fixed percentage of salary at the crediting rate during membership time of T-t (defined contribution).

Death, disability, or withdrawal benefits need not be based on the maximum of these two benefits. The assumptions made here are that the "greater of" benefit is paid on death and disability. The benefit paid on withdrawal is assumed to be the defined contribution benefit, which is assumed to be the accumulation of a fixed percentage of salary. Other assumptions for these benefits can be readily incorporated.

Benefit amounts are the value of the benefit payments on the date of exit so that in a pension fund this value is assumed to be equal to a fixed multiple of salary determined from a commutation factor reflecting the present value of the pension annuity. Thus, the multiples used here for the defined benefits equal the value of the pension benefit at the date of exit and not the annual pension payments. This also assumes that the accumulated contributions, if used to purchase an annuity, are applied at market rates for the purchase of a life annuity at the time of exit. Pension payments after the exit date are valued implicitly using the defined benefit multiples so that valuation is only required for benefit payments to the age at exit.

The salary-related benefit is assumed to be based on the final salary at the date of exit. The benefit as a multiple of final salary is D(x,t,T) = (T-t) k S(x,t,T), where T-t is membership in years, k is the salary multiple for the value of the benefit at exit, and S(x,t,T) is the salary at time T for a member who entered at time t aged x. In practice, the defined benefit is based on a multiple of final average salary, often in the three years before exit, which means that value of the benefit will depend on the salary over the three years before retirement. The salary at exit is a function of the salary growth rate state variable s from the age at entry to the age at exit, since

$$dS(x,t,T) = s(T)S(x,t,T)dT, \tag{4}$$

where s(T) is determined from equation (1).

The cash flows for A(x,t,T), the accumulation benefit, are analogous to those of a notional security that has a negative continuous dividend equal to the contribution rate times salary. The contribution amount added to the benefit is the contribution rate times the salary. This amount is added to the benefit value, and not subtracted as in the case of a dividend, so it is equivalent to a negative dividend payment on the notional security. The capital value of this notional security grows at the interest crediting rate since the accumulated contributions grow at this rate. If the fund is invested in a diversified portfolio, as is generally the case for such retirement funds, then the growth in the capital value of the notional security will be the same as the growth rate for the index value of a diversified portfolio. Valuing the A(x,t,T) benefit is therefore equivalent to valuing a security whose capital value grows like a market index and whose (negative) dividend is a percentage of salary. This benefit depends on the history of salary growth rates as well as the fund earning rates during membership and is path dependent. The value of A(x,t,T) is determined from

$$dA(x,t,T) = [f(T)A(x,t,T) + cS(x,t,T)]dT, \tag{5}$$

where f(T) is given by equation (2), S(x,t,T) is given by equation (4), and c is the defined contribution rate as a multiple of salary.

The intensities of resignations, deaths, and retirements—or the forces of decrement, in actuarial terminology—are assumed to be nonrandom functions of age and are denoted by $\mu_i(x)$, with i = w to denote resignation, i = d to denote death or disability, and i = r to denote retirement. The benefit values D(x,t,T) and A(x,t,T) are functions of the state variables s and f so that Ito's lemma implies that D(x,t,T) and A(x,t,T) also follow diffusion processes. The form of these diffusions are given by equations (4) and (5).

The assumptions of Cox, Ingersoll, and Ross (1985a), the results of Shimko (1989, 1992a), and a straightforward application of Ito's lemma can be used to develop a partial differential equation for the value of the benefits from the retirement fund, denoted by V(D,A,x,t,T). For convenience of exposition, the benefit value is expressed as a function of the value of the defined benefit, D(x,t,T), and the value of the accumulated contributions, A(x,t,T). Both values are functions of the state variables s and f through the differential equations (4) and (5). This partial differential equation for the value of the benefits takes the form given by equations (6), (7), (8), and (9) and is subject to the boundary conditions given by equations (10), (11), (12), and (13).

$$\begin{aligned} &\tfrac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma^2(D,T) + \tfrac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2(A,T) + \frac{\partial^2 V}{\partial D \partial A}\rho\,\sigma(D,T)\sigma(A,T) \\ &+ \frac{\partial V}{\partial D}\alpha(D,T) + \frac{\partial V}{\partial A}\alpha(A,T) + \frac{\partial V}{\partial T} - rV + \sum_i \mu_i(x+T-t)B_i(x,t,T) = 0. \end{aligned} \tag{6}$$

Expressions for $\alpha(D,t)$, $\alpha(A,t)$, $\sigma(D,t)$, and $\sigma(A,t)$ in the partial differential equation (6) are found from the diffusion equations for A(x,t,T) and D(x,t,T) and from assumptions of no-arbitrage and the form of preferences and the market price of risk. Because the expressions are complex to evaluate analytically, this article evaluates them numerically using risk-adjusted diffusions for the salary growth rate and the crediting rate. This procedure is similar to Hull's (1993, chap. 12) and is discussed in more detail in the numerical techniques section below. Equations (7), (8), and (9) define the form of the benefit payable for each decrement.

$$B_d(x,t,T) = \max\ [D(x,t,T),\ A(x,t,T)] - V(A,D,x,t,T). \tag{7}$$

$$B_w(x,t,T) = A(x,t,T)] - V(A,D,x,t,T). \tag{8}$$

$$B_r(x,t,T) = \max\ [D(x,t,T),\ A(x,t,T)] - V(A,D,x,t,T). \tag{9}$$

The form of the benefit for decrement i, equal to d, w, or r, given in equations (7), (8), and (9), equals the benefit payable for the decrement less the value of the benefit, V, at the time of payment reflecting the terminating nature of the decrement in a similar fashion to the treatment of insurance claims in Shimko (1992a). These equations apply to exits from age at entry x to the ultimate age of membership in the fund, which is taken as the retirement age.

The ultimate age of membership in the retirement fund is assumed to be 65, which, for a new entrant at time t aged x, will occur at time $\omega = t + 65 - x$. At this time, the benefit paid on survival and then retirement is the maximum of $D(x,t,\omega)$ and $A(x,t,\omega)$, and this is reflected in boundary condition (10).

$$V(D,A,x,t,\omega) = \max[D(x,t,\omega),\ A(x,t,\omega)]. \tag{10}$$

Boundary conditions (11) and (12) ensure that the total benefit value is bounded for large values of the defined benefit and the accumulated contributions. Condition (13) ensures that the benefit value is zero when both the defined benefit and accumulated contributions are both zero. These conditions are checked when numerical techniques are used to value the benefit.

$$\left|\frac{\partial V}{\partial A}(D,\infty,T)\right| < \infty. \tag{11}$$

$$\left|\frac{\partial V}{\partial D}(\infty,A,T)\right| < \infty. \tag{12}$$

$$V(0,0,T) = 0. \tag{13}$$

It is informative for computation of the benefit value to consider what happens if decrements other than the retirement decrement are ignored. In this case, the partial differential equation can be rewritten as

$$\begin{gathered}\tfrac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma^2(D,T) + \tfrac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2(A,T) + \frac{\partial^2 V}{\partial D \partial A}\rho\,\sigma(D,T)\sigma(A,T) \\ + \frac{\partial V}{\partial D}\alpha(D,T) + \frac{\partial V}{\partial A}\alpha(A,T) + \frac{\partial V}{\partial T} - (r(T) + \mu_r(x+T-t))V \;=\; -\mu_r(x+T-t)\max[D(x,t,T),A(x,t,T)].\end{gathered} \tag{14}$$

This can be interpreted as the partial differential equation for an option on the maximum of $D(x,t,T)$ and $A(x,t,T)$ with risk-free rate $(r(T)+\mu_r(x+T-t))$ and continuous dividend $\mu_r(x+T-t)\max[D(x,t,T),\ A(x,t,T)]$. If $D(x,t,T)$ and $A(x,t,T)$ are assumed to have a bivariate lognormal distribution, then this partial differential equation could be solved to produce a complicated time integral of bivariate normal densities that would be a closed-form solution.

For practical purposes, numerical techniques allow a more flexible and more efficient calculation of benefit values, and the closed-form solution is primarily of theoretical interest. The closed-form solution does, however, provide a useful approximation for use in variance reduction techniques, such as the control variate approach, when simulation is used to value the benefit under more general assumptions than bivariate lognormality.

A model of optimal early retirement is similar to early exercise of an American-style option. Early retirement is not a significant factor in valuing these benefits since the benefit increases with additional membership. It is also computationally difficult to incorporate early retirement in the valuation.

Numerical Techniques

A numerical technique is required to solve the general partial differential equation for the benefit values. The use of numerical techniques for financial options is discussed in Hull and White (1990) and Hull (1993). These techniques include discrete approximations to the partial differential equation in the form of lattices for the state variables. An equivalent approach adapts the contingent claims general arbitrage-free valuation results outlined in Hull (1993) for the valuation of the "greater of" benefits and implements this approach using either discrete lattices or simulation. This latter approach is more general in the stochastic process that can be assumed for the state variables. The important consideration for any computational method is the efficiency and rapid convergence for practical applications of the technique. An important practical consideration is the selection of an efficient computational technique since values are required for a range of current ages and membership classes.

An additional consideration in this problem not normally found in contingent claims in financial markets is that the benefit $A(x,t,T)$ depends on the complete history of the salary growth rate, s, so the boundary condition for age at exit $x+T-t$ is not a simple function of the then-current values of the state variables.

The adaptation of the contingent claims general arbitrage-free approach to the "greater of" benefit was first suggested in Sherris (1993). The procedure involves adjusting the drift terms of the state variables for the relevant market prices of risk. These transformed dynamics are used to calculate the expected value of the benefit for each benefit payment age, and this expected value is discounted to the valuation date using the market-determined risk-free spot rate for bonds maturing on the date of the benefit payment. These spot rates are assumed to be available from market data on traded bonds or from a term structure model. The expected value of these benefit payments is then calculated to obtain the value of the benefit with respect to the age at exit.

For financial options, this approach is usually implemented using a discrete lattice model for the state variables. Discrete time approximations for the continuous time dynamics are examined in Boyle (1990), Boyle, Evnine, and

Gibbs (1989), He (1990), and Rajasingham (1990). If a complete market is assumed for the state variables used in the valuation model, then with two state variables the lattice structure at each node in the lattice will consist of three branches. The general form of the lattice for the salary growth rate s and the crediting rate f are as shown in Table 1, where δ_{ij} is the proportionate change in the current value of state variable j for the ith branch. Each of the lattice branches can be written in terms of an increase at the risk-adjusted expected change plus a random jump. The jumps may be determined in a number of ways, but the method outlined below and experimented with in this article is based on He (1990).

Table 1

Complete Markets Trinomial Lattice for Two State Variables

Current Node Values	*Next Time Period Node Values*	*Probability*
	$[(1+\delta_{1s})s_t, (1+\delta_{1f})f_t]$	p
$[s_t, f_t]$	$[(1+\delta_{2s})s_t, (1+\delta_{2f})f_t]$	q
	$[(1+\delta_{3s})s_t, (1+\delta_{3f})f_t]$	1-p-q

Note: s_t = salary growth rate at time t. $_t$ = crediting rate at time t. The proportionate increase in s and f over any time period is given by $(1+\delta_{ij})$, where the subscript i {=1, 2, 3} indicates the branch of the lattice, and subscript j {=s, f} indicates the state variable for which the jump applies.

The method to determine the jumps uses equal probabilities of one-third for each branch and selects the jumps to have the required means, variances, and correlations by expressing the jumps as a function of a basis of uncorrelated discrete processes. The basis has the property that the means are zero, the variances are unity, and the processes are uncorrelated—that is, they are orthogonal martingales. The orthogonal basis for the two-state variable case is shown in Table 2.

Table 2

Orthogonal Basis, Z_{ij}, for the Trinomial Lattice

	Z_{ij}		
Branch i	*j=s*	*j=f*	*Probability of Branch*
1	$\sqrt{(3/2)}$	$\sqrt{(1/2)}$	$^1/_3$
2	0	$-2\sqrt{2}$	$^1/_3$
3	$-\sqrt{(3/2)}$	$\sqrt{(1/2)}$	$^1/_3$

Note: s = salary growth rate. f = crediting rate.

For each branch i = 1, 2, and 3, the corresponding value for δ_{ij}, for j = s and f, are given by equations (15) and (16), respectively.

$$\delta_{is} = \mu_s h + \sigma_s \sqrt{h} Z_{is}, \text{ and} \tag{15}$$

$$\delta_{if} = \mu_f h + \sigma_f \sqrt{h} (\rho Z_{is} + \sqrt{1-\rho^2} Z_{if}), \tag{16}$$

where μ is the annualized risk-adjusted expected change, σ is the annualized standard deviation, and ρ is the correlation coefficient. Using such a lattice ensures convergence to the theoretical continuous time complete markets value of the benefit.

It is not necessary to use the complete market's lattice for computational purposes, and increasing the number of branches could reduce the number of calculations required to obtain a given accuracy since the number of steps for calculation in the discrete approximation can be reduced. In the "greater of" benefits case, the lattice will not recombine since the benefit value is path dependent. This leads to a very large number of nodes and computational difficulties even if a coarse grid is used.

Britt (1991) experiments with simulation to estimate the value of the "greater of" benefit. This technique involves sampling from the total possible paths for the continuous time multivariate state variables. Simulation is readily applied to the "greater of" benefits case since optimal dynamic decisions can reasonably be ignored in the retirements benefit fund at least as a good approximation. Monte Carlo variance reduction techniques speed the computation and improve the accuracy of benefit value estimates for any given sample of paths. Simulation is used extensively in practice in the valuation of mortgage-backed securities where prepayment rates are modeled as state dependent.

The valuation is implemented using simulation paths by calculating benefit values along individual paths for the salary growth rate and the crediting rate, multiplying the benefit values by decrement rates along the path, and then discounting these values at the spot rates. The benefit values for each path are then averaged to obtain an estimate of the present value of the benefit payments.

Both the discrete lattice approach and a naive simulation approach are used in the valuation of the "greater of" benefits for a range of ages. The lattice approach is implemented with the complete markets trinomial lattice using equations (15) and (16) to generate values at the nodes for the salary growth rate and the crediting rate. This approach is used because of its popularity for the evaluation of financial options.

