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Insurance and Precautionary Capital Accumulation in a Continuous-Time Model

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ABSTRACT

This article analyzes the optimal dynamic strategy of a risk-averse agent bearing an insurable risk to determine whether precautionary saving is superior to insurance in the long run. The risk of loss is described by a Poisson process. Policyholders can purchase insurance and/or accumulate precautionary savings in order to forearm themselves in the face of uncertainty. It is shown that the demand for insurance vanishes in the long run if the loading factor exceeds a critical value which is strictly positive. However, insurance may be a transitory strategy to protect capital accumulation. The optimal strategy for capital accumulation and insurance demand is derived in the constant relative risk aversion case. It is also shown that compulsory full insurance reduces the rate of consumption if and only if risk aversion exceeds one.

Introduction

The theory of optimal response to risk is one of the most lively areas of research in economics. Two types of response to risk have been studied intensively. The first type of response is the transfer of risk through insurance. A premium is paid today to receive an additional revenue in case of a future accident. This behavior is related to "risk aversion," which measures how much one dislikes uncertainty and how much one would be willing to pay to escape risk. In the expected utility framework, risk aversion originates from the negativity of the second derivative of the utility function. It explains why risk averse people are willing to buy insurance, even at unfavorable prices.

The second type of response to risk is capital accumulation in the form of "precautionary" deposits. Money is saved today for consumption tomorrow regