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An Equilibrium Model of Insurance Pricing and Capitalization

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ABSTRACT

This article is concerned with insurer capitalization and its implications for insurance pricing. It establishes an economy-wide one-period model of the dependencies between the prices of insurance and other stocks, real assets, and insurance policies, as well as consumer preferences for consumption relative to saving and insurance security, and insurer capitalization. All ingredients except the last were included in the model of Turner (1981). Turner's framework is used here but is augmented to incorporate insurer capitalization as a free variable. The relation between total risk and capitalization answers a fundamental question that arises in the context of fair pricing.

Introduction

This article is concerned with insurer capitalization and its implications for insurance pricing. The theoretical approaches to insurance pricing now most widely accepted are the internal rate of return approach (Cummins, 1990) and that of Myers and Cohn (1981). In each of these cases (and others), the required capitalization of the line of business being priced is taken as the minimum regulatory requirement or determined by reference to market practice.

The degree of capitalization of an insurer will, however, reflect consumer preferences. Increased capital provides policyholders with increased security, but at a cost in the form of an increase of the premium loadings that enable the capital to be serviced. It is therefore of interest to investigate the equilibrium capitalization of various insurers in the total economy.

Turner (1981) investigated the simultaneous equilibrium of stock prices, real asset prices, and insurance prices in a one-period model but without reference to insurer capitalization. Turner's framework is used here but is supplemented by analysis of the manner in which an insurer's surplus affects the security of policyholders and the value of shares in the insurer.

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The equilibrium in this augmented one-period model is then investigated in terms of the same variables as in Turner's model and in terms of the equilibrium amount of capital to be held by each insurer in the economy. Of course, equilibrium capitalization affects equilibrium pricing.

Notation and Assumptions

Initially, the model developed here will follow closely Turner's (1981), and the same notation is used to facilitate comparison. Many of the following notational definitions are extracted verbatim from Turner. They relate to an economy observed over a single period.

Subscripts

f = 1, ..., F = insurers.
h = 1, ..., H = individuals or households.
i = 1, ..., I = firms engaged in production (productive firms).
k = 1, ..., K = types of real assets.

Variables

V_i = initial (beginning of period) price of one share in productive firm i.
R_i = cash flow per share of stock i.
ν_{hi} = number of shares in productive firm i owned by household h.
ν_{fi} = number of shares in productive firm i owned by insurer f.
A_k = initial (beginning of period) price of one real asset of type k.
R_k = cash flow per unit real asset of type k.
X_{hk} = individual losses, that is, negative cash flow of real assets, per unit of asset of type k due to individual hazards of household h.
a_{hk} = number of real assets of type k owned by household h.
S_f = initial (beginning of period) price of one share in insurer f.
Y_f = cash flow per share of insurance firm f when K_f, as defined below, is zero (negative values of Y_f are permitted, that is, limited liability is not reflected in this variable).
s_{hf} = number of shares in insurer f owned by household h.
P_{fk} = premium per unit of insurance in asset type k charged by insurer f.
n_{hfk} = number of units of insurance in asset type k purchased by household h from insurer f.
W_h = initial (beginning of period) wealth of household h.
T_h = terminal (end of period) wealth of household h.
C_h = current consumption of household h.
μ_h = expected value of terminal wealth of household h.
ν_h = variance of terminal wealth of household h.
K_f = initial (beginning of period) equity (net assets) per share of insurer f.

A number of quantities, derived from those defined above, play a fundamental role in the following analysis. It is useful to assign abbreviated symbols to these. Some of the algebraic definitions of these quantities are conveniently deferred to the relevant section of analysis. In this spirit, the following additional definitions are given.

ν_{hk}	$= V[X_{hk}]$.
j_i	$= R_i/V_i = 1 +$ rate of return on shares in firm i.
j_o	$= 1 +$ risk-free rate of return.
$X_f = \sum_{hk} n_{hfk} X_{hk}$	= total claims on insurer f.
$\gamma_f = V^{1/2}[X_f]/E[X_f]$	= coefficient of variation of total claims on insurer f.
$s_f = \sum_h s_{hf}$	= number of shares of insurer f on issue.
$P_f = \sum_{hk} n_{hfk} P_{fk}$	= total premium payable to insurer f.
$R_f = \sum_i \nu_{fi} R_i / \sum \nu_{fi} V_i$	$= 1 +$ rate of return on investments held by insurer f.
ω_f	= coefficient of variation of R_f.
E_f	$= Y_f + K_f R_f$ = closing net assets per share of insurer f for the period.
Q_f	= payout ratio of insurer f, that is, proportion of each claim paid taking into account its available funds.
δ_f	= ratio of equity in insurer f to expected value of liabilities, as measured at the end of the period of underwriting $= s_f K_f j_o/\overline{X}_f$.
η_f	= average risk loading factor in premiums charged by insurer f, that is, ratio of total premium written to expected present value of claims $= P_f j_o/\overline{X}_f$.

Note that R_i, R_k, X_{hk}, Y_f, T_h, and Q_f are random variables. The expected value of any random variable will be denoted below by an overbar; for example, $\overline{R}_i = E[R_i]$.

Certain variables will be assumed normal or lognormal later in the article. The standard normal distribution function and probability density function will be denoted by $\Phi(.)$ and $\phi(.)$, respectively. In addition, $\phi^{(n)}(.)$ will denote the nth derivative of $\phi(.)$.

Also, let $[X]_+$ denote max $(0,X)$ for any quantity X, and let $I(.)$ be the indicator function

$$I(x) = 1 \text{ if } x \geq 0;$$
$$= 0 \text{ if } x < 0.$$

The following economy-wide variables are defined:

$V_M = \sum_{hi} \nu_{hi} V_i + \sum_{fi} \nu_{fi} V_i$ = opening total stock market value.

$R_M = \sum_{hi} \nu_{hi} R_i + \sum_{fi} \nu_{fi} R_i$ = closing total stock market value.

$R_N = \sum_{hk} a_{hk} R_k$ = opening total real assets.

$X_N = \sum_{hk} a_{hk} X_{hk}$ = total losses for the period.

$j_M = R_M/V_M$ = 1 + rate of return on total stock market.

Certain assumptions are made at various points in the subsequent analysis. First, the assumptions supporting Turner's earlier analysis are retained. In brief form, they are perfect competition, no transaction costs or taxes, no restrictions on short sales, homogeneous expectations across all agents, and all households are risk-averse, one-period utility maximizers whose preferences can be described in terms of means and variances.

The following additional assumptions are made: the number of households H in the economy is large, and, as a result, s_{hf}/s_f is small for each h, and the dependence of E_f on each n_{hfk} is small; R_f and X_f are log normally distributed; share market returns are not correlated with losses of real assets, that is, $\text{cov}[R_M, X_{hk}] = 0$ for each h,k;[1] and for each asset type, the losses per unit of those assets are identically (but not necessarily independently) distributed for each household, that is, for each fixed k, all X_{hk} are identically distributed.

Equilibrium Pricing of Insurance

In the Absence of Insolvency Risk

Turner (1981) assumes that each household h seeks to maximize a utility function U_h, which depends on the mean and variance of terminal wealth as well as consumption, that is, $U_h = U_h(\mu_h, \nu_h, C_h)$. Each individual is subject to a budget constraint on initial wealth:

$$W_h = \sum_i \nu_{hi} V_i + \sum_k a_{hk} A_k + \sum_f s_{hf} S_f + \sum_{fk} n_{hfk} P_{fk} + C_h. \tag{1}$$

The optimization problem to be solved is, therefore,

$$\max\ U_h(\mu_h, \nu_h, C_h)$$

[1] This assumption is something of an approximation. Some studies (e.g., Fairley, 1979) find some negative covariance for losses aggregated over all lines. Others find no significant covariances (e.g., Cummins and Harrington, 1985). In practice, covariances may be nonzero, perhaps even with sign varying from one line of business to another.

$$- \psi_h\left(\sum_i \nu_{hi}V_i + \sum_k a_{hk}A_k + \sum_f s_{hf}S_f + \sum_{fk} n_{hfk}P_{fk} + C_h - W_h\right), \tag{2}$$

where Ψ_h is a Lagrange multiplier.

The following treatment of this problem runs completely parallel to Turner's but omits a couple of aspects. In particular, Turner allows for social losses L_k representing the uncertainty common to all owners of type k assets. Such losses are ignored here. In addition, Turner allows coverage ratios c_{fk} other than 100 percent representing the percentage of losses X_{hk} carried by insurer f. This refinement is also ignored here. These simplifications of Turner's work do not produce any fundamental change in the nature of the equilibrium but do lighten the notation. Reinstatement of Turner's refinements would be quite straightforward.

Within the above framework, the terminal wealth of household h (a random variable) is

$$T_h = \sum_i \nu_{hi}R_i + \sum_k a_{hk}(R_k - X_{hk}) + \sum_f s_{hf}Y_f + \sum_{fk} n_{hfk}X_{hk}. \tag{3}$$

It follows that expected terminal wealth is

$$\mu_h = \sum_i \nu_{hi}\overline{R}_i + \sum_k a_{hk}(\overline{R}_k - \overline{X}_{hk}) + \sum_f s_{hf}\overline{Y}_f + \sum_{fk} n_{hfk}\overline{X}_{hk}. \tag{4}$$

Note that equations (3) and (4) assume that claims on insurers are paid in full, that is, insurer insolvency risk is assumed away.

Turner obtains the following equilibrium:

$$V_i = \frac{1}{\theta}[\overline{R}_i - \lambda \mathrm{cov}(R_i, R_M + R_N)], \tag{5}$$

$$S_f = \frac{1}{\theta}[\overline{Y}_f - \lambda\ \mathrm{cov}(Y_f, R_M + R_N - X_N)], \tag{6}$$

$$A_k = \frac{1}{\theta}[\overline{R}_k - \overline{X}_k - \lambda \mathrm{cov}[R_k - X_k, R_M + R_N - X_N)], \tag{7}$$

$$P_{fk} = \frac{1}{\theta}[\overline{X}_k + \lambda \mathrm{cov}(X_k, R_M + R_N - X_N)], \tag{8}$$

where

$$\lambda = -2[\sum_h \lambda_h^{-1}]^{-1}, \tag{9}$$

$$\theta = -\tfrac{1}{2}\lambda \sum_h \theta_h, \tag{10}$$

$$\lambda_h = [\partial U_h/\partial v_h]/[\partial U_h/\partial \mu_h], \tag{11}$$

$$\theta_h = [\partial U_h/\partial C_h]/[\partial U_h/\partial v_h], \text{ and} \tag{12}$$

$$X_k = \sum_h \lambda_h^{-1} X_{hk} \,/\, \sum_h \lambda_h^{-1}. \tag{13}$$

Turner shows that, in the presence of a riskless asset, λ_h becomes independent of h, and equation (10) reduces to

$$\theta = \sum_h \{[\partial U_h/\partial C_h]/[\partial U_h/\partial v_h]\}, \tag{14}$$

representing the aggregate rate of substitution of consumption for savings.

In the present context, the key result is equation (8), which indicates that an equilibrium premium rate is obtained by adjusting expected losses in two ways: adding a quantity related to the variance of the risks being rated and adjusting further to allow for the economy's propensity to save.

Turner simplifies equation (5) to equation (8) by means of an assumption that distinct risks are stochastically independent, that is, X_{hk} and X_{gl} are independent if $(h,k) \neq (g,l)$. Here, this assumption is weakened. It is assumed, in fact, that risks involving different asset types are independent; specifically, that X_{hk} and X_{gl} are stochastically independent if $k \neq l$. Thus, unlike Turner's model, X_{hk}, X_{gk}, $h \neq g$ are *not* assumed independent.

Although Turner's assumption will usually be appropriate for X_{hk} and X_{gl} conditioned on the parameters governing the claims cost process, there will usually be uncertainty in these parameters, inducing a correlation between the unconditional X_{hk} and X_{gl}. For example, in the case of motor insurance, X_{hk} and X_{gl} will depend on collision frequency α. Although it may be reasonable to assume that $\text{cov}(X_{hk}, X_{gk} \mid \alpha) = 0$ for $h \neq g$, it is likely that $\text{cov}(X_{hk}, X_{gk}) \neq 0$.