The simulation is implemented using a monthly time interval. The state variables are simulated using the following equations. The rate of salary growth at time t+h, s_{t+h} is derived from the risk-adjusted stochastic difference equation,

$$s_{t+h} = s_t + (\alpha_s + b_s s_t - \lambda_s \sigma_s s_t^{\gamma_s - 1.0}) h + \sigma_s s_t^{\gamma_s - 1.0} \sqrt{h} Z_s, \tag{17}$$

where $Z_s \sim N(0,1)$. This formulation allows several alternative distributions to be generated for the salary growth rate depending on the choice of the parameter γ_s. If γ_s is set equal to one, then the unconditional distribution of s_t is normal; if γ_s is set equal to 1.5, then the unconditional distribution of s_t is gamma; and if γ_s is set equal to two, then the unconditional distribution of s_t is lognormal.

This model has been used for interest rates with γ_s equal to one in Vasicek (1977) and with γ_s equal to 1.5 in Cox, Ingersoll, and Ross (1985b). The market price of risk for the salary growth rate is given by λ_s, and the drift term in the difference equation is risk adjusted by the market price of risk times the conditional standard deviation. The parameters α_s and b_s allow the generation of a mean-reverting process.

The crediting rate at time t+h is generated from the equation

$$f_{t+h} = f_t + (\alpha_f + b_f f_t - \lambda_f \sigma_f f_t^{\gamma_f - 1.0})h + \sigma_f f_t^{\gamma_f - 1.0}\sqrt{h}\,(\rho Z_s + \sqrt{1-\rho^2}Z_f), \tag{18}$$

where $Z_f \sim N(0,1)$ and Z_s and Z_f are independent. The correlation between the rate of salary growth and the crediting rate is given by ρ.

The salary was generated using the difference equation

$$\text{Salary}_{t+h} = \text{Salary}_t(1+hs_{t+h}). \tag{19}$$

All calculations assume a salary at entry of \$10,000.

The defined benefit is then calculated as

$$D_t = k^*\text{Service}^*\text{Salary}_t, \tag{20}$$

where k is the benefit accrual rate for each year of service, and service is calculated from age at entry to age at death or retirement. The accumulated contribution was determined using the difference equation

$$A_{t+h} = A_t(1+hf_t) + c^*h^*\text{Salary}_t, \tag{21}$$

where c is the member's defined contribution rate.

The benefit at any time for death, disability, and retirement is taken as the greater of the two benefits defined in equations (20) and (21). In the case of withdrawal, the accumulated contributions given by equation (21) are assumed paid. The decrement rates used in the calculations, on an annual basis, are presented in Table 3. These rates are typical for an actuarial valuation of a retirement fund. For monthly calculations, the decrement rates were assumed uniformly distributed over the year of age.

Results

Because of its popularity with financial options, the discrete lattice model was used initially to value the "greater of" benefits for the simplified fund. Results for the discrete lattice models demonstrate the impracticality of using this approach for this path dependent problem for ages earlier than 50 at entry

Table 3
Annual Decrement Rates for Retirement Benefit Valuations

Age	*Resignation*	*Death and Disability*	*Retirement*
20	0.16650	0.00090	
21	0.16020	0.00089	
22	0.15390	0.00083	
23	0.14760	0.00075	
24	0.14130	0.00067	
25	0.13500	0.00063	
26	0.12870	0.00063	
27	0.12240	0.00064	
28	0.11610	0.00066	
29	0.10980	0.00068	
30	0.10350	0.00072	
31	0.09720	0.00077	
32	0.09090	0.00082	
33	0.08460	0.00087	
34	0.07830	0.00093	
35	0.07200	0.00101	
36	0.06795	0.00110	
37	0.06390	0.00121	
38	0.05985	0.00132	
39	0.05580	0.00146	
40	0.05175	0.00163	
41	0.04770	0.00183	
42	0.04365	0.00205	
43	0.03960	0.00231	
44	0.03555	0.00262	
45	0.03150	0.00297	
46	0.02745	0.00338	
47	0.02340	0.00385	
48	0.01935	0.00439	
49	0.01530	0.00501	
50	0.01125	0.00572	
51	0.00900	0.00655	
52	0.00675	0.00751	
53	0.00450	0.00860	
54	0.00225	0.00983	
55	0.00000	0.01123	
56	0.00000	0.01282	
57	0.00000	0.01462	
58	0.00000	0.01665	
59	0.00000	0.01895	
60	0.00000	0.02154	
61	0.00000	0.02443	
62	0.00000	0.02784	
63	0.00000	0.03185	
64	0.00000	0.03659	
65	0.00000	0.04215	0.95785

Note: These rates are applied to the number of members at the start of the month at the age of last birthday in the first column. Monthly rates are equal to the annual rates in this table divided by 12. For example, the number of resignations for members aged 50 and n months is equal to the number in force at the start of the month times 0.01125/12.

and for less-than-annual time intervals.[2] The valuation results obtained with this technique are similar to those obtained using simulation.

Deterministic Scenarios

For a comparison to the stochastic simulation valuations, traditional deterministic valuations were performed using a range of salary growth rates and crediting rates. The assumptions (presented in Table 4) also are used to determine if the retirement benefit valued using the traditional approach is the defined benefit or the accumulated contributions.

Table 4
Assumptions for Deterministic Valuations

Deterministic Scenario	*Parameter Values* α_s	b_s	α_f	b_f	*Initial Values* s_0	f_0	*Discount Rate*	*Contribution Rate c*	*Benefit Accrual Rate k*
1	0.0720	-1.20	0.0960	-0.96	0.06	0.10	0.10	0.125	0.150
2	0.0720	-1.20	0.0768	-0.96	0.06	0.08	0.08	0.125	0.150
3	0.0720	-1.20	0.1296	-0.96	0.06	0.135	0.10	0.125	0.150
4	0.0720	-1.20	0.0768	-0.96	0.06	0.08	0.08	0.125	0.175
5	0.0720	-1.20	0.1296	-0.96	0.06	0.135	0.10	0.125	0.175

Note: For deterministic valuations, the values of σ_s and σ_f are set equal to zero. The salary growth rate is equal to $-\alpha_s/b_s$, and the crediting rate is equal to $-\alpha_f/b_f$. The discount rate used for present valuing the deterministic benefit amounts is an annual effective interest rate. Deterministic scenario 1 assumes a salary growth rate of 6 percent per year and a crediting rate of 10 percent per year so that the gap is 4 percent. The discount rate for present valuing benefit payments is 10 percent per year. The benefit accrual rate is 0.15 per year so that a 20-year-old member with 45 years membership will receive a defined benefit of 6.75 times final salary on retirement. The contribution rate for the accumulation benefit is 12.5 percent of salary payable monthly. Deterministic scenario 2 assumes an 8 percent per year crediting rate and an 8 percent per year discount rate but is otherwise the same as Scenario 1. This corresponds to a 2 percent gap. Deterministic scenario 3 assumes a 13.5 percent per year crediting rate and a 10 percent per year discount rate. This is equivalent to Scenario 5 in the stochastic valuations. Deterministic scenario 4 uses the same assumptions as Scenario 2 but with a benefit accrual rate of 0.175 per year. Deterministic scenario 5 uses the same assumptions as Scenario 3 but with a benefit accrual rate of 0.175 per year. α_s and b_s = parameters for salary growth in equation (17). α_f and b_f = parameters for the crediting rate in equation (18). s_0 = salary growth rate at time zero. f_0 = crediting rate at time zero.

