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The Value of Information in Insurance Pricing: Comment

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Introduction

In estimating the expected losses underlying an insurance premium, the insurer must consider both actuarial issues, such as the accuracy of prior information and historical data, and market effects, such as the impact of adverse selection. To analyze these types of issues, Gogol (1993) proposes a sequential model of the estimation of expected losses.

Gogol's analysis is based largely upon two mathematical propositions, which he uses to provide insights into various aspects of insurance pricing. In this note it is shown that Gogol's Proposition 2 is incorrect. After providing a corrected version of this result, it is further demonstrated that two general statements made by the author are also not true.

The Basic Model and Results

To summarize Gogol's basic model of the estimation process, let
Λ = expected losses (the unknown quantity to be estimated),[1]
m = the prior estimate of Λ,
X = a subsequent estimator of Λ,
$X_1 = \ln(\Lambda/m)$,
$X_2 = \ln(\Lambda/X)$,
X_1, X_2 ~ Bivariate Normal with $E[X_1] = E[X_2] = 0$, $Var[X_1] = \sigma_1^2$, $Var[X_2] = \sigma_2^2$, and $Corr[X_1, X_2] = \rho$.
Using this framework, and the notation $v^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$, the author's two propositions may be stated as follows.

Proposition 1

Given that X = x, the posterior probability distribution of Λ has mean

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[1] The symbol Λ is introduced here for ease of exposition. In Gogol (1993), the random variable representing expected losses is simply called "expected losses."

$$E[\Lambda | X = x] = m \exp(\mu + \sigma^2/2)$$

and variance

$$\mathrm{Var}[\Lambda | X = x] = m^2 \exp(2\mu + \sigma^2) [\exp(\sigma^2) - 1],$$

where $\mu = [(\sigma_1^2 - \rho\sigma_1\sigma_2)\ln(x/m)]/\nu^2$ and $\sigma^2 = (1-\rho^2)\sigma_1^2\sigma_2^2/\nu^2$.

Proposition 2

Given that $X \leq E$, where E is a positive constant, the posterior distribution of Λ has mean

$$E[\Lambda | X \leq E] = m \exp[(\sigma^2+a^2)/2]\ \Phi[(1/\nu)\ln(E/m) - a]/\Phi[(1/\nu)\ln(E/m)],$$

where $a = (\sigma_1 - \rho\sigma_2)^2/\nu$. Also,

$$\Pr\{X \leq E\} = \Phi[(1/\nu)\ln(E/m)].$$

As noted above, Proposition 2 is incorrect. The following is a corrected version of this result.

Proposition 2'

Given that $X \leq E$, where E is a positive constant, the posterior distribution of Λ has mean

$$E[\Lambda | X \leq E] = m \exp(\sigma_1^2/2)\ \Phi[(1/\nu)\ln(E/m) - \alpha]/\Phi[(1/\nu)\ln(E/m)],$$

where $\alpha = (\sigma_1^2 - \rho\sigma_1\sigma_2)/\nu$. Also,

$$\Pr\{X \leq E\} = \Phi[(1/\nu)\ln(E/m)].$$

The source of error in Proposition 2 is found in its proof, offered in the Appendix to Gogol's article. In this proof, he makes the substitution $t = (1/\nu)\ln(x/m)$ in the expression for $E[\Lambda | X = x]$ given by Proposition 1 and finds that

$$\mu = \nu t\ (\sigma_1 - \rho\sigma_2)^2/\nu^2,$$

when in fact

$$\mu = \nu t\ (\sigma_1^2 - \rho\sigma_1\sigma_2)/\nu^2.$$

By correcting this error, and following all of the remaining steps of the author's proof, one easily obtains the result stated in Proposition 2'.[2]

New Insights

Following his presentation of Propositions 1 and 2, Gogol provides examples in which he applies these mathematical results to the analysis of pricing

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[2] It should be noted that the factor $(2\pi)^{-1/2}$ was left out of the numerator of the expression for $E[\Lambda | X \leq E]$ in several steps of this proof.

problems. In the course of these examples, he makes two general statements that may be disproved by counterexamples.

In Example 1, Gogol considers the effect of the accuracy of the estimator X on an insurer's pricing. Assuming that a given insurer will accept a risk with unknown expected losses Λ if and only if $X \leq E$, and that $E = 0.82m$, $\sigma_1 = 0.5$, and $\rho = 0.5$, Gogol calculates that if $\sigma_2 = 0.667$, then $E[\Lambda | X \leq E] = 1.072m$, and if $\sigma_2 = 0.333$, then $E[\Lambda | X \leq E] = 0.802m$. He then states, without proof, that "an increase in accuracy [of the estimator X] decreases both the mean and the variance of the expected losses of the risks chosen."