In the Presence of Insolvency Risk

The above reasoning involves an implicit assumption that the payout ratio Q_f equals one. Consider now the case in which Q_f is less than one with nonzero probability. Note first that funds held by insurer f to pay claims of X_f at the end of the period amount to $X_f + s_f E_f$. Claims will be paid in full provided that $E_f \geq 0$. It follows that

$$Q_f = 1 + \min\,[0,\, s_f\, E_f/X_f]. \tag{15}$$

It is useful to note that

$$s_f\, Y_f = P_f\, R_f - X_f, \tag{16}$$

so that the definition of E_f gives

$$s_f E_f = (P_f + s_f K_f) R_f - X_f. \tag{17}$$

Substitution of equation (17) in equation (15) gives

$$Q_f = (P_f + s_f K_f) R_f / X_f, \text{ when } E_f < 0. \tag{18}$$

The presence of the payout ratios Q_f leads directly to a modification of the equilibrium described above, but this equilibrium is further affected by the fact that the K_f become decision variables in the modified optimization problem corresponding to equation (2). The new optimization problem may be set up as follows. The budget constraint reflects the allocation of capital to insurers

$$W_h = \sum_i \nu_{hi} V_i + \sum_k a_{hk} A_k + \sum_f s_{hf} S_f(K_f) + \sum_{fk} n_{hfk} P_{fk} + C_h, \tag{19}$$

which is the same as equation (1) except that S_f now depends on K_f. Terminal wealth reflects this capital, the investment income generated by it, and the effect of an insurer insolvency on claim payout:

$$\mu_h = \sum_i \nu_{hi} \overline{R}_i + \sum_f s_{hf} E[E_f]_+ + \sum_k a_{hk}(\overline{R}_k - \overline{X}_{hk}) + \sum_{fk} n_{hfk} E[Q_f X_{hk}]. \tag{20}$$

The optimization problem becomes

$$\max_{\{\nu_{hi}, s_{hf}, a_{hk}, n_{hfk}, K_f, C_h\}} U_h(\mu_h, \nu_h, C_h)$$

$$-\psi_h[\sum_i \nu_{hi} V_i + \sum_k a_{hk} A_k + \sum_f s_{hf} S_f(K_f) + \sum_{fk} n_{hfk} P_{fk} + C_h - W_h], \tag{21}$$

where ν_h denotes variance of terminal wealth as before, but now reflects the different structure of terminal wealth as set out in equation (20). The solution of this problem is tedious and awkward; it is derived in Appendix A.

Corresponding to the four sets of equations of Turner's solution (equations [5] through [8]), there are now five, reflecting the additional set of decision variables K_f. Three of these equations are of prime interest and only these three are considered in Appendix A: the asset pricing equation, corresponding to equation (5) in Turner's solution; the insurance pricing equation, corresponding to equation (8) in Turner's solution; and the new equation setting the allocation of capital to insurers.

Asset Pricing Equation

The asset pricing equation is unchanged from Turner's results (see equation (A26):

$$\theta V_i = \overline{R}_i - \lambda \text{Cov } [R_i, R_M + R_N - X_N]. \tag{22}$$

It is shown in Appendix A that the capital asset pricing model equation is obtained as a special case of equation (22). If real assets and insurance are eliminated from the model, then equation (22) reduces (see equations [A36] through [A38]) to

$$\bar{j}_i = j_o + \beta_i(\bar{j}_M - j_o), \tag{23}$$

where β_i is the usual capital asset pricing model beta.

Insurance Pricing Equation

The insurance pricing equation is computed in Appendix A. The complete result is equation (A15). However, Appendix A argues that the last member of equation (A15) will be small. If it is dropped, and the values of θ and λ are taken from equations (A37) and (A38), then the following approximate form of the insurance pricing equation results:

$$\begin{aligned} P_{fk} = j_o^{-1}E[Q_f X_k] + \frac{\bar{j}_M - j_o}{j_o V_M \sigma_M^2} \Big\{ & \sum_h a_{hk}\text{cov}[X_k, X_{hk}] \\ & - \sum_{hl} a_{hl}\text{cov}[(1-Q_f)X_k, X_{hl}] - \text{cov}[Q_f, R_M]\overline{X}_k \Big\}, \end{aligned} \tag{24}$$

where

$$\sigma_M^2 = V[j_M]. \tag{25}$$

There is no assumption here of independence between distinct risks within the same asset type, a point discussed above.

Each member of equation (24) is capable of interpretation. That involving an expectation is a risk premium (pure premium), adjusted for insolvency risk. The remaining members constitute the risk loading. Consider the three members involved in the risk loading. The first is independent of insolvency risk. It is, in fact, the loading that would arise *in the absence of insolvency risk*, when the last two members of equation (24) vanish. It depends on the variability of losses of asset type k.

The penultimate member of equation (24) corrects (reduces) this risk loading for insolvency risk. In fact, the corrected risk loading then becomes

$$\sum_h a_{hk}\ \text{cov}\ [Q_f X_k, X_{hk}] - \sum_{\substack{hl \\ l \neq k}} a_{hl}\ \text{cov}\ [(1 - Q_f)\ X_k, X_{hl}].$$

The first term here indicates the direct correction to the earlier risk loading. The last expression is still mainly dependent on the variability of losses, particularly of asset type k.

The final member of equation (24) represents a further correction (reduction) to premium recognizing the *systematic part of the insolvency risk*. The premium is reduced to allow for the extent to which the policyholder's inability to collect on losses incurred is likely to coincide with poor returns on his or her investments. Similar pricing adjustments, though in slightly different contexts, have been made in other works (e.g., Doherty and Garven, 1986).

Capital Allocation Equation

The calculation of the capital allocation equation is rendered rather tedious by the option-like truncation of the statistical distributions involved to allow for the possibility that policyholders' claims will be uncollectable. Nevertheless, the result can be expressed in reasonably succinct form. If equation (A25) is supplemented by equations (A29), (A37), and (A38), the result is

$$\partial S_f/\partial K_f = j_o^{-1}\{E[R_f\epsilon_f] - (\bar{j}_M - j_o)/V_M\sigma_M^2 x \\ [\mathrm{cov}(R_f\epsilon_f, R_M + R_N - X_N) - F_f(K_f)]\}, \tag{26}$$

where $F_f(.)$ is a function depending on the parameters of the total insurance market (see equation [A29]), and

$$\begin{aligned} \varepsilon_f &= 1 \text{ if } E_f \geq 0, \\ &= \hat{x}_f/X_f, \text{ if } E_f < 0, \end{aligned} \tag{27}$$

and

$$\hat{X}_f = s_f \sum_{hk} (\lambda_h^{-1} \, n_{hfk} \, X_{hk}/s_{hf}) \; / \; \sum_h \lambda_h^{-1}. \tag{28}$$

Equation (26) is a statement about the change in share price of insurer f resulting from the addition of a marginal unit of capital. Appendix A simplifies equation (26) further in the case

$$\hat{X}_f = X_f. \tag{29}$$

The result is equation (A40):

$$\partial P_f \, (K_f)/\partial K_f = F_f \, (K_f)(\bar{j}_M - j_o)/j_o V_M \sigma_M^2, \tag{30}$$

where

$P_f(K_f)$ = the value of goodwill associated with one share of insurer f with capital K_f. (31)

If equation (29) fails to hold, the error in equation (30) is of order

$[1 - \hat{X}_f \, / \, X_f]$ x probability of insolvency of insurer f (see equation [A33]). (32)

Equation (30) has an advantage over equation (26) in that the awkward term ε_f has been eliminated. It states the change in value of goodwill in insurer f per marginal unit of capital in terms of market conditions. Now the value of goodwill equation (31), and hence its derivative in equation (30), can be calculated under suitable assumptions. The calculation is ultimately based on Fisher (1978) and is carried out in Appendix B.

The assumptions made are that

$$R_f \sim \log N(\mu_{Rf}, \sigma_{Rf}^2), \tag{33}$$

$$X_f \sim \log N\ (\mu_{Xf}, \sigma_{Xf}^2), \tag{34}$$

where

$$\sigma_{Rf}^2 = \log\ (1 + \omega_f^2), \tag{35}$$

and

$$\sigma_{Xf}^2 = \log\ (1 + \gamma_f^2). \tag{36}$$

It is then shown (see equations [B12] and [B18]) that

$$\partial P_f(K_f)/\partial K_f = -[1 - \Phi(d_f)], \tag{37}$$

where

$$d_f = [\log\ (\eta_f + \delta_f) + \tfrac{1}{2}\sigma_{Xf}^2]\ /\ \sigma_f + \tfrac{1}{2}\sigma_f, \tag{38}$$

and

$$\sigma_f^2 = \sigma_{Rf}^2 + \sigma_{Xf}^2. \tag{39}$$

Substitution of equation (37) into equation (30) yields

$$1 - \Phi(d_f) = -F_f(K_f)(j_M - j_o)/j_o V_M \sigma_M^2, \tag{40}$$

which is an equation involving capitalization of insurer f on both sides. Note, however, that in equilibrium,

$$\partial P_f(K_f)/\partial K_f = \partial P_g(K_g)/\partial K_g \text{ for all f and g;} \tag{41}$$

otherwise investors would find it profitable to shift capital between insurers f and g. It follows from equations (30) and (41) that, in equilibrium,

$$F_f(K_f) = F, \text{ independent of f.} \tag{42}$$

Then equations (40) and (42) yield

$$1 - \Phi(d_f) \text{ independent of f,} \tag{43}$$

that is,

$$d_f = d, \text{ independent of f,} \tag{44}$$

in equilibrium.

Combination of equations (38) and (44) yields

$$\log(\eta + \delta) = d\sigma - \tfrac{1}{2}\sigma - \tfrac{1}{2}\sigma^2{}_X, \tag{45}$$

where the subscripts f have been deleted, and d is a fixed parameter dependent on market conditions. Equation (45) describes how the equilibrium capitalization of insurers (the left side) will vary with their asset and liability risks (the σ, σ^2, and σ_X^2 terms on the right) under fixed market conditions (the parameter d). To first approximation, log $(\eta + \delta)$ is *linear in total risk* σ. It follows, therefore, that α is effectively a measure of the (market determined) security level. It will be referred to as such in the diagrams below.

Insurance Pricing

Market Equilibrium

The basic pricing equation appears as equation (24). In considering how premiums vary with capital, note that the second of the four members on the right is independent of capitalization. In fact, the second and third of the four members will tend to be small relative to the first and last. For the explanation of this, it is useful to express the two members concerned in the form

$$\left\{\sum_{h} a_{hk}\mathrm{cov}[Q_f X_k, X_{hk}] - \sum_{\substack{h \\ l \neq k}} a_{hl}\mathrm{cov}[1 - Q_f)X_k, X_{hl}]\right\} x(\bar{j}_M - j_o)/j_o V_M \sigma_M^2. \quad (46)$$

Since both Q_f and X_k are less than unity, the first summation in equation (46) is of lesser order than $\sum_h a_{hk}$ = expected losses of assets of type k from the period.

Now

$$\begin{aligned} &\text{expected losses of assets of type k} \\ &\cdot \text{expected proportion of those losses insured} \\ &= \text{expected insured losses of all types} \\ &\cdot \text{expected proportion of these relating to assets of type k.} \end{aligned} \quad (47)$$

The first factor on the right side of equation (47) is roughly 5 percent of V_M in Australia at present. Hence, equation (47) suggests that expected losses of assets of just type k will be a small percentage of V_M. Similar reasoning can be applied to the second summation in equation (46). This time losses of all types of assets, instead of just type k, are considered. But the order of the summation is reduced by a factor of $(1\text{-}Q_f)$ < probability of insolvency of insurer f. In addition, only covariances between losses of different types of assets $(l \neq k)$ are included, and many of these will be small. By this reasoning, the second summation in equation (46) will also be a small percentage of V_M. Thus, when the factor outside the braces is taken into account, equation (46) reduces to an order probably less than a single percentage point. This will be small in relation to the first term (of order one) and the last term (where R_M is of order V_M).