The results of the deterministic valuations are given in Table 5. Deterministic scenario 1 assumes a long-term salary growth rate of 6 percent per year, given by $-\alpha_s/b_s = 0.072/1.2$; a long-term crediting rate of 10 percent per year, given by $-\alpha_f/b_f = 0.096/0.96$; and a discount rate for present valuing benefit payments of 10 percent per year. A flat term structure is assumed for all calculations for simplicity. In practice, current market interest rates would be used to discount the expected benefit payments for different ages depending on the

[2] For example, with ten time intervals for a 55-year-old new entrant, the computation of the cost of the benefits as a lump sum present value and as a percentage of salary took over one hour on a personal computer operating with a 486 processor and at 66 MHz. Time and memory requirements increase exponentially with the number of time intervals. As a result, it was not practical to value benefits for younger ages using the lattice approach.

time to payment. Initial values were 6 percent for the salary growth rate and 10 percent for the crediting rate. The defined contribution as a percentage of the monthly salary is 12.5 percent per year, and the benefit accrual rate is 15 percent per year. Thus, deterministic scenario 1 is equivalent to a traditional actuarial basis with a 4 percent gap between the valuation interest rate and the salary growth rate.

Table 5
New Entrant Cost (as a Percentage of Salary)
of Retirement Benefit Option Feature
for Deterministic Valuation Scenarios

Deterministic Scenario	*Age at Entry*			
	20	*30*	*40*	*50*
1	12.6	12.6	12.5	12.6
2	12.6	12.6	12.6	12.2
3	21.1	22.3	20.0	16.6
4	12.7	12.6	13.6	15.1
5	21.1	22.3	20.0	16.6

Note: Percentages equal the monthly cost of the benefits as a percentage of monthly salary. Details of the deterministic scenarios are found in Table 4. Under deterministic scenario 1, the defined benefit is greater than the accumulation benefit for the first nine years of membership. Thereafter, the larger benefit amount is the accumulation benefit. The cost basically reflects the cost of providing the accumulation benefit of 12.5 percent of salary. Under deterministic scenario 2, the defined benefit is greater than the accumulation benefit for the first 17 years and 10 months. Thereafter, the larger benefit amount is the accumulation benefit. For age 50, the larger benefit at retirement is the defined benefit. Under deterministic scenario 3, the defined benefit is greater than the accumulation benefit only for the first 4 years and 10 months of membership. The discount rate is lower than the crediting rate, resulting in an inflated and misleading indication of the cost. Under deterministic scenario 4, the defined benefit is greater than the accumulation benefit for the first 32 years and 2 months of membership. Although the benefit accrual rate has increased for the defined benefit, the larger benefit on retirement is the accumulation benefit for ages 20 and 30, and this is reflected in the cost. The defined benefit is larger on retirement for ages 40 and 50. Under deterministic scenario 5, the defined benefit is greater than the accumulation benefit for the first 8 years and 7 months of membership.

The costs of the benefits as a percentage of salary for a new entrant at ages 20, 30, 40, and 50 also are given in Table 5. The cost of the benefits does not vary much by age and is approximately 12.6 percent of salary. The retirement benefit valued is the accumulated contributions under this deterministic scenario, and the cost of the benefit is approximately equal to the contribution rate.

Deterministic scenario 2 is the same as deterministic scenario 1 but with a 2 percent gap between the crediting rate, 8 percent per year, and an annual salary growth rate of 6 percent. The discount rate, which is usually referred to as the valuation interest rate in an actuarial valuation, is 8 percent per year. The cost of the benefits, as a percentage of salary, is the same as for the gap of 4 percent used in deterministic scenario 1 except for ages 40 and 50.

Deterministic scenario 3 assumes an annual crediting rate of 13.5 percent and an annual discount rate of 10 percent. The cost is significantly higher at around 21 percent of salary for a 20-year-old new entrant since a benefit that is growing at the crediting rate is being discounted at a lower interest rate. Clearly, the basis for deciding the discount rate must be consistent with that used for the crediting rate. This basis is also used in the stochastic simulations for comparison purposes with similar conclusions.

Deterministic scenarios 4 and 5 are the same as deterministic scenarios 1 and 2 but with a higher benefit accrual rate for the defined benefit of 17.5 percent per year of membership. Since the main form of benefit that is valued under these deterministic scenarios is the accumulated contributions and not the defined benefit, this alteration makes little difference to the cost of the benefits as a percentage of salary.

Stochastic Scenarios

Table 6 summarizes the assumptions used in the stochastic scenarios. Stochastic scenario 1 is equivalent to deterministic scenario 1. The salary growth rate and the crediting rate are mean reverting, uncorrelated, and bivariate normally distributed with ρ equal to zero and γ_s and γ_f in equations (16) and (17) set to one. The unconditional or long-term mean salary growth rate is 6 percent per year, given by $-(\alpha_s - \lambda_s\sigma_s)/b_s$, and the long-term mean crediting rate is 10 percent per year, given by $-(\alpha_f - \lambda_f\sigma_f)/b_f$. The long-term standard deviation of the annual salary growth rate is 6.7 percent, given by σ_s/b_s, and the long-term standard deviation of the annual crediting rate is 20.8 percent, given by σ_f/b_f. Values for the parameters b_s, σ_s, b_f, and σ_f are based on estimates from fitting autoregressive time series models to annual inflation and investment return data over the period 1971 through 1991.

Each valuation used 10,000 simulation paths. The simulated distributions for the salary growth rate, the crediting rate, and the salary for stochastic scenario 1 after 45 years are shown in Figures 1, 2, and 3, respectively, along with a fitted normal distribution. The number of simulations is selected in order to reduce the size of the standard error of the expected cost of the benefits, as a percentage of salary, so that the contribution rate is accurate to the nearest percent. For convenience, all valuations assume a flat term structure with an annual interest rate of 10 percent for present valuing along the simulated paths.

The valuation results for ages at entry of 20, 30, 40, 50, and 55 are presented in Table 7. The expected cost of the benefits for stochastic scenario 1 are between 15.6 percent (age 50) and 17.2 percent (age 30) of salary, which compares with the deterministic cost for the equivalent scenario of 12.6 percent. Thus, the option feature in the retirement benefit under consideration adds 3 to 4.6 percent of salary to the cost of the benefits determined using a traditional deterministic actuarial valuation; the traditional valuation understates the cost of the benefits by a significant 24 to 36 percent.

Stochastic scenarios 2 and 3 examine the sensitivity of the results to a variation in the correlation between the salary growth rate and the crediting rate. The results indicate that this assumption has very little effect on the cost of the benefit, probably because of the relatively high variance of the crediting rate.