After revising these calculations in light of Proposition 2', it can be seen that, if $\sigma_2 = 0.667$, then $E[\Lambda | X \leq E] = 0.977m$, and if $\sigma_2 = 0.333$, then $E[\Lambda | X \leq E] = 0.707m$. Hence, for the parameter values selected, the above quotation is true in that an increase in the accuracy of X is associated with a decrease in $E[\Lambda | X \leq E]$.[3] However, for other values of the parameters, the quotation is false. For example, letting $E = 1.25m$, $\sigma_1 = 0.05$, and $\rho = -0.99$, it follows that if $\sigma_2 = 0.667$, then $E[\Lambda | X \leq E] = 0.971m$, but if $\sigma_2 = 0.333$, then $E[\Lambda | X \leq E] = 0.978m$. Thus, as the accuracy of X increases from $\sigma_2 = 0.667$ to $\sigma_2 = 0.333$, the corresponding value of $E[\Lambda | X \leq E]$ also increases.

Noting that

$$\begin{aligned} \partial E[\Lambda | X \leq E]/\partial\sigma_2 &= \{m \exp(\sigma_1^2/2)/\{\Phi[(1/v)\ln(E/m)]\}^2\} \\ &\times \{\Phi[(1/v)\ln(E/m) - \alpha]\Phi'[(1/v)\ln(E/m)]\ [\ln(E/m)\ \partial v/\partial\sigma_2]/v^2 \\ &- \Phi[(1/v)\ln(E/m)]\Phi'[(1/v)\ln(E/m) - \alpha]\ \{[\ln(E/m)\ \partial v/\partial\sigma_2]/v^2 + \partial\alpha/\partial\sigma_2\}\}, \end{aligned}$$

it is not difficult to show (see the Appendix) that each of the following two sets of conditions is sufficient (but not necessary) for $\partial E[\Lambda | X \leq E]/\partial\sigma_2 > 0$ (i.e., for Gogol's assertion to be true):

(1) $\sigma_1 < \sigma_2$, $\rho > \sigma_1/\sigma_2$, and $\ln(E/m) > 0$, and

(2) $\sigma_2 < \sigma_1$, $\rho > \sigma_2/\sigma_1$, and $\ln(E/m) > \sigma_1^2 - \rho\sigma_1\sigma_2$.

However, for cases in which $\rho \leq 0$, sufficient conditions for $\partial E[\Lambda | X \leq E]/\partial\sigma_2 > 0$ cannot be so neatly expressed; in fact, it can be shown (see the Appendix) that, if $\rho = -1$, then the condition $\ln(E/m) > \sigma_1^2 + \sigma_1\sigma_2$ is sufficient for $\partial E[\Lambda | X \leq E]/\partial\sigma_2 < 0$ (i.e., for Gogol's assertion to be false). Note that the counterexample provided above meets this last condition, with $\rho = -0.99$.

To see that it is quite possible for the correlation coefficient ρ to be negative, consider what would happen if the actuary responsible for constructing the estimator X believed that the prior estimate m had been generated in a manner that tended to produce low estimates of Λ. The actuary might choose to compensate for the initial underestimation of Λ by deliberately using methods that tended to overestimate Λ, in which case $X_2 = \ln(\Lambda/X)$ would be nega-

[3] That is, the quotation is true (for the particular parameter values selected) if the "increase in accuracy" of X is measured by a decrease in $\sigma_2^2 = \mathrm{Var}[\ln(\Lambda/X)]$. Note that the quotation is also true if the increase in accuracy of X is measured by a decrease in either $\mathrm{Var}[\ln(X) | \Lambda = \lambda] = \mathrm{Var}[X_2 | X_1 = x_1] = (1-\rho^2)\sigma_2^2$ or $\mathrm{Var}[\ln(X)] = \mathrm{Var}[X_1 - X_2] = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = v^2$.

tively correlated with $X_1 = \ln(\Lambda/m)$. The greater the actuary's concern and intervention, the closer ρ could be to -1.