The dominant terms in equation (24) thus yield the approximation

$$P_{fk} = j_o^{-1}\ E\ [Q_f\ X_k] - \text{cov}\ [Q_f, \overline{R}_m]\ \overline{X}_k\ (\bar{j}_M - j_o)\ /\ j_o\ V_M\ \sigma_M^2\ . \tag{48}$$

The premium rate consists of the discounted expected cost of claims with two main adjustments: one regarding the insolvency risk, and the other regarding the correlation between insolvency and share market risks.

Appendix D evaluates each of these components of premium rate in terms of fundamental quantities. By equation (D7),

$$E[Q_f X_k] = \overline{X}_k\left\{\Phi(q_f - \sigma_f) + \left[(\eta_f + \delta_f)\overline{R}_f / j_o\right][1 - \Phi(q_f)]\right\}, \tag{49}$$

where

$$\begin{aligned} q_f &= \left[\log(\eta_f + \delta_f)\overline{R}_f / j_o\right] / \sigma_f + ½\sigma_f \\ &= d_f + \left[\log[\overline{R}_f / j_o] - ½\sigma_{Xf}^2\right] / \sigma_f \end{aligned} \tag{50}$$

by equation (38). Further, by equation (D16),

$$\begin{aligned} \text{cov}[Q_f, R_M] / V_M &= \beta_f^{-1}\overline{R}_f[\Phi(q_f) - \Phi(q_f - \sigma_{Rf}^2 / \sigma_f)] \\ &+ \beta_f^{-1}\overline{R}_f^2 j_o^{-1}(\eta_f + \delta_f)(1 + \gamma_f^2)\{(1 + \omega_f^2)[1 - \Phi(q_f + \sigma_f)] \\ &\quad - [1 - \Phi(q_f + \sigma_{Xf}^2 / \sigma_f)]\} \end{aligned} \tag{51}$$

with β_f the capital asset pricing model beta of the asset portfolio held by insurer f. Equation (51) holds unless $\beta_f = 0$, in which the covariance reduces to zero.

The simplest observation that can be made on the premium rate equation (24), taking into account equations (49) and (51), as capitalization δ_f increases, is that

$$\lim_{\delta_f \to \infty} P_{fk} = j_0^{-1}\ \overline{X}_k. \tag{52}$$

In particular, continued increase in insurer capitalization eventually generates negligible increases at the margin in equilibrium premium rates. This is simply an alternative statement of the fact that the market will not be prepared to compensate inefficiency in the form of excessive use of capital.

Numerical Example

The variation of premium rate with the different influential parameters is illustrated by means of a numerical example, in which the standard case comprises certain parameter values (see Table 1). The first three parameters—risk-free rate of return, share market risk premium (pure premium), and share market volatility—are economy-wide parameters. The last two—liability risk and proportion of total assets held in equities—are parameters specific to the insurer. Together, they determine $\overline{R}_f$ and ω_f, which are required in equations (49) and (51).

Table 1
Variation of the Premium Rate with Five Influential Parameters

Parameter	*Symbol*	*Value (%)*
Risk-Free Rate of Return	j_o-1	8
Share Market Risk Premium (Pure Premium)	j_M-j_o	6
Share Market Volatility	σ_M	18
Liability Risk	γ_M	10
Proportion of Total Assets Held in Equities	β_f	15

The capital asset pricing model equation gives

$$\overline{R}_f = j_o + \beta_f\ (\bar{j}_M - j_o), \tag{53}$$

it being assumed that all assets other than equities are risk free. Similarly,

$$\sigma_{Rf} = \beta_f \sigma_M. \tag{54}$$

This determines ω_f via equation (35). Similarly, γ_f determines σ_{Xf} via equation (36). Together, σ_{Rf} and σ_{Xf} determine σ_f via equation (39). For the standard case, σ_f = 13 percent.

The parameter σ_f is referred to in the figures as the *total risk*. When it and β_f are fixed in variations from the above standard case, σ_{Xf} (equivalently γ_f) is determined by inversion of equation (39). All of the figures use d (i.e., d_f) as the independent variable because its constancy from one insurer to another makes it a descriptor of market conditions.

Figure 1 plots the so-called "premium basis" against d for the standard case and a number of variations in the proportion of assets held in equities. Each variation's curve is labeled at its left end with the equity proportion. The total risk is held constant at 13 percent throughout. In this context, the premium basis comprises the approximation equation (48) per unit of $\overline{X}_k$, but *before* application of the discount factor j_o^{-1}. Thus, it is *not* equal to the premium rate, differing by the omission of this discount factor and the intrinsic uncertainty associated with the line being rated, represented by the first of the two covariances in equation (24).

Figure 1 also examines the effect of varying the asset risk (by varying the equity proportion). Total risk is maintained at 13 percent throughout so that an increase in the asset risk is accompanied by a reduction in liability risk (variation in total risk is considered in Figures 3 through 6), which causes slight variation in premium rate, with the higher share market exposures leading to higher premium rates because of reduced chance of insolvency (see Figure 2).

The magnitude of the covariance term of equation (48) is illustrated in Table 2. As equity exposure becomes large, the covariance-related premium

Figure 1
Premium Basis vs. Equity Proportion
Total Risk = 13 Percent

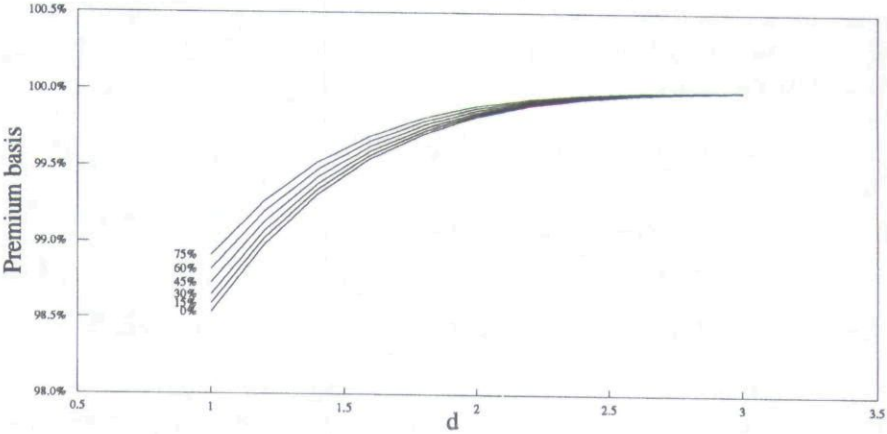
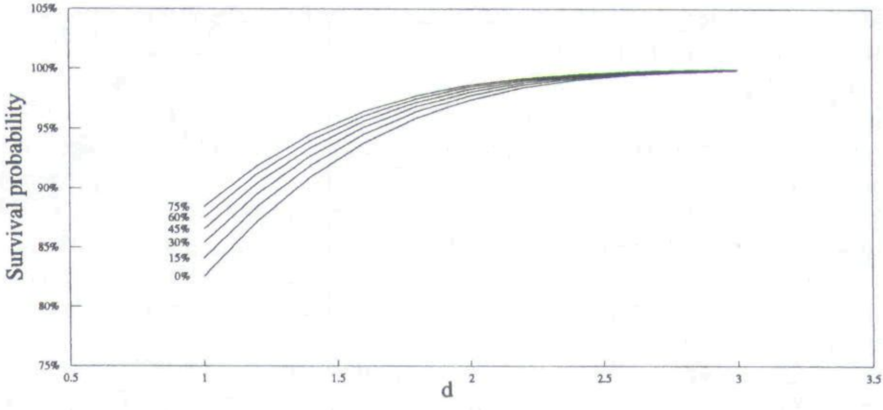


Figure 2
Solvency vs. Equity Proportion
Total Risk = 13 Percent



adjustment approaches the same order of magnitude as that which allows for the chance of insolvency.

In Figure 2 and all subsequent figures, each vertical cut (constant d) represents a selection of cases occurring simultaneously in a market equilibrium. Figure 2 plots the (one-year) survival probability of the insurer under the same scenarios as considered by Figure 1. Higher equity exposure results in a lower chance of insolvency, as anticipated in the discussion of Figure 1.

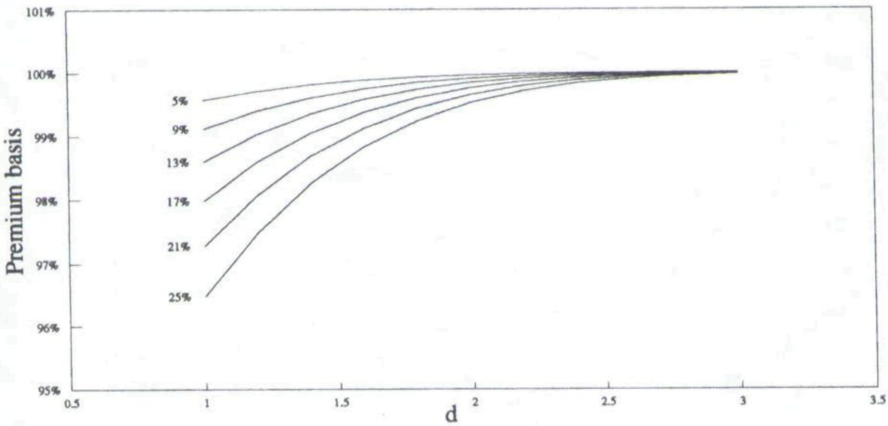
Table 2
The Effect of Equity Exposure on the Magnitude of the Covariance

	Total Risk = 13 percent			
	Equity Exposure = 15 percent		*Equity Exposure = 75 percent*	
	Premium Deduction for		*Premium Deduction for*	
d	*Insolvency %*	*Covariance %*	*Insolvency %*	*Covariance %*
1	1.3	0.1	0.7	0.4
1.6	0.4	0.04	0.2	0.1
2.2	0.09	0.01	0.04	0.03
3	0.007	0.001	0.002	0.002

Note: Premium deduction is expressed in terms of percentage of discounted expected claim cost. Insolvency is $1 - E[Q_f X_k]/\bar{X}_k$. The relevance of this quantity is apparent in equation (48). Covariance refers to the covariance term of equation (48) with the factor $\bar{X}_k/j_o$ removed.

Figures 3 and 4 correspond to Figures 1 and 2 except that they hold equity exposure constant at 15 percent and vary total risk (by varying liability risk σ_{Xf}). They illustrate how the survival probability and the premium rate fall parallel to each other as total risk increases. The falls become steep at the lower values of d.

Figure 3
Premium Basis vs. Total Risk
Equity Proportion = 15 Percent



Figures 5 and 6 repeat this analysis but hold equity exposure constant at 45 percent instead of 15 percent. The results are similar to those exhibited in Figures 3 and 4, but with the increased equity exposure causing slightly increased survival probability and premium rate (with total risk held constant), as was found in Figures 1 and 2.