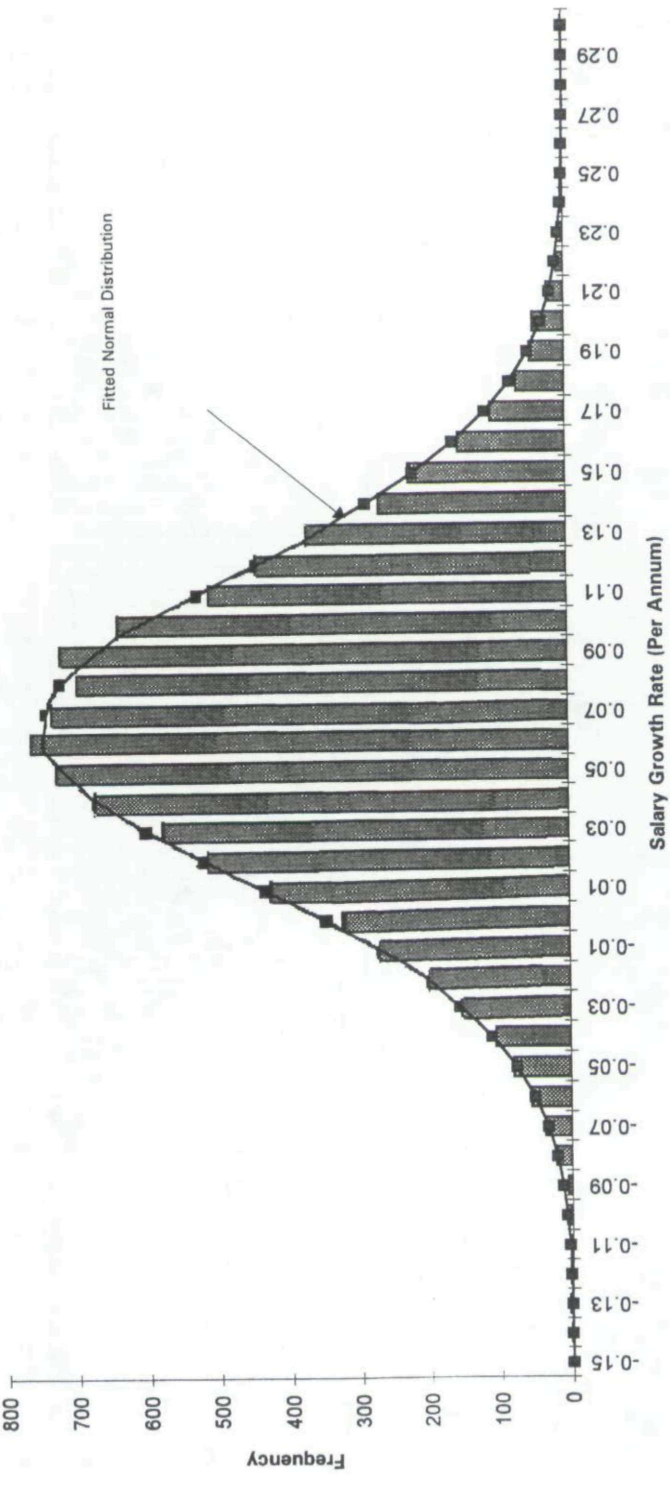
Using a normal distribution for the salary growth rate and the crediting rate has the disadvantage of allowing negative salary growth rates. Negative crediting rates also can be considered unacceptable. Stochastic scenario 4 sets γ_s and γ_f equal to 1.5 in order to examine the effect of excluding negative values for

Table 6
Assumptions for Stochastic Simulation Valuations

Scenario	Parameter Values											Initial Values		Discount
	α_s	b_s	σ_s	λ_s	γ_s	α_f	b_f	σ_f	λ_f	γ_f	ρ	s_0	f_0	Rate
1	0.072	-1.2	0.08	0.0	1.0	0.1296	-0.96	0.2	0.168	1.0	0.00	0.06	0.10	0.10
2	0.072	-1.2	0.08	0.0	1.0	0.1296	-0.96	0.2	0.168	1.0	0.5	0.06	0.10	0.10
3	0.072	-1.2	0.08	0.0	1.0	0.1296	-0.96	0.2	0.168	1.0	-0.5	0.06	0.10	0.10
4	0.072	-1.2	0.08	0.0	1.5	0.1296	-0.96	0.2	0.168	1.5	0.0	0.06	0.10	0.10
5	0.072	-1.2	0.08	0.0	1.0	0.1296	-0.96	0.2	0.000	1.0	0.0	0.06	0.135	0.10
6	0.072	-1.2	0.08	-0.1	1.0	0.1296	-0.96	0.2	0.000	1.0	0.0	0.06	0.135	0.10
7	0.072	-1.2	0.08	0.0	1.0	0.1296	-0.96	0.2	0.100	1.0	0.0	0.06	0.135	0.10

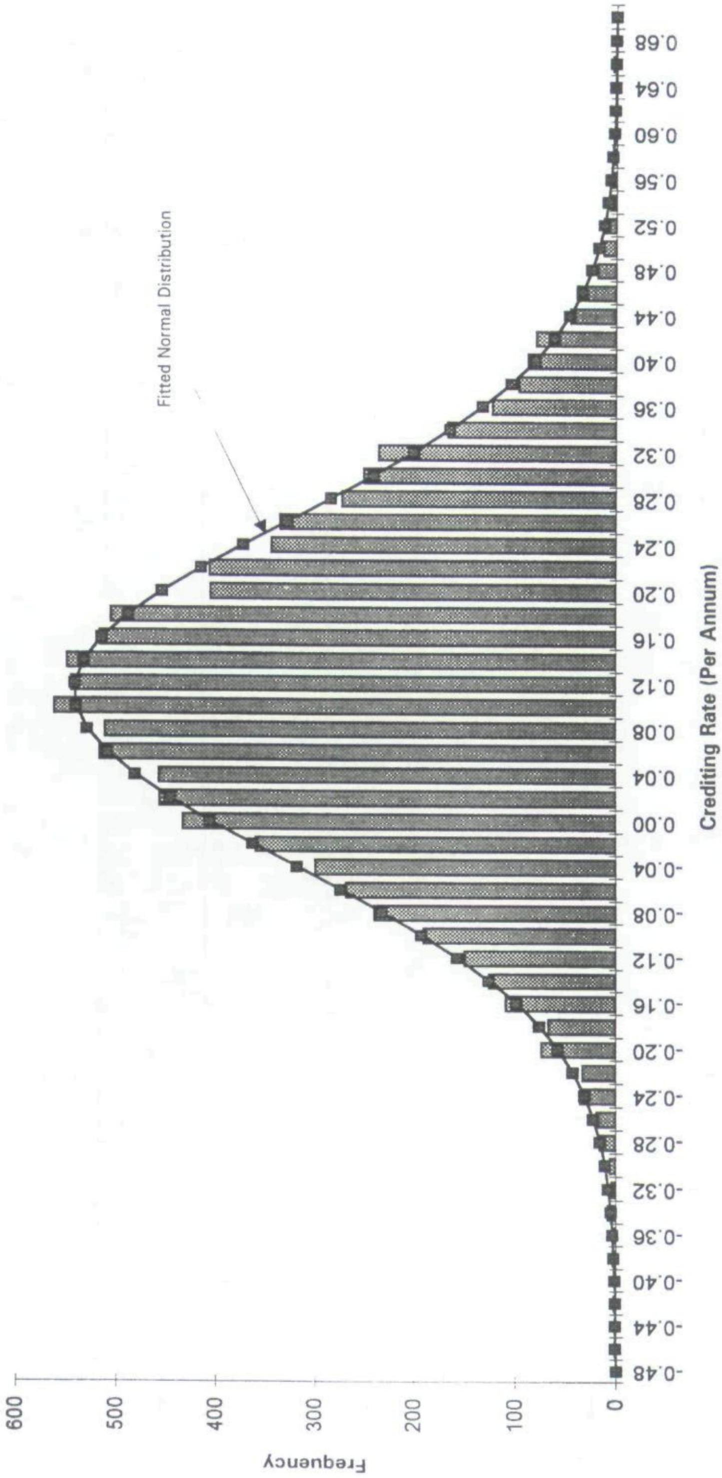
Note: This table presents the parameter values for equations (17) and (18). Stochastic scenario 1 is equivalent to deterministic scenario 1. Salary growth rates and crediting rates are uncorrelated and normally distributed. The expected change in the salary growth rate is not risk adjusted. The crediting rate is risk adjusted so that the long-term expected crediting rate is equal to the risk-free rate. Stochastic scenario 2 is the same as stochastic scenario 1 but with a +0.5 correlation between the salary growth rate and the crediting rate. Stochastic scenario 3 is the same as stochastic scenario 1 but with a -0.5 correlation between the salary growth rate and the crediting rate. For stochastic scenario 4, the salary growth rate and the crediting rate are uncorrelated and each have a gamma distribution. The parameter values were otherwise unchanged from stochastic scenario 1. Stochastic scenario 5 is equivalent to deterministic scenario 3. Salary growth rates and crediting rates are uncorrelated and normally distributed. The expected crediting rate is not risk adjusted. Stochastic scenario 6 is equivalent to stochastic scenario 5 but with the market price of risk for the salary growth rate equal to -0.1. Stochastic scenario 7 is equivalent to stochastic scenario 5 but with the market price of risk for the crediting rate equal to +0.1. α_s and b_s = parameters for salary growth in equation (17). α_f and b_f = parameters for the crediting rates in equation (18). σ_s and σ_f = annualized standard deviations for salary growth rate and crediting rate, respectively. λ_s and λ_f = market prices of risk for salary growth rate and crediting rate, respectively. γ_s and γ_f = ρ = correlation coefficient between salary growth rate and crediting rate. s_0 = salary growth rate at time zero. f_0 = crediting rate at time zero.

Figure 1
Salary Growth Rate Distribution (After 45 Years)
Simulation Scenario 1



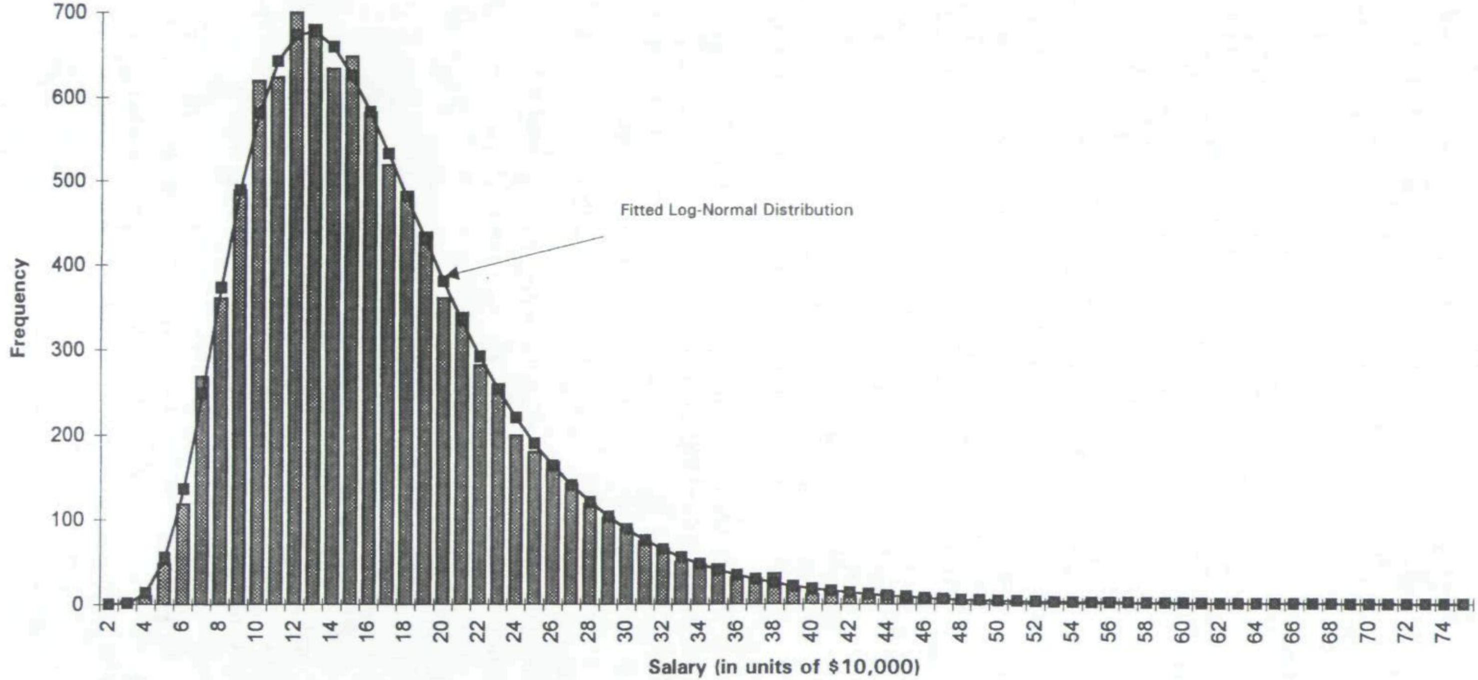
Note: This distribution was generated from equation (17) using the parameter values for stochastic scenario 1 in Table 6.

Figure 2
Crediting Rate Distribution (After 45 Years)
Simulation Scenario 1



Note: This distribution was generated from equation (18) using the parameter values for stochastic scenario 1 in Table 6.

Figure 3
Salary Distribution (After 45 Years with an Initial Value of $10,000)
Simulation Scenario 1



Note: This distribution was generated from equation (19) using the parameter values for stochastic scenario 1 in Table 6.

Table 7
New Entrant Expected Cost (as a Percentage of Salary) and Standard Errors of Retirement Benefit Option Features for Stochastic Valuation Simulation Scenarios

Stochastic Scenario	*Age at Entry*				
	20	*30*	*40*	*50*	*55*
1	16.3	17.2	16.8	15.6	16.1
	(0.1)	(0.1)	(0.1)	(0.1)	(0.0)
2	16.4	17.2	16.6	15.4	16.0
	(0.1)	(0.1)	(0.1)	(0.1)	(0.0)
3	16.8	17.3	16.9	15.8	16.4
	(0.1)	(0.1)	(0.1)	(0.1)	(0.0)
4	17.9	19.9	18.4	15.7	14.9
	(0.0)	(0.0)	(0.0)	(0.1)	(0.0)
5	32.0	32.5	26.6	19.9	17.3
	(0.4)	(0.2)	(0.2)	(0.1)	(0.0)
6	31.9	32.2	26.6	19.8	17.3
	(0.4)	(0.4)	(0.2)	(0.1)	(0.0)
7	20.9	21.8	19.9	17.2	15.8
	(0.1)	(0.1)	(0.1)	(0.1)	(0.0)