In Example 2, Gogol considers the effect on insurance pricing of adverse selection caused by competition. In this example, two insurers, A and B, each offer to sell insurance to a risk with unknown expected losses Λ. Insurer A possesses the prior probability distribution of Λ (i.e., $\ln(\Lambda/m) \sim N(0, \sigma_1^2)$), and insurer B possesses the estimator X. Assuming that B will make a better offer to the risk only if $X \leq E$ (where E is some positive constant), the author analyzes the relationship among

$$\mu_1 = E[\Lambda\,|\,\text{insurer B makes a better offer}] = E[\Lambda\,|\,X \leq E],$$
$$\mu_2 = E[\Lambda] = E[\exp[X_1 + \ln(m)]] = m\exp(\sigma_1^2/2), \text{ and}$$
$$C = E[\Lambda\,|\,\text{insurer B does not make a better offer}] = E[\Lambda\,|\,X > E].$$

After correctly observing that

$$\mu_2 = p\mu_1 + (1-p)C,$$

where $p = \Pr\{X \leq E\} = \Phi[(1/v)\ln(E/m)]$, the author incorrectly states that $\mu_1 < \mu_2$ (and therefore that $C > \mu_2$). To see that μ_1 can be greater than μ_2, consider Proposition 2', which implies that

$$\mu_1 = \mu_2\, \Phi[(1/v)\ln(E/m) - \alpha]/\Phi[(1/v)\ln(E/m)].$$

Since $\Phi[(1/v)\ln(E/m) - \alpha]/\Phi[(1/v)\ln(E/m)] > 1$ if and only if $\alpha < 0$ (or equivalently, if and only if $\rho > \sigma_1/\sigma_2$), it follows that $\mu_1 > \mu_2 \Leftrightarrow \rho > \sigma_1/\sigma_2$. Using the parameter values $E = 0.82m$, $\sigma_1 = 0.5$, and $\rho = 0.5$ from Example 1, it can be seen that, if $\sigma_2 = 1.5$, then $\mu_1 = 1.230m$, $\mu_2 = 1.133m$, and $C = 1.057m$; that is, there are parameter values for which the event that "insurer B makes a better offer" actually raises the expected value of Λ.

For the condition $\rho > \sigma_1/\sigma_2$ to hold, both $\sigma_1 < \sigma_2$ and $\rho > 0$ must be true. The inequality $\sigma_1 < \sigma_2$ is reasonable because the prior estimate m may have been based upon many years of historical experience, whereas the subsequent estimator X may derive from only very limited recent experience. The inequality $\rho > 0$ is just as plausible as $\rho < 0$, which was discussed above. If the actuary responsible for constructing the estimator X were also responsible for generating the prior estimate m, and if the actuary used methods that tended to over- or underestimate Λ in both cases, then $X_2 = \ln(\Lambda/X)$ would be positively correlated with $X_1 = \ln(\Lambda/m)$.

Conclusions

After correcting Gogol's Proposition 2, one can readily see that two statements made by the author, (1) that $E[\Lambda\,|\,X \leq E]$ decreases as the accuracy of X increases, and (2) that $E[\Lambda\,|\,X \leq E] < E[\Lambda]$, are not generally true. This observation is important primarily because these two claims are, *a priori,* intuitively plausible. It is hoped that the counterexamples and discussion provided in this note will assist practitioners in evaluating the applicability of the underlying model of the estimation process.

Appendix

Sufficient Conditions for $\partial E[\Lambda|X \leq E]/\partial\sigma_2 > 0$

From the expression for $\partial E[\Lambda|X \leq E]/\partial\sigma_2$ provided in the text, it is clear that $\partial E[\Lambda|X \leq E]/\partial\sigma_2 > 0$ if and only if

$$\Phi[(1/\nu)\ln(E/m) - \alpha]\Phi'[(1/\nu)\ln(E/m)]\ [\ln(E/m)\ \partial\nu/\partial\sigma_2]/\nu^2$$
$$> \Phi[(1/\nu)\ln(E/m)]\Phi'[(1/\nu)\ln(E/m) - \alpha]\ \{[\ln(E/m)\ \partial\nu/\partial\sigma_2]/\nu^2 + \partial\alpha/\partial\sigma_2\}, \quad (A1)$$

where $\partial\nu/\partial\sigma_2 = (\sigma_2 - \rho\sigma_1)/\nu$ and
$\partial\alpha/\partial\sigma_2 = -\ (1-\rho^2)(\sigma_1^2\sigma_2/\nu^3) \leq 0.$