Figure 4
Solvency vs. Total Risk
Equity Proportion = 15 Percent

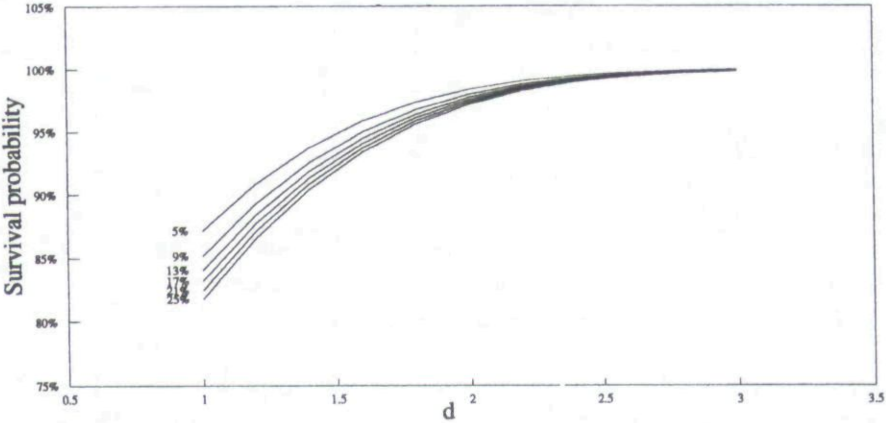
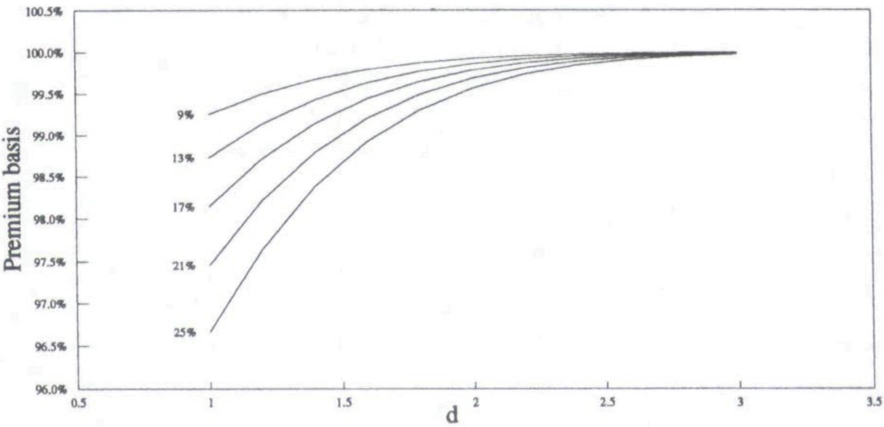


Figure 5
Premium Basis vs. Total Risk
Equity Proportion = 45 Percent

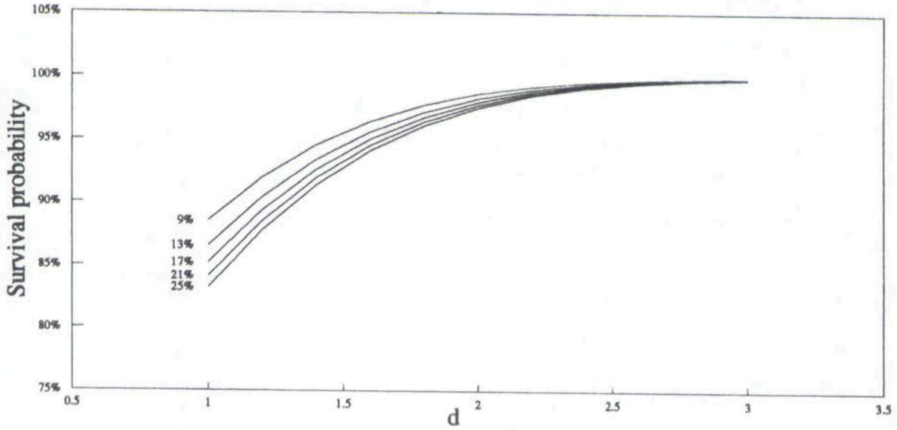


Insurer Capitalization

Market Equilibrium

The basic capital allocation equation (45) indicates that capital (the quantity $\eta+\delta$) depends on d, σ, and, to a minor extent, σ^2_X.

Figure 6
Solvency vs. Total Risk
Equity Proportion = 45 Percent



Numerical Example

The example described above is worked out in terms of its consequences for insurer capitilization in market equilibrium. Figure 7 corresponds to the situation addressed in Figures 1 and 2; Figure 8 corresponds to Figures 3 and 4. Figure 7 comprises six almost indistinguishable curves. Because of their near identity they have not been labeled with the associated equity exposures.

Figure 7
Capital vs. Equity Proportion
Total Risk = 13 Percent

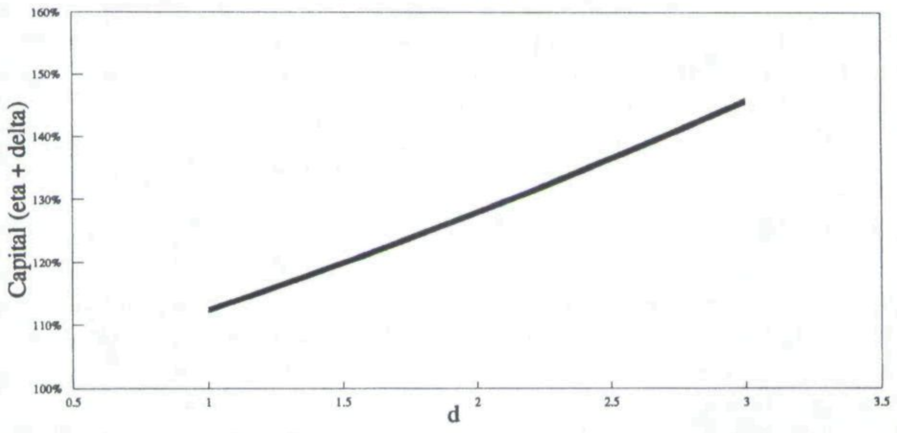
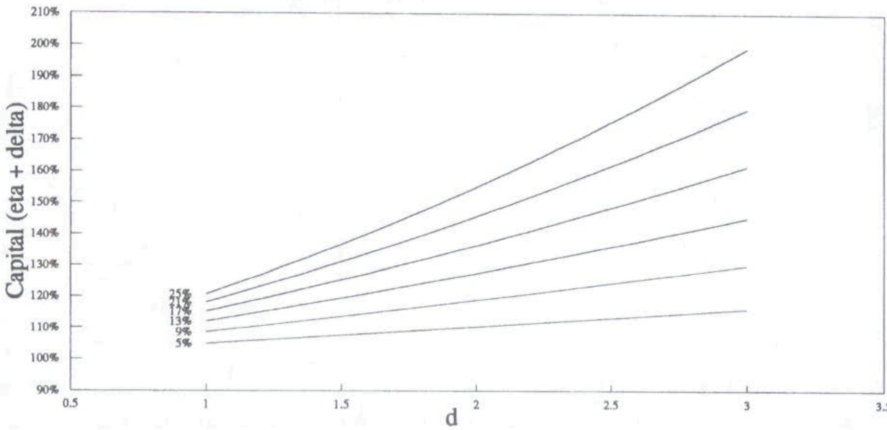


Figure 8 plots the variation of $(\eta+\delta)$ with d, with a separate curve for each value of total risk σ considered. A similar plot could have been made to correspond to Figures 5 and 6 but, to the naked eye, it would have been indistinguishable from Figure 8. For most practical purposes, insurer capitalization is linear in d. Moreover, the roughly equal vertical spacing of the curves in Figure 8 indicates that equilibrium capitalization is approximately proportional to total risk.

Figure 8
Capital vs. Total Risk
Equity Proportion = 15 Percent



Both of these simple results derive from the simple form of equation (45). The second of them is of particular interest in relation to questions of fair pricing discussed below. It is interesting to relate equilibrium capitalization and pricing by comparing Figure 7 or 8 with its corresponding figures in the previous section. For example, comparison of Figures 3 and 8 indicates that, as total risk is increased, equilibrium capitalization *increases* but premium rates *decrease*. This is contrary to conventional wisdom on pricing, but the paradox is resolved by a recognition of the increasing value of the insolvency put (indicated by Figure 4) with increasing total risk. This observation is also relevant to fair pricing.

Commentary on the Results

Market Dynamics

It is useful to comment on the robustness and relevance of equilibrium results in what is a manifestly dynamic market. To a large extent, though subject to the exception discussed below, such results provide no more than a backdrop of understanding, against which the more realistic market dynamics may be placed in perspective.

There are, of course, a number of tacit assumptions in the equilibrium framework, of varying degrees of artificiality. Among these assumptions are that there are no entries to or exits from the sample of insurers; that the different insurers are identical in a number of respects; and that, to the extent that differences exist between them, consumers will be perfectly informed. Violations of the first assumption can prevent the market from settling to an equilibrium. Violations of the second and third assumptions skew the market's equilibrium point away from its theoretical value, as found above.

Easily identifiable breaches of the assumption that insurers are identical would include differences between insurers in product design; claims control; expense control; investment expertise; customer service, or the consumer's strength of identification with the insurer (as in the case of some mutuals); and strategic response to market situations, especially in relation to capitalization.

To elaborate on just one of these items, consider a direct marketing insurer with a correspondingly lower expense structure than the market average. Since the total cost of delivery of a defined product by this insurer will be less than for its average competitor, its profit margin from a specific premium rate will be greater. It follows that the insurer with a lower expense structure can service a larger capital base from the same premium rate. At the same time, other insurers might possess less easily quantifiable but countervailing advantages that tend to redress this imbalance. Thus, the idealized equilibrium calculated above should be regarded as a less than literal interpretation of the market, and rather as providing a broad qualitative understanding of market determinants of capitalization.

It is interesting to note that, even within this broad interpretation, the equilibrium solution of capitalization provides some reconciliation between free market economists and the so-called "supply-siders." The latter group includes those who view the appropriate level of an insurer's capitalization as that which reduces the chance of insolvency to an "acceptable" level. The objection to this viewpoint is that it ignores the cost of capital. This basic objection remains. However, equilibrium capitalization is not much different from a multiple of standard deviation σ of the total stochastic risk to which the insurer is subject (see equation [45] and the discussion in the previous section).

This functional relation between standard deviation and capitalization is the same as used by the supply-siders. The two approaches are at least reconciled to this extent. Disagreement between the two camps then focuses on the multipliers of standard deviation. This is the parameter d in equation (45), which appears there as a function of market conditions, whereas supply-siders would view it as determined by the insurer's management.

There is one area in which this equilibrium may occupy a more central role than just broad understanding. In the area of fair pricing, capitalization is usually a free, but highly influential, variable. There is a reasonably compelling argument that the "fair" level of capitalization that consumers should be required to support is that which would emerge spontaneously from a market like that assumed in the preceding sections. That market was composed of insurers that were largely undifferentiated except in their respective mixes by line of business. In other respects, the parameters describing those insurers could be chosen as broadly representative of the observable market. Fair pricing is discussed in greater detail below.

Fair Pricing

Fair pricing is discussed in the various contributions to Cummins and Harrington (1987). The main methods of fair pricing recognized in practice are the Myers-Cohn (1981) net present value method and the internal rate of return method (Cummins, 1990).

Each method emphasizes the fact that cash flows of an insurance operation, including premiums, must be sufficient to service the necessary capital at a rate commensurate with the risk to insurance shareholders. Otherwise, those shareholders would invest elsewhere. Tax questions aside (which considerably complicate the issue), the principle implies a loading to increase premiums above the expected present value of claims and expenses, because insurance assets will be of lower average volatility than shares and therefore will provide a lower expected return. The balance (since insurer betas tend to be close to unity) will need to be made up from premiums.

None of the foregoing sections departs from the basic principle of fair return on capital invested in the insurance operation. Indeed, this is explicitly recognized by the current model in the section dealing with the asset pricing equation. Equation (23) shows this to be related to the capital asset pricing model.

However, approaches to fair pricing are usually bedevilled in two areas. First, although fair pricing calculations require an explicit statement of the volume of capital supporting the business being priced, there is nothing internal to these methods determining that volume. It needs to be determined by external references, commonly by reference to market practice. Second, while fair pricing methods recognize the stochastic nature of liability payments in terms of their volatility, they treat these variable amounts as payable *with certainty.* That is, the possibility of insurer insolvency goes unrecognized.

The results of the section on equilibrium pricing of insurance in the presence of insolvency risk provide at least a partial resolution of these two issues. Equation (45) shows how equilibrium capital commitment varies with total risk, serving to establish the relationship between the respective capital commitments required by different lines of business. This does *not* establish the absolute capital commitment required. This depends on the global parameter d (see equation [44]) which, by equation (40), depends on all the market conditions summarized by the functions $F_f(.)$, as defined by equation (A29).

No practical method of analytical evaluation of these functions exists. However, a reasonable practical expedient might be to adopt a medium term average level of empirical capitalization of the entire market as appropriate to fair pricing of the market's aggregate underwriting, and then allocate this level of capital to individual lines in accordance with equation (45).

The insurance pricing equation (24) recognizes the possibility of insurer insolvency by two specific deductions from premium, namely the two dealt with in the "premium basis" used in the numerical examples of insurance pricing above. These deductions are relatively small. Note that the first term on the right side of equation (24) is (up to a constant multiplier)

$$E[Q_fX_k] = \text{Prob}\ [Q_f = 1]\ E[X_k \mid Q_f = 1]$$
$$+ \text{Prob}\ [Q_f < 1]\ E[Q_fX_k \mid Q_f<1] \quad (55)$$
$$= \text{Prob}\ [Q_f = 1]\bar{x}_k + \text{Prob}\ [Q_f < 1]\ E[Q_fX_k \mid Q_f < 1],$$

as a reasonable approximation when Q_f is close to unity.

The proportionate premium reduction inherent in this term is

$$1\text{-}E[Q_fX_k]/\bar{x}_k = \{1 - \text{Prob}\ [Q_f = 1]\}$$
$$- \text{Prob}\ [Q_f < 1] \cdot E[Q_fX_k \mid Q_f < 1]/\bar{x}_k. \quad (56)$$

The quantity within braces is the probability of insolvency. Consequently, the premium reduction is substantially less than the probability of insolvency.

The numerical examples of insurance pricing above confirm this to be the case even when the additional premium deduction, related to $\text{cov}[Q_f, R_M]$, is taken into account. Further, these premium deductions are partly offset by the two premium loadings that appear in equation (24) related to the variability of liabilities. The result is that, at the level of insolvency probability likely to be acceptable to regulators (perhaps of the order of 1 percent), the corresponding premium deduction is likely to be a relatively small fraction of 1 percent. The political motivation for disregarding it completely at this level is obvious.

In conclusion, therefore, this analysis leaves accepted fair pricing practices intact for all practical purposes but adds rigor to the choice of the level of insurer capitalization to be reflected in that pricing.

The Role of Consumer Diversity

Consider the earlier optimization problem in the presence of insolvency risk, when all households in the economy are identical in wealth and preferences. By symmetry, all households will choose the same insurances and the same shareholdings (including shareholdings in insurers). In this case, insurance across the economy will amount to just a pooling of risks. The contribution of insurance and insurance shareholding to an individual household's terminal wealth will be equal to a proportion H^{-1} of all premiums paid, plus investment income earned by insurers, less total losses across the economy; that is,

$$H^{-1}\left[\sum_f (P_f+s_fK_f)R_f - \sum_{fhk} n_{hfk}X_{hk}\right].$$

Then terminal wealth becomes

$$T_h = \sum_i v_{hi}R_i + \sum_k a_{hk}R_k + H^{-1}[\sum_f (P_f + s_fK_f)R_f - \sum_{fhk} n_{hfk}X_{hk}]. \quad (57)$$

It is evident that, under this simple pooling arrangement, terminal wealth of a household is affected by total losses across the economy, but not by insurer insolvencies *per se*. Capital allocation to insurers then becomes irrelevant, and so there can be no solution to the optimization under examination. The formal analysis leading to this conclusion is nevertheless instructive. It is found that the formal solution to the optimization problem requires infinite insurance capital.[2]

Although this result is not meaningful in itself (since the level of capitalization of insurers has already been noted irrelevant in this special case), it is useful as a limiting result. It indicates that, when consumers are not quite identical but nearly so, the optimal level of capitalization will be high, and growing with increasing similarity of consumers.

The Insurance Cycle

The insurance cycle is characterized by periodic fluctuations in premium rates and inversely related fluctuations in insurance capital. This must reflect market disequilibrium, or at least periodic shifts in equilibrium. The degree of explanation of the market cycle to be expected of any equilibrium model is therefore limited.

Nevertheless, some useful comment can be made. Again, reference is made to equation (A29), this time to the first term on the right, which reflects the presence of different insurers in the market (subscript g). As g increases, the number of terms contributing to the relevant term of equation (A29) increases, but the contribution of each decreases as business is dispersed over a greater number of insurers causing $[E_g]_+$ to decrease.

It is difficult to deal with this situation rigorously. However, a heuristic argument runs as follows. Consider an existing equilibrium and suppose that additional insurers are introduced to the market. First, underwriting will be dispersed more widely in terms of premium per unit of capital, assuming any increase in the volume of business underwritten does not keep pace with the influx of capital. This will reduce the equilibrium capital requirement of each of the old insurers. Equivalently, the same capital as before leads to an increase in d, as defined by equation (38) and discussed in previous sections. The new equilibrium will therefore involve lower capitalization and lower premium rates. The generalization of this is that any cycle in market capacity measured by aggregate insurance capital per premium dollar will induce a corresponding, inversely related cycle in premium rates.

Figure 7 illustrates the fact that a modest change in capacity of 10 percent can affect the value of d radically, changing it from about 1.6 to about 2.3. Nevertheless, Figures 1, 3, and 5 indicate that the resulting change in equilibrium premium rates is likely to be only fractional.

Two questions remain about the market cycle: Why do premium rates vary so much more than their equilibrium values? And what causes the cycle in market capacity described above to be the cause of cycling premium rates? The amplitude of the premium rate cycle on the down side is probably due to competition in the presence of increased capacity. New entrants are likely to price below equilibrium rates in an attempt to gain market share. Other insur-

[2] The analysis is available in an appendix, available from the author upon request.

ers, who find themselves with an excess of capital because of loss of market share, are likely to use that excess to lower prices as a defense against a further loss of market share.

This scenario is consistent with Winter's (1988) model of the insurance cycle, which leads to a protracted period of subeconomic prices and slow erosion of each insurer's capital base. Winter's model may also answer the question about cycles. It takes into account regulatory solvency constraints and their interaction with competitive pricing. In that model, the reduction in capital base can proceed only to the point where it is about to breach solvency requirements. Winter shows that a market "crunch" occurs at this point with premium rates and capital base increasing sharply. The crunch is exacerbated by the exit of inefficient and capital-poor insurers. With the consequent reduction in market capacity, the cycle recommences.

Conclusion

This article establishes an economy-wide one-period model of the dependencies between the prices of insurance and other stocks, real assets, and insurance policies, as well as consumer preferences for consumption relative to saving and insurance security, and insurer capitalization. All ingredients other than the last were included in Turner's (1981) model, which is used here, but augmented to incorporate insurer capitalization as a free variable.

No market equilibrium exists unless there is diversity between consumers in terms of risk aversion. Equilibrium capitalization will depend on the *extent* of such diversity. A free market will differentiate premium rates to recognize insolvency risk, but in practical terms the differences in rates will usually be small (measured in fractions of a percentage point). It is shown that equilibrium premium rates also contain loadings reflecting the inherent variability of the losses underwritten, but that these loadings are also small, probably of even lesser magnitude.

Equilibrium capitalization of an insurer will be directly related to the *total risk* assumed by the insurer, incorporating both asset and liability risk. The relation is not linear, but is close enough for most practical purposes. This last result determines the *differences* between the capital bases required to support various lines of business. The *absolute levels* of these capital bases depend, however, on total market conditions, most specifically the number and size of the insurers of which the market is composed and the schedule of consumers' risk aversions.

The relation between total risk and capitalization is useful in the context of fair pricing. Without such a result, the level of capitalization to be reflected in any fair pricing exercise will be settled by dubious means, including guesswork, possibly unfounded tradition or, worse, prejudice and specious argument.

The model presented here cannot be expected to explain the insurance cycle, which is not an equilibrium phenomenon. However, some aspects of the model are consistent with plausible explanations of the cycle.

Appendix A
Maximization of Expected Utility

Optimal Solution

The variance of terminal wealth, corresponding to expected terminal wealth in equation (20), is

$$
\begin{aligned}
v_h = {} & \sum_{ij} v_{hi} v_{hj} \operatorname{cov}(R_i, R_j) + 2\sum_{if} v_{hi} s_{hf} \operatorname{cov}[R_i, [E_f]_+] \\
& + 2\sum_{ik} v_{hi} a_{hk} \operatorname{cov}(R_i, R_k - X_{hk}) \\
& + 2\sum_{fik} v_{hi} n_{hfk} \operatorname{cov}(R_i, Q_f X_{hk}) \\
& + \sum_{fg} s_{hf} s_{hg} \operatorname{cov}[[E_f]_+, [E_g]_+] \\
& + 2\sum_{fk} s_{hf} a_{hk} \operatorname{cov}[[E_f]_+, R_k - X_{hk}] \\
& + 2\sum_{gfk} s_{hg} n_{hfk} \operatorname{cov}[[E_g]_+, Q_f X_{hk}] \\
& + \sum_{kl} a_{hl} a_{hk} \operatorname{cov}(R_k - X_{hk}, R_l - X_{hl}) \\
& + 2\sum_{lfk} a_{hl} n_{hfk} \operatorname{cov}(R_l - X_{hl}, Q_f X_{hk}) \\
& + \sum_{fkgl} n_{hfk} n_{hgl} \operatorname{cov}(Q_f X_{hk}, Q_g X_{hl}). \qquad \text{(A1)}
\end{aligned}
$$

The first-order conditions of the optimization problem of equation (21) may now be derived taking into account equations (20) and (A1). The six sets of decision variables will lead to six sets of equations, instead of five as in Turner (1981). This article concentrates on just two of the six, namely those relating to optimization with respect to n_{hfk} and K_f, respectively. These will lead to the equilibrium equations for pricing and allocation of capital to insurers.

As in Turner (1981), let U_μ' , U_v', and U_C' denote $\partial U_h/\partial \mu_h$, $\partial U_h/\partial v_h$, and $\partial U_h/\partial C_h$, respectively. The two first-order conditions of interest are then obtained by differentiating the objective function in equation (21) with respect to n_{hfk} and K_f, respectively. The first of these differentiations requires evaluation of the quantities $\partial \mu_h/\partial n_{hfk}$ and $\partial v_h/\partial n_{hfk}$. By equation (20),

$$\partial\mu_h/\partial n_{hfk} = E[Q_f X_{hk}] + \sum_f s_{hf}\partial E[E_f]_+/\partial n_{hfk}.$$

By equation (17),

$$\begin{aligned} s_f\partial E_f/\partial n_{hfk} &= (\partial/\partial n_{hfk})(P_f R_f - X_f) \\ &= P_{fk}R_f - X_{hk}, \end{aligned}$$

so that

$$\partial\mu_h/\partial n_{hfk} = E[Q_f X_{hk}] + (s_{hf}/s_f)(P_{fk} - X_{hk})I(E_f).$$

It usually will be the case that H is large and that, as a result, the factor s_{hf}/s_f is negligibly small. If it is assumed that this is the case, then $\partial\mu_h/\partial n_{hfk}$ reduces to just $\partial\mu_h/\partial n_{hfk} = E[Q_f X_{hk}]$.