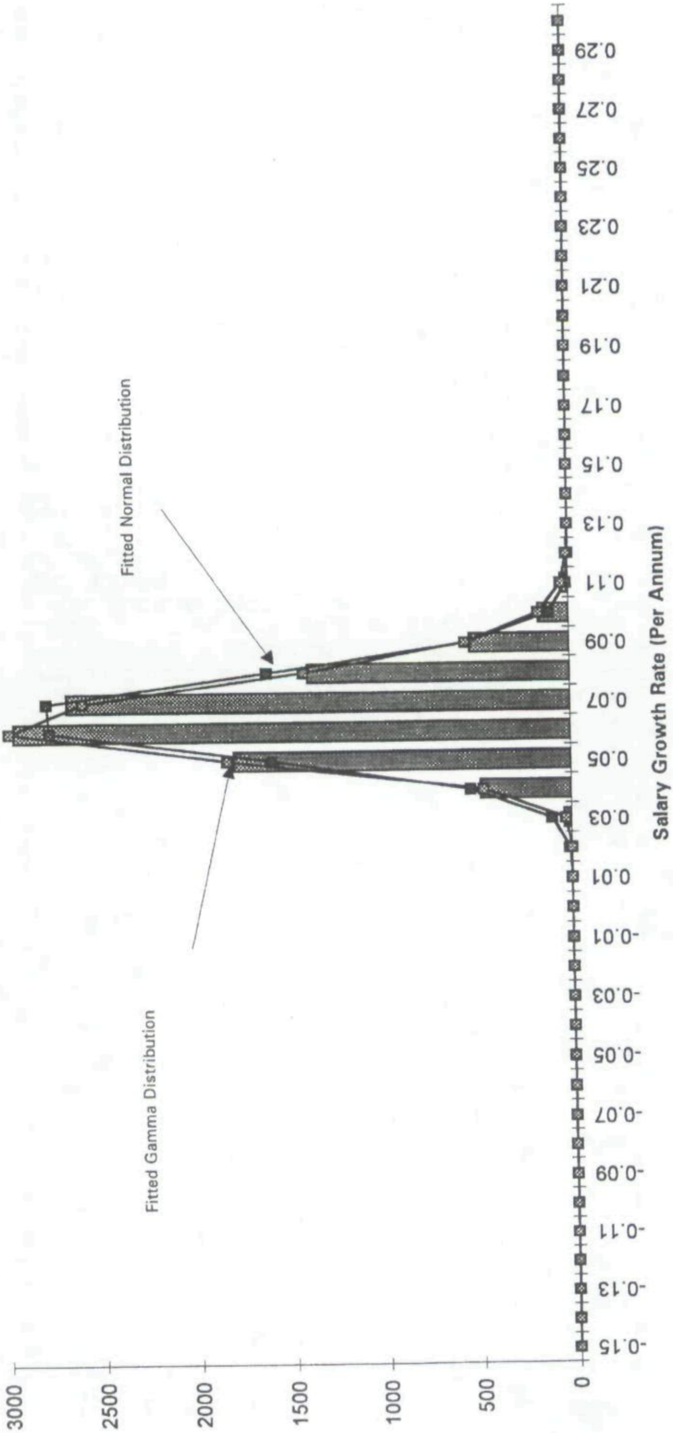
Note: Percentages equal the monthly expected cost of the benefits as a percentage of monthly salary using 10,000 simulations. The standard error of the expected cost is shown in parentheses.

these rates. The simulation parameters are not estimated from historical data for these altered distributional assumptions so that the results are not necessarily consistent with those generated using a normal distribution for the rates. Figures 4, 5, and 6 show the distribution of 10,000 simulated values for the salary growth rate, the crediting rate, and the salary under stochastic scenario 4 after 45 years.

Surprisingly, the expected cost of the benefits does not change dramatically from the normal distribution scenarios with the same parameters. An increase of approximately 1.6 percent of salary occurs at age 20, and a fall of 1.2 percent of salary occurs at age 55.

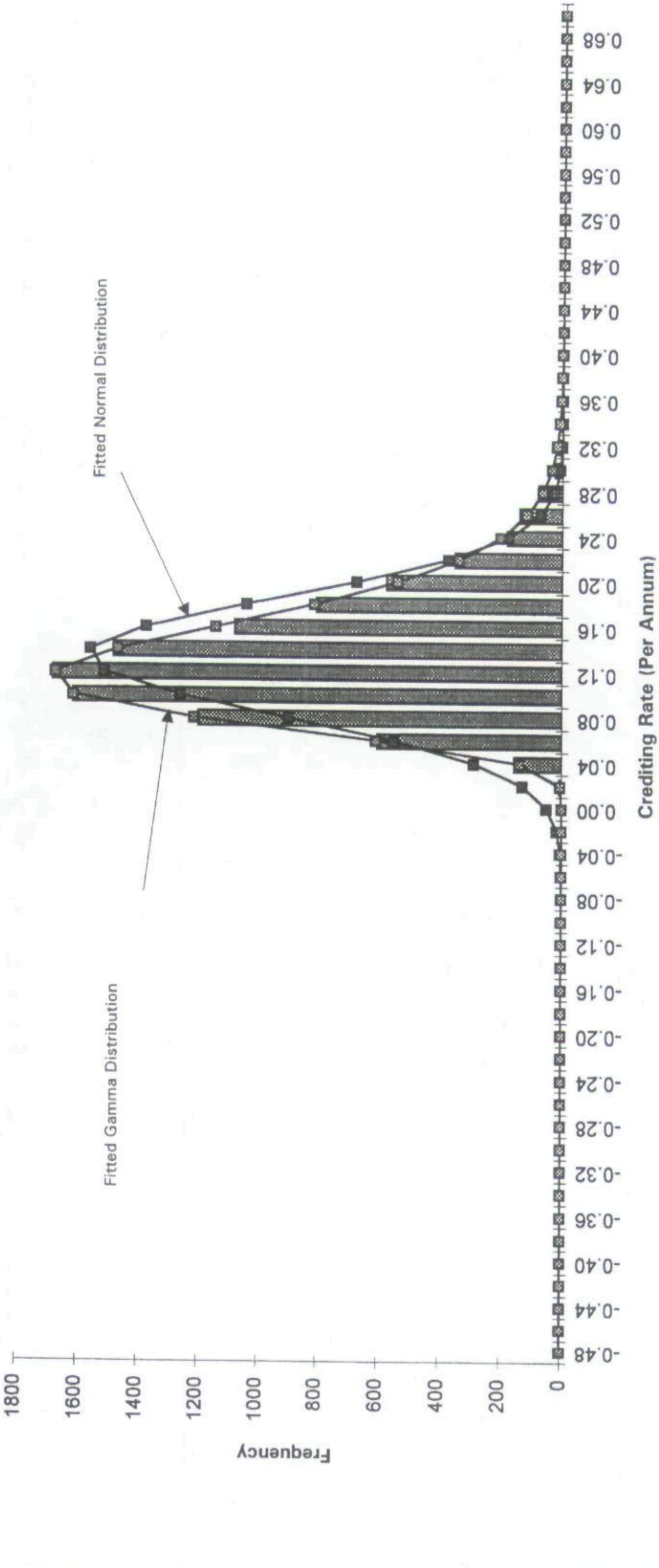
In all simulations, the value for λ_f is chosen to produce a risk-adjusted long-term crediting rate equal to the risk-free discount. Because the fund assets trade, it was considered reasonable to make this adjustment. No risk adjustment is made to the salary growth rate. The values for the market prices of risk are the most difficult assumption to determine for these nontraded state variables. It is possible that these factors are traded in financial markets and these prices of risk could be estimated from traded assets, but, given the empirical difficulties of doing so, no attempt was made to estimate these market prices of risk. However, sensitivity of the cost to the market prices of risk was assessed in stochastic scenarios 5, 6, and 7.

Figure 4
Salary Growth Rate Distribution (After 45 Years)
Simulation Scenario 4



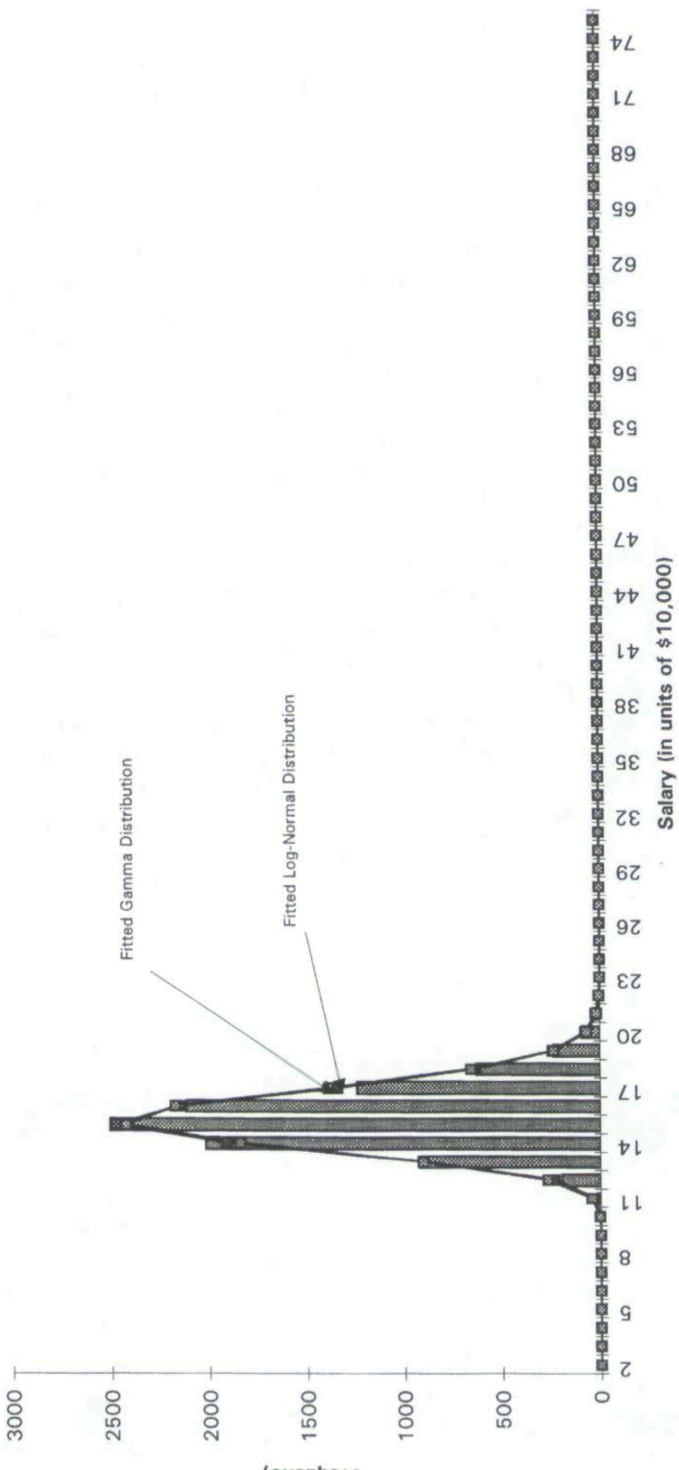
Note: This distribution was generated from equation (17) using the parameter values for stochastic scenario 4 in Table 6.

Figure 5
Crediting Rate Distribution (After 45 Years)
Simulation Scenario 4



Note: This distribution was generated from equation (18) using the parameter values for stochastic scenario 4 in Table 6.

Figure 6
Salary Distribution (After 45 Years with an Initial Value of $10,000)
Simulation Scenario 4



Note: This distribution was generated from equation (19) using the parameter values for stochastic scenario 4 in Table 6.

Stochastic simulation 5 indicates the effect of not making a risk adjustment to the crediting rate. This leads to an inconsistency between the discount rate and the long-term crediting rate and inflated and misleading expected costs similar to the deterministic scenario 3 valuation.

Stochastic simulation 6 uses the same input parameters as stochastic simulation 5, except that the market price of risk for the salary growth rate is set equal to -0.1. This has very little effect on the expected cost. Stochastic scenario 7 sets the market price of risk for the crediting rate equal to +0.1, which has a more significant effect on the expected cost of the benefits. A careful determination of the market price of risk for the crediting rate is required when implementing this method in practice. The market price of risk for the salary growth rate does not appear to be as critical.

Conclusion

This article applies the contingent claims valuation approach developed in the finance literature to the valuation of option features in the form of "greater of" benefits in retirement funds. The research demonstrates that lattice models, as currently used in contingent claims valuation of financial options, are not computationally feasible for these retirement fund benefits. This approach is implemented using crude simulation. The article demonstrates how traditional deterministic valuations of these benefits significantly understate the cost of providing such option features in retirement benefits by as much as 35 percent.

Better estimates of the parameters for the salary growth rate and the crediting rate based on empirical data are needed. The use of variance reduction techniques in the simulations need investigation since they will speed up the calculations, and for large retirement funds such techniques will prove essential.

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Examining Changes in Reserves Using Stochastic Interest Models

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ABSTRACT

A fundamental problem in actuarial science is the determination of the reserves necessary to meet future obligations. Reserves are useful quantities because they summarize a vector of discounted cash flows. However, through this summarization, they mask the dynamic nature of interest rates. To study the effects on reserves of the dynamic nature of a stochastic interest environment, we look at a change in discounted reserves and study the potential short-term consequences of changes in the interest environment. Both the traditional linear ARIMA models and newer nonlinear autoregressive conditionally heteroskedastic (ARCH) processes are used to model the force of interest stochastically. We find that, in general, the next period reserve is a function of the previous interest rate. However, this is not true when the force of interest can be modeled as a white noise process. Explicit formulas compute changes in discounted reserves for linear interest rate processes. For nonlinear processes, we describe some approximations and exact simulation algorithms for these computations. An empirical example using a U.S. Treasury security series is presented and evidence of ARCH behavior is found.

Introduction

On balance sheets, assets must equal liabilities plus net worth of the firm. An important component of these liabilities for insurers is the reserve, the portion of a firms' assets set aside to meet future uncertain obligations arising from insurance contracts. Although the obligations of each contract are contingent upon uncertain future events, and thus may be modeled stochastically, the reserve set aside is a single number. There are limitations when using a single number to summarize a stochastic quantity. However, reserves play a promi-

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