Now consider the first set of conditions: $\sigma_1 < \sigma_2$, $\rho > \sigma_1/\sigma_2$, and $\ln(E/m) > 0$. Given $\sigma_1 < \sigma_2$, it follows that $\partial\nu/\partial\sigma_2 = (\sigma_2 - \rho\sigma_1)/\nu > 0$. Furthermore, $\rho > \sigma_1/\sigma_2$ implies that $\alpha = (\sigma_1^2 - \rho\sigma_1\sigma_2)/\nu < 0$, and, therefore,

$$(1/\nu)\ln(E/m) - \alpha > (1/\nu)\ln(E/m) > 0. \quad (A2)$$

Inequality (A2) then implies that

$$\Phi[(1/\nu)\ln(E/m) - \alpha]\Phi'[(1/\nu)\ln(E/m)] >$$
$$\Phi[(1/\nu)\ln(E/m)]\Phi'[(1/\nu)\ln(E/m) - \alpha] > 0,$$

and (A1) follows from $[\ln(E/m)\ \partial\nu/\partial\sigma_2]/\nu^2 > 0$ and $\partial\alpha/\partial\sigma_2 \leq 0$.

Now consider the second set of conditions: $\sigma_2 < \sigma_1$, $\rho > \sigma_2/\sigma_1$, and $\ln(E/m) > \sigma_1^2 - \rho\sigma_1\sigma_2$. Given $\rho > \sigma_2/\sigma_1$, it follows that $\partial\nu/\partial\sigma_2 = (\sigma_2 - \rho\sigma_1)/\nu < 0$. Furthermore, $\sigma_2 < \sigma_1$ implies that $\alpha = (\sigma_1^2 - \rho\sigma_1\sigma_2)/\nu > 0$, and, therefore,

$$(1/\nu)\ln(E/m) - \alpha < (1/\nu)\ln(E/m). \quad (A3)$$

The condition $\ln(E/m) > \sigma_1^2 - \rho\sigma_1\sigma_2$ implies that the left-hand side of inequality (A3) is positive, so that

$$0 < \Phi[(1/\nu)\ln(E/m) - \alpha]\Phi'[(1/\nu)\ln(E/m)] <$$
$$\Phi[(1/\nu)\ln(E/m)]\Phi'[(1/\nu)\ln(E/m) - \alpha].$$

The inequality (A1) then follows from $[\ln(E/m)\ \partial\nu/\partial\sigma_2]/\nu^2 < 0$ and $\partial\alpha/\partial\sigma_2 \leq 0$.

Sufficient Conditions for $\partial E[\Lambda|X \leq E]/\partial\sigma_2 < 0$

First, note that, if $\rho = -1$, then

$$\partial\nu/\partial\sigma_2 = (\sigma_2 - \rho\sigma_1)/\nu = (\sigma_2 + \sigma_1)/\nu > 0, \text{ and}$$
$$\partial\alpha/\partial\sigma_2 = -\ (1-\rho^2)(\sigma_1^2\sigma_2/\nu^3) = 0.$$

Furthermore, $\alpha = (\sigma_1^2 - \rho\sigma_1\sigma_2)/\nu = (\sigma_1^2 + \sigma_1\sigma_2)/\nu > 0$, which implies that

$$(1/\nu)\ln(E/m) - \alpha < (1/\nu)\ln(E/m). \quad (A4)$$

The condition $\ln(E/m) > \sigma_1^2 + \sigma_1\sigma_2$ implies that the left-hand side of (A4) is positive, so that

$$0 < \Phi[(1/v)\ln(E/m) - \alpha]\Phi'[(1/v)\ln(E/m)] < \Phi[(1/v)\ln(E/m)]\Phi'[(1/v)\ln(E/m) - \alpha].$$

The inequality

$$\Phi[(1/v)\ln(E/m) - \alpha]\Phi'[(1/v)\ln(E/m)]\,[\ln(E/m)\,\partial v/\partial\sigma_2]/v^2 < \Phi[(1/v)\ln(E/m)]\Phi'[(1/v)\ln(E/m) - \alpha]\,\{[\ln(E/m)\,\partial v/\partial\sigma_2]/v^2 + \partial\alpha/\partial\sigma_2\},$$

and its consequence, $\partial E[\Lambda | X \leq E]/\partial\sigma_2 < 0$, then follow from $[\ln(E/m)\,\partial v/\partial\sigma_2]/v^2 > 0$ and $\partial\alpha/\partial\sigma_2 = 0$.

Reference

Gogol, Daniel, 1993, The Value of Information in Insurance Pricing, *Journal of Risk and Insurance,* 60: 119-128.