By the same argument, the evaluation of $\partial v_h/\partial n_{hfk}$ may be carried out by reference to equation (A1) with the dependency of $[E_f]_+$ on n_{hfk} ignored. In this case, differentiation of equation (21) with respect to n_{hfk} yields the following result:

$$\begin{aligned} 0 = U_\mu' E[Q_f X_{hk}] + 2U_v'\{&\sum_i v_{hi}\operatorname{cov}[R_i, Q_f X_{hk}] \\ &+ \sum_g s_{hg}\operatorname{cov}[[E_g]_+, Q_f X_{hk}] \\ &+ \sum_l a_{hl}\operatorname{cov}[R_l - X_{hl}, Q_f X_{hk}] \\ &+ \sum_{gl} n_{hgl}\operatorname{cov}[Q_f X_{hk}, Q_g X_{hl}]\} - \Psi_h P_{fk}, \end{aligned} \tag{A2}$$

Differentiation of equation (21) with respect to K_f yields the following result:

$$\begin{aligned} 0 = {} & U_\mu' \frac{\partial}{\partial K_f} E[s_{hf}[E_f]_+ + \sum_k n_{hfk} Q_f X_{hk}] \\ & + 2U_v' \frac{\partial}{\partial K_f}\left\{ s_{hf}\operatorname{cov}\left[[E_f]_+, \sum_i v_{hi}R_i + \sum_g s_{hg}[E_g]_+ + \sum_k a_{hk}(R_k - X_{hk}) + \sum_{gk} n_{hgk}Q_g X_{hk}\right]\right. \\ & \left. + \sum_k n_{hfk}\operatorname{cov}[Q_f X_{hk}, \sum_i v_{hi}R_i + \sum_l a_{hl}(R_l - X_{hl}) + \sum_{gl} n_{hgl} Q_g X_{hl}]\right\} - \Psi_h s_{hf}(\partial S_f/\partial K_f). \end{aligned} \tag{A3}$$

In addition, differentiation of equation (21) with respect to C_h yields

$$U_C' - \psi_h = 0. \tag{A4}$$

Following Turner (1981), use equation (A4) to substitute for ψ_h in equations (A2) and (A3), divide each of equations (A2) and (A3) through by U_v', and recall definitions from equations (11) and (12) to obtain

$$\theta_h P_{fk} = \lambda_h^{-1} E[Q_f X_{hk}] + 2\,\mathrm{cov}[Q_f X_{hk}, \sum_i \nu_{hi} R_i + \sum_g s_{hg}[E_g]_+ + \sum_l a_{hl}(R_l - X_{hl}) + \sum_{gl} n_{hgl} Q_g X_{hl}], \tag{A5}$$

and

$$\begin{aligned}\theta_h s_{hf}(\partial S_f/\partial K_f) &= \lambda_h^{-1} \frac{\partial}{\partial K_f} E[s_{hf}[E_f]_+ + \sum_k n_{hfk} Q_f X_{hk}] \\ &+ 2\frac{\partial}{\partial K_f}\Bigg\{ s_{hf}\,\mathrm{cov}\Big[[E_f]_+, \sum_i \nu_{hi} R_i + \sum_g s_{hg}[E_g]_+ + \sum_k a_{hk}(R_k - X_{hk}) + \sum_{gk} n_{hgk} Q_g X_{hk}\Big] \\ &+ \sum_k n_{hfk}\,\mathrm{cov}\Big[Q_f X_{hk}, \sum_i \nu_{hi} R_i + \sum_l a_{hl}(R_l - X_{hl}) + \sum_{gl} n_{hgl} Q_g X_{hl}\Big]\Bigg\}.\end{aligned} \tag{A6}$$

Define

$$\hat{X}_f = s_f \sum_{hk} (\lambda_h^{-1} n_{hfk} X_{hk}/s_{hf}) / \sum_h \lambda_h^{-1}. \tag{A7}$$

Now divide equation (A6) through by s_{hf}, aggregate both equations (A5) and (A6) over h, use equation (A7), and recall equations (9), (10), and (13) to obtain

$$\theta P_{fk} = E[Q_f X_k] - \lambda \sum_h \mathrm{cov}[Q_f X_{hk}, \sum_i \nu_{hi} R_i + \sum_g s_{hg}[E_g]_+ + \sum_l a_{hl}(R_l - X_{hl}) + \sum_{gl} n_{hgl} Q_g X_{hl}], \tag{A8}$$

$$\begin{aligned}\theta(\partial S_f/\partial K_f) &= \frac{\partial}{\partial K_f} E\big[[E_f]_+ + s_f^{-1} Q_f \hat{X}_f\big] \\ &- \lambda \frac{\partial}{\partial K_f}\Bigg\{ \sum_h \mathrm{cov}\Big[[E_f]_+, \sum_i \nu_{hi} R_i + \sum_g s_{hg}[E_g]_+ + \sum_k a_{hk}(R_k - X_{hk}) + \sum_{gk} n_{hgk} Q_g X_{hk}\Big] \\ &+ \sum_{hk} (n_{hfk}/s_{hf})\,\mathrm{cov}\Big[Q_f X_{hk}, \sum_i \nu_{hi} R_i + \sum_l a_{hl}(R_l - X_{hl}) + \sum_{gl} n_{hgl} Q_g X_{hl}\Big]\Bigg\}.\end{aligned} \tag{A9}$$

It is now necessary to simplify the summed covariance terms in equations (A8) and (A9).

Pricing Equation

Consider the pricing equation (A8). It may be written in the form

$$\theta P_{fk} = E[Q_f X_k] - \lambda \sum_h \text{cov}[Q_f X_k, ...] - \lambda \sum_h \text{cov}[Q_f(X_{hk} - X_k), ...], \quad \text{(A10)}$$

where the dots represent the second argument in the covariance term in equation (A8). The first covariance in equation (A10) is

$$\sum_h \text{cov}[Q_f X_k, ...] = \text{cov}\left[Q_f X_k, \sum_{hi} \nu_{hi} R_i + \sum_{hg} S_{hg}[E_g]_+ + \sum_{hl} a_{hl}(R_l - X_{hl}) + \sum_{hgl} n_{hgl} Q_g X_{hl}\right]. \quad \text{(A11)}$$

The second argument of the covariance here represents the aggregate wealth of all households. It is therefore equal to $R_M + R_N - X_N$, where these three terms were defined in the notation and assumptions section. Thus, equation (A11) becomes

$$\sum_h \text{cov}[Q_f X_k, ...] = \text{cov}[Q_f X_k, R_M + R_N - X_N] = \text{cov}[Q_f, R_M]\bar{X}_k - \text{cov}[Q_f X_k, X_N], \quad \text{(A12)}$$

since $Q_f X_k$ and R_N are independent, and so are X_k and R_M.

Now substitute equation (A12) into (A10), recalling at the same time that $X_N = \sum_{jl} a_{jl} X_{jl}$ and that cov $[Q_f X_k, R_l] = 0$ to obtain

$$\begin{aligned}\theta P_{fk} &= E[Q_f X_k] + \lambda \sum_{jl} a_{jl} \text{ cov}[Q_f X_k, X_{jl}] - \lambda \text{cov}[Q_f R_M]\bar{X}_k \\ &- \lambda \sum_h \text{cov}\left[Q_f(X_{hk} - X_k), \sum_i \nu_{hi} R_i + \sum_g S_{hg}[E_g]_+ + \sum_l \left(\sum_g n_{hgl} Q_g - a_{hl}\right) X_{hl}\right].\end{aligned} \quad \text{(A13)}$$

Note that

$$\begin{aligned}\text{cov}[Q_f X_k, X_{jl}], &= \text{cov}\ [X_k, X_{jl}] - \text{cov}[(1-Q_f)X_k, X_{jl}] \\ &= \delta_{kl}\ \text{cov}[X_k, X_{jk}] - \text{cov}\ [(1-Q_f)X_k, X_{jl}],\end{aligned} \quad \text{(A14)}$$

if claims in respect of asset types k, l are assumed stochastically independent, and δ_{kl} denotes the Kronecker delta.

Substitution of equation (A14) in (A13) yields

$$\begin{aligned}\theta P_{fk} &= E[Q_f X_k] + \lambda \sum_h a_{hk} \text{cov}[X_k, X_{hk}] - \lambda \sum_{hl} a_{hl} \text{cov}[(1-Q_f)X_k, X_{hl}] - \lambda\ \text{cov}[Q_f R_M]\bar{X}_k \\ &- \lambda \sum_h \text{cov}\left[Q_f(X_{hk} - X_k), \sum_i \nu_{hi} R_i + \sum + \sum_g S_{hg}[E_g]_+ + \sum_l \left(\sum_g n_{hgl} Q_g - a_{hl}\right) X_{hl}\right].\end{aligned} \quad \text{(A15)}$$

Note that the right side of equation (A15) separates the fundamental claims risk cov $[X_k, X_{hk}]$ from the insolvency risk involving $(1-Q_f)$. A similar treatment could be applied to the final covariance of equation (A15). However, this leads to tedious algebra which is not very rewarding since this term, involving a sum of covariances $(X_{hk}-X_k)$ about a weighted mean, is likely to be small. Note also that cov$[X_k, X_{hk}]$ can be simplified further if independence between individual risks (different values of h) is assumed, as is done by Turner.

Capital Allocation Equation

Consider the various terms of equation (A9) one by one.

Expectation term of equation (A9). The argument of the expectation is

$$= \frac{\partial}{\partial K_f}\left\{[E_f]_+ + s_f^{-1}Q_f\hat{X}_f\right\}$$
$$= \frac{\partial}{\partial K_f}\left\{\max\ (0,E_f) + s_f^{-1}\hat{X}_f\ \min\ [0,s_fE_f/X_f]\right\} \tag{A16}$$
$$= [\partial E_f/\partial K_f]\varepsilon_f,$$

where

$$\varepsilon_f = 1 \text{ if } E_f \geq 0,$$
$$\varepsilon_f = \hat{X}_f/X_f \text{ if } E_f < 0. \tag{A17}$$

From the definition of E_f,

$$\partial E_f/\partial K_f = R_f, \tag{A18}$$

and so substitution of equations (A16) and (A18) in equation (A9) yields

$$E[R_f\varepsilon_f], \tag{A19}$$

with ε_f defined by equation (A17).

First covariance term of equation (A9). By the same argument that led to equation (A12), the first term of equation (A9) involving a covariance is

$$(\partial/\partial K_f)\ \mathrm{cov}\ [[E_f]_+,R_M+R_N-X_N] = \mathrm{cov}\ [(\partial/\partial K_f)\ [E_f]_+,R_M+R_N-X_N], \tag{A20}$$

since the second argument is aggregated wealth of the economy and is therefore independent of K_f.

It is convenient to write equation (A20) in the form

$$\mathrm{cov}\ [(\partial/\partial K_f)\ [[E_f]_+ + s_f^{-1}Q_f\hat{X}_f],\ R_M+R_N-X_N]$$
$$-\ \mathrm{cov}\ [(\partial/\partial K_f)\ [s_f^{-1}Q_f\hat{X}_f],\ R_M+R_N-X_N]$$
$$= \mathrm{cov}\ [R_f\varepsilon_f,\ R_M+R_N-X_N]$$
$$-\ \mathrm{cov}\ [(\partial/\partial K_f)\ [s_f^{-1}Q_f\hat{X}_f],\ R_M+R_N-X_N], \tag{A21}$$

by the same argument as led to equation (A19). By further use of the fact that $[R_M+R_N-X_N]$ is independent of K_f, the fact that $R_M+R_N-X_N = \sum_h T_h$, and equation (A7), equation (A21) may be written as

$$\mathrm{cov}\ [R_f\varepsilon_f,\ R_M+R_N-X_N]$$
$$-(\partial/\partial K_f)\sum_h \mathrm{cov}\left\{Q_f\left(\lambda_h^{-1}/\sum_j \lambda_j^{-1}\right)\sum_k (n_{hfk}/s_{hf})X_{hk},\ T_h\right\}. \tag{A22}$$

There is a small approximation involved in the second term of equation (A22) in that the cross-covariances $\mathrm{cov}[Q_f\ X_{gk},\ T_h]$ from equation (A21) have been ignored for $g \neq h$.

Second covariance term of equation (A9). This is equal to

$$
\begin{aligned}
&(\partial/\partial K_f)\sum_{hk}(n_{hfk}/s_{hf})\ \text{cov}\left[Q_f X_{hk},\ T_h - \sum_g s_{hg}[E_g]_+\right] \\
&\quad = (\partial/\partial K_f)\sum_h \text{cov}\left[Q_f,\ \sum_k (n_{hfk}/s_{hf})X_{hk}, T_h\right] \\
&\quad - (\partial/\partial K_f)\sum_{hgk}(n_{hfk}s_{hg}/s_{hf})\ \text{cov}[Q_f X_{hk}, [E_g]_+].
\end{aligned} \tag{A23}
$$

The second of the two members on the right may be rewritten as

$$
\begin{aligned}
&\sum_{hgk}(n_{hfk}s_{hg}/s_{hf})\ \{\text{cov}[(\partial/\partial K_f)(Q_f X_{hk}),[E_g]_+] + \text{cov}[Q_f X_{hk},(\partial/\partial K_f)[E_g]_+]\} \\
&\quad = \sum_{hgk}(n_{hfk}s_{hg}/s_{hf})\ \text{cov}[(\partial/\partial K_f)(Q_f X_{hk}),[E_g]_+] \\
&\quad + \sum_{hk} n_{hfk}\ \text{cov}[Q_f X_{hk}, R_f\ I(E_f)],
\end{aligned} \tag{A24}
$$

because of the independence of E_g from K_f when $g \neq f$, and with I(.) as defined in the notation and assumptions section. The second argument of the covariance in equation (A24) is obtained from the definition of $[E_g]_+$.

Whole equation. Substitution of equations (A19) and (A22) through (A24) in equation (A9) gives

$$
\begin{aligned}
\theta(\partial S_f/\partial K_f) &= E[R_f \varepsilon_f] - \lambda\ \text{cov}\ [R_f \varepsilon_f,\ R_M + R_N - X_N] \\
&+ \lambda \sum_{hgk}(n_{hfk}s_{hg}/s_{hf})\ \text{cov}\left[(\partial/\partial K_f)(Q_f X_{hk},[E_g]_+\right] \\
&+ \lambda \sum_{hk} n_{hfk}\ \text{cov}\left[Q_f X_{hk}, R_f I(E_f)\right] \\
&+ \lambda(\partial/\partial K_f)\ \sum_h \text{cov}\left\{Q_f \sum_k (n_{hfk}/s_{hf})\left[\lambda_h^{-1}/\sum_j \lambda_j^{-1} - 1\right] X_{hk}, T_h\right\}.
\end{aligned} \tag{A25}
$$

Equation (A25) is related to the equilibrium equation for pricing stocks, as shown by Turner (1981, equation [4.47]) to be

$$\theta V_i = \bar{R}_i - \lambda\ \text{Cov}\ [\bar{R}_i,\ R_M + R_N - X_N]. \tag{A26}$$

It is straightforward to retrace Turner's working in the current model to show that equation (A26) still holds. Now multiply through by $\nu_{fi}/\sum_i \nu_{fi} V_i$ and sum over i to obtain

$$\theta = \bar{R}_f - \lambda\ \text{Cov}\ [R_f,\ R_M + R_N - X_N], \tag{A27}$$

where R_f is defined as in the notation and assumptions section.

Equation (A27) enables θ to be eliminated from equation (A25) to give

$$E[R_f(\partial S_f/\partial K_f - \varepsilon_f)] - \lambda\ \text{cov}\ [R_f(\partial S_f/\partial K_f - \varepsilon_f),\ R_M + R_N - X_N] = \lambda F_f(K_f), \tag{A28}$$

where

$$F_f(K_f) = \sum_{hgk} (n_{hfk}s_{hg}/s_{hf}) \text{ cov } [(\partial/\partial K_f)(Q_f X_{hk}), [E_g]_+]$$
$$+ \sum_{hk} n_{hfk} \text{ cov } [Q_f X_{hk}, R_f I(E_f)] \tag{A29}$$
$$+ (\partial/\partial K_f)\sum_h \text{cov} \left\{ Q_f \sum_k (n_{hfk}/s_{hf}) \left[\lambda_h^{-1} / \sum_j \lambda_j^{-1} - 1 \right] X_{hk}, T_h \right\}.$$

The left side of equation (A28) has the same form as the right side of equation (A26) and must therefore be equal to θ times the value of an asset which pays a cash flow of R_f $(\partial S_f/\partial K_f - \varepsilon_f)$. If S_f^* denotes the value of such an asset, then equation (A28) becomes

$$S_f^*(K_f) = (\lambda/\theta) \, F_f(K_f), \tag{A30}$$

where the dependency of S_f^* on K_f has been shown explicitly.

It is possible to interpret S_f^* further:

$$S_f^*(K_f) = \frac{\partial}{\partial K_f} P_f(K_f), \tag{A31}$$

where $P_f(K_f)$ is the value of an asset with a payoff of

$$[S_f - \varepsilon_f K_f + \text{terms independent of } K_f] R_f$$
$$= (S_f - K_f)R_f + (1-\varepsilon_f) \, K_f R_f + \text{terms independent of } K_f. \tag{A32}$$

Note that S_f-K_f is the value of goodwill in the insurer at time zero.

By equation (A17),

$$1\text{-}\varepsilon_f = 1 - \hat{X}_f/X_f \text{ if insurer f is insolvent,}$$
$$1\text{-}\varepsilon_f = 0 \text{ if solvent.} \tag{A33}$$

By the definitions of X_f and $\hat{X}_f$, each is a weighted sum of the terms n_{hfk} X_{hk}, and the two are likely to differ little. The term of equation (A32) involving 1-ε_f can then be neglected. In addition, the terms independent of K_f can be neglected. Although they will contribute to the value $P_f(K_f)$, they will not affect its derivative in equation (A31), which is the quantity sought.

Thus, to sufficient approximation, $P_f(K_f)$ may be taken as the value of an asset with payoff $(S_f$-$K_f)R_f$, which is the value of the goodwill associated with one share of insurer f. Then equations (A30) and (A31) give

$$\frac{\partial}{\partial K_f} P_f(K_f) = (\lambda/\theta) F_f(K_f), \tag{A34}$$

with

$P_f(K_f)$ = value of goodwill associated with one share of insurer f. (A35)

The parameter λ/θ may be interpreted from equation (A26) as the price of investment risk. If real assets and insurance are eliminated from the model, then equation (A26) collapses to a capital asset pricing model equation:

$$\begin{aligned}\overline{R}_i/V_i &= \theta + \lambda \text{ cov } [R_i,R_M]/V_i \\ \bar{j}_i &= \theta + \lambda\, V_M \text{ cov } [j_i,j_M],\end{aligned}$$

where j_i and j_M are defined as in the notation and assumptions section. Thus,

$$\bar{j}_i = \theta + \lambda V_M \beta_i \sigma_M^2, \tag{A36}$$

where

$$\begin{aligned}\sigma_M^2 &= V[j_M], \\ \beta_i &= \text{cov } [j_i,j_M]/\sigma_M^2 \\ &= \text{capital asset pricing model } \beta \text{ of stock i.}\end{aligned}$$

As special cases of equation (A36),

$$\begin{aligned}j_o &= \theta \\ \bar{j}_M &= \theta + \lambda\, V_M \sigma_M^2,\end{aligned}$$

where j_o is defined as in the notation and assumptions section. Thus,

$$\theta = j_o. \tag{A37}$$

$$\lambda = (\bar{j}_M - j_o)/V_M \sigma_M^2, \tag{A38}$$

$$\lambda/\theta = (\bar{j}_M - j_o)/j_o V_M \sigma_M^2. \tag{A39}$$

Substitution of this result into equation (A34) gives

$$\frac{\partial}{\partial K_f} P_f(K_f) = F_f(K_f)(\bar{j}_M - j_o)/j_o V_M \sigma_M^2. \tag{A40}$$

Appendix B
Value of Equity and Goodwill

Consider the option price in equation (A40). A related price has been calculated by Fisher (1978) and used by Cummins (1991). Define

A_f = total invested funds of the insurer (just after premium receipt)

$$= P_f + s_f K_f. \tag{B1}$$

Recall that X_f denotes the total liabilities of insurer f, and let L_f be the expected value of this quantity at time zero, conditional on information current at that time. It is assumed that log L_f evolves over time according to a Wiener process, realizing log X_f at time one. The instantaneous variance of the process is σ^2_{Xf}, as defined in equations (34) and (36). Thus,

$$X_f/L_f \sim \log N(0,\sigma^2_{Xf}). \tag{B2}$$

By equations (33), (39), and (B2),

$$A_f R_f/X_f = (A_f/L_f)Z_f,$$

where

$$Z_f \sim \log N(\mu_{Rf}, \sigma^2_f). \tag{B3}$$

Fisher (1978) gives the value of equity (of insurer f) as

$$C(A,L) = A\Phi(d_1) - Lj^{-1}_o \Phi(d_2), \tag{B4}$$

where the subscript f has been suppressed for brevity, and

$$d_1 = \log(Aj_0/L)/\sigma + \tfrac{1}{2}\sigma, \tag{B5}$$

$$d_2 = d_1 - \sigma. \tag{B6}$$

Then the value of goodwill immediately before receipt of premium P is

$$C(A,L) - sK = sP(K), \tag{B7}$$

by the definition of P(K) in equation (A35).

By equations (B1), (B4), and (B7),

$$sP(K) = (P+sK)\Phi(d_1) - Lj_o^{-1}\Phi(d_2) - sK. \tag{B8}$$

By the definitions of η_f and δ_f,

$$P = \bar{X}j_o^{-1}\eta, \tag{B9}$$

$$sK = \bar{X}j_o^{-1}\delta, \tag{B10}$$

$$P + sK = \bar{X}j_o^{-1}(\eta+\delta). \tag{B11}$$

Substitution of equations (B1), (B2), and (B11) in equation (B5) gives

$$d_1 = \log\left[\frac{(P+sK)j_o}{\bar{X}}\ \frac{\bar{X}}{L}\right]/\sigma + \tfrac{1}{2}\sigma \qquad \text{(B12)}$$
$$= \left[\log(\eta+\delta) + \tfrac{1}{2}\sigma_X^2\right]/\sigma + \tfrac{1}{2}\sigma.$$

Substitution of equations (B2), (B10), and (B11) in equation (B8) gives

$$sP(K) = j_o^{-1}\bar{X}\{(\eta+\delta)\Phi(d_1) - \exp(-\tfrac{1}{2}\sigma_X^2)\Phi(d_2) - \delta\}. \qquad \text{(B13)}$$

To differentiate P(K) with respect to K, first note that, by equation (B12),

$$\partial d_1/\partial K = sj_o/\bar{X}(\eta+\delta)\sigma = s/\sigma(P+sK). \qquad \text{(B14)}$$

Also, by equation (B6),

$$\partial d_2/\partial K = \partial d_1/\partial K. \qquad \text{(B15)}$$

Now differentiate equation (B8) with respect to K:

$$s\partial P(K)/\partial K = -s[1-\Phi(d_1)] + (P+sK)\phi(d_1)(\partial d_1/\partial K) - Lj_o^{-1}\phi(d_2)(\partial d_2/\partial K).$$

Applying equations (B14) and (B15) to this yields

$$\partial P(K)/\partial K = -[1-\Phi(d_1)] + \phi(d_1)/\sigma - (\eta+\delta)^{-1}\exp(-\tfrac{1}{2}\sigma_X^2)\phi(d_2)/\sigma. \qquad \text{(B16)}$$

Now

$$\phi(d_2)/\phi(d_1) = \exp \tfrac{1}{2}(d_1^2 - d_2^2)$$
$$= \exp \tfrac{1}{2}\sigma(d_1 + d_2) \text{ (by equation [B6])}$$
$$= (\eta+\delta)\exp(\tfrac{1}{2}\sigma_X^2) \qquad \text{(B17)}$$

by equations (B12) and (B6). Substitution of equation (B17) in equation (B16) gives

$$\partial P(K)/\partial K = -[1-\Phi(d_1)]. \qquad \text{(B18)}$$

Appendix C
Some Results for the Normal Distribution

Lemma C1

$$\int_{-\infty}^{+\infty} \phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{x+\nu}{\tau}\right)dx = \Phi\left(\frac{\mu+\nu}{[\sigma^2+\tau^2]^{1/2}}\right). \tag{C1}$$

Corollary

$$\int_{-\infty}^{+\infty} \phi\left(\frac{x-\mu}{\sigma}\right)\left[1-\Phi\left(\frac{x+\nu}{\tau}\right)\right]dx = 1-\Phi\left(\frac{\mu+\nu}{(\sigma^2+\tau^2)^{1/2}}\right).$$

Lemma C2

$$\int_{-\infty}^{y} e^{x}\phi\left(\frac{x-\mu}{\sigma}\right)dx = \exp[\mu+\tfrac{1}{2}\sigma^2]\,\Phi\left(\frac{y-(\mu+\sigma^2)}{\sigma}\right). \tag{C2}$$

Lemma C3

$$\int_{-\infty}^{+\infty} e^{r}\phi\left(\frac{r-\mu}{\sigma}\right)dr\int_{-\infty}^{r+c}\phi\left(\frac{x-\nu}{\tau}\right)dx = \exp[\mu+\tfrac{1}{2}\sigma^2]\,\Phi\left(\frac{c+\mu-\nu+\sigma^2}{(\sigma^2+\tau^2)^{1/2}}\right). \tag{C3}$$

Lemma C4

$$\int_{-\infty}^{+\infty} e^{x}\phi\left(\frac{x-\mu}{\sigma}\right)\left[1-\Phi\left(\frac{x+\nu}{\tau}\right)\right]dx = \exp[\mu+\tfrac{1}{2}\sigma^2]\cdot\left[1-\Phi\left(\frac{\mu+\nu+\sigma^2}{[\sigma^2+\tau^2]^{1/2}}\right)\right]. \tag{C4}$$

Lemma C5

$$\int_{-\infty}^{+\infty} e^{2r}\Phi\left(\frac{r-\mu}{\sigma}\right)dr\int_{-\infty}^{r+c}e^{-x}\phi\left(\frac{x-\nu}{\tau}\right)dx = \exp[2\mu-\nu+2\sigma^2+\tfrac{1}{2}\tau^2]\,\Phi\left(\frac{c+(\mu-\nu)+(2\sigma^2+\tau^2)}{[\sigma^2+\tau^2]^{1/2}}\right). \tag{C5}$$

Lemma C6

$$\int_{-\infty}^{+\infty} e^{r}\Phi\left(\frac{r-\mu}{\sigma}\right)dr\int_{-\infty}^{r+c}e^{-x}\phi\left(\frac{x-\nu}{\tau}\right)dx = \exp[\mu-\nu+\tfrac{1}{2}\sigma^2+\tfrac{1}{2}\tau^2]\,\Phi\left(\frac{c+(\mu-\nu)+\sigma^2+\tau^2}{[\sigma^2+\tau^2]^{1/2}}\right). \tag{C6}$$

Appendix D
Further Development of Pricing Equation

Consider the pricing equation (24). By definition of X_f,

$$\sum_{hk} n_{hfk} E[Q_f X_k] = E[Q_f X_f]; \tag{D1}$$

equivalently,

$$\sum_{hk} n_{hfk} \bar{X}_k E[Q_f X_k/\bar{X}_k] = E[Q_f X_f]. \tag{D2}$$

Assume that, for each k separately, all X_{hk} are identically distributed, that is, the distribution of losses *per unit asset of a particular* type does not vary from one household to another. This assumption will be required later. For the moment, it is possible to use the weaker assumption that the $X_{hk}/\bar{X}_{hk}$ (k fixed) are identically distributed. By equation (13),

$$X_k/\bar{X}_k = \sum_h \lambda_h^{-1} \bar{X}_{hk}(X_{hk}/\bar{X}_{hk})/\sum_h \lambda_h^{-1} \bar{X}_{hk},$$

a weighted average of the $X_{hk}/\bar{X}_{hk}$. Thus, the $X_k/\bar{X}_k$ are all identically distributed (with the same distribution as the $X_{hk}/\bar{X}_{hk}$, k fixed), and the expectation on the left side of equation (D2) becomes independent of k, and so is given by

$$E[Q_f X_k/\bar{X}_k] = E[Q_f X_f]/\sum_{hl} n_{hfl} \bar{X}_l.$$

Thus,

$$E[Q_f X_k] = E[Q_f X_f]\bar{X}_k/\sum_{hl} n_{hfl} \bar{X}_l. \tag{D3}$$

To evaluate $E[Q_f X_f]$, note from equations (15), (17), and (18) that

$$\begin{aligned} Q_f X_f &= X_f, \text{ if } (P_f + s_f K_f)R_f \geq X_f, \\ &= (P_f + s_f K_f) R_f, \text{ if } (P_f + s_f K_f) R_f < X_f. \end{aligned}$$

Therefore,

$$\begin{aligned} E[Q_f X_f] &= \int_{-\infty}^{+\infty} e^x \phi\left(\frac{x-\mu_{Xf}}{\sigma_{Xf}}\right)\left[1-\Phi\left(\frac{x-\log(P_f+s_f K_f)-\mu_{Rf}}{\sigma_{Rf}}\right)\right]dx \\ &+ (P_f+s_f K_f) \int_{-\infty}^{+\infty} e^r \Phi\left(\frac{r-\mu_{Rf}}{\sigma_{Rf}}\right)\left[1-\Phi\left(\frac{r+\log(P_f+s_f K_f)-\mu_{Xf}}{\sigma_{Xf}}\right)\right]dr \\ &= \bar{X}_f[1-\Phi(-q_f+\sigma_f)] + (P_f+s_f K_f)\bar{R}_f[1-\Phi(q_f)] \end{aligned}$$

by Lemma C4, with q_f defined as

$$q_f = \left[\mu_{Rf} - \mu_{Xf} + \log(P_f + s_f K_f) + \sigma_{Rf}^2\right] / \sigma_f;$$

that is,

$$E[Q_f X_f] = \bar{X}_f \Phi(q_f - \sigma_f) + (P_f + s_f K_f) \bar{R}_f [1 - \Phi(q_f)], \tag{D4}$$

where

$$q_f = \left[\log(\eta_f + \delta_f) \bar{R}_f / j_o\right] / \sigma_f + \tfrac{1}{2}\sigma_f,$$

with σ_f defined in equation (39).

By definition of η_f and δ_f, equation (D4) becomes

$$E[Q_f X_f] = \bar{X}_f \left\{ \Phi(q_f - \sigma_f) + \left[(\eta_f + \delta_f) \bar{R}_f / j_o\right]\left[1 - \Phi(q_f)\right] \right\}. \tag{D5}$$

Now note that

$$\bar{X}_f / \sum_{hl} n_{hfl} \bar{X}_l = \sum_{hl} n_{hfl} \bar{X}_{hl} / \sum_{hl} n_{hfl} \bar{X}_l = 1, \tag{D6}$$

on the basis of the earlier assumption that all X_{hl} (l fixed) are identically distributed. Then substituting equation (D5) in (D3), and making use of equation (D6), yields

$$E[Q_f X_k] = \bar{X}_k \left\{ \Phi(q_f - \sigma_f) + \left[(\eta_f + \delta_f) \bar{R}_f / j_o\right]\left[1 - \Phi(q_f)\right] \right\}. \tag{D7}$$

Now consider the final term of equation (24), which involves cov $[Q_f, R_M]/V_M$. Note that

$$R_f = j_o + \beta_f [R_M / V_M - j_o] + \text{unsystematic risk}, \tag{D8}$$

where β_f is the capital asset pricing model beta of the asset portfolio held by insurer f. Inversion of equation (D8) gives

$$R_M / V_M = \beta_f^{-1} R_f + \text{nonstochastic and unsystematic terms}, \tag{D9}$$

whence

$$\text{cov } [Q_f, R_M] / V_m = \beta_f^{-1} \text{cov } [Q_f, R_f]. \tag{D10}$$

This result will be invalid if $\beta_f = 0$, but in this case, $\text{cov}[Q_f, R_M] = 0$.

By equations (15) and (18),

$$\begin{aligned} E[Q_f\, R_f] &= E[R_f \mid E_f > 0] \cdot \text{Prob } [E_f > 0] \\ &+ (P_f + s_f K_f)\, E[R_f^2 / X_f \mid E_f < 0] \cdot \text{Prob } [E_f < 0]. \end{aligned} \tag{D11}$$

The first of the two terms on the right is evaluated by putting $c = \log(P_f + s_f K_f)$ in Lemma C3, giving

$$E[R_f | E_f > 0] \cdot \text{Prob}[E_f > 0] = \bar{R}_f \Phi(q_f). \tag{D12}$$

Now consider the second term. By equation (18),

$$
\begin{aligned}
& E[R_f^2/X_f | E_f<0] \cdot \text{Prob}[E_f<0] \\
&= E[R_f^2/X_f | X_f > (P_f+s_fK_f)R_f] \cdot \text{Prob}[X_f>(P_f+s_fK_f)R_f] \\
&= \int_{-\infty}^{+\infty} e^{2r}\phi\left(\frac{r-\mu_{Rf}}{\sigma_{Rf}}\right)dr\left[\int_{-\infty}^{+\infty} - \int_{-\infty}^{r+\log(P_f+s_fK_f)}\right]e^{-x}\phi\left(\frac{x-\mu_{Xf}}{\sigma_{Xf}}\right)dx
\end{aligned}
\tag{D13}
$$

(by equations [33] and [34]).

Substitution of equations (D12) and (D13) into equation (D11) and use of Lemma C5 yields

$$E[Q_fR_f] = \bar{R}_f\left\{\Phi(q_f)+(\eta_f+\delta_f)\bar{R}_fj_o^{-1}\exp(\sigma_f^2)[1-\Phi(q_f+\sigma_f)]\right\}. \tag{D14}$$

Now $E[Q_f]$ is obtained by replacing R_f by unity in the left side of equation (D12) and replacing R^2_f by R_f in equation (D13). The latter case may be evaluated by means of Lemma C6. The former may be evaluated in a manner parallel to that leading to equation (D12), giving

$$
\begin{aligned}
& E\,[1|E_f > 0] \cdot \text{Prob}\,[E_f > 0] \\
&= \text{Prob}\,[X_f/R_f < P_f+s_fK_f] \qquad \text{(by equation [18])} \\
&= \text{Prob}\,[\log(X_f/R_f) < \log \bar{X}_f\,(\eta_f+\delta_f)j_o^{-1}], \qquad \text{(by equation [B11])}
\end{aligned}
$$

where $\log (X_f/R_f) \sim N(\mu_{Xf}-\mu_{Rf}, \sigma_f^2)$, by equations (33), (34), and (39). Thus,

$$
\begin{aligned}
\text{Prob}\,[E_f>0] &= \Phi\left([\log \bar{X}_f(\eta_f+\delta_f)j_o^{-1} - (\mu_{Xf}-\mu_{Rf})]/\sigma_f\right) \\
&= \Phi\left(q_f-\sigma_{Rf}^2/\sigma_f\right).
\end{aligned}
$$

In this case, equation (D14) becomes

$$E[Q_f] = \Phi(q_f-\sigma_{Rf}^2/\sigma_f) + (\eta_f+\delta_f)\bar{R}_fj_o^{-1}\exp(\sigma_{Xf}^2)[1-\Phi(q_f+\sigma_{Xf}^2/\sigma_f)]. \tag{D15}$$

By equations (D10), (D14), and (D15),

$$
\begin{aligned}
& \text{cov}\,[Q_f, R_M]/V_M \\
&= \beta_f^{-1}\bar{R}_f[\Phi(q_f) - \Phi(q_f-\sigma_{Rf}^2/\sigma_f)] \\
&+ \beta_f^{-1}\bar{R}_f^2(\eta_f+\delta_f)j_o^{-1}\exp(\sigma_{Xf}^2)\left\{\exp(\sigma_{Rf}^2)\left[1-\Phi(q_f+\sigma_f)\right] - \left[1-\Phi(q_f+\sigma_{Xf}^2/\sigma_f)\right]\right\} \\
&= \beta_f^{-1}\bar{R}_f[\Phi(q_f)-\Phi(q_f-\sigma_{Rf}^2/\sigma_f)] \\
&+ \beta_f^{-1}\bar{R}_f^2(\eta_f+\delta_f)j_o^{-1}(1+\gamma_f^2)\left\{(1+\omega_f^2)\left[1-\Phi(q_f+\sigma_f)\right] - \left[1-\Phi(q_f+\sigma_{Xf}^2/\sigma_f)\right]\right\},
\end{aligned}
\tag{D16}
$$

by equations (35) and (36). As noted just below equation (D10), this result holds unless $\beta_f = 0$, in which case $\text{cov}\,[Q_f, R_M] = 0$.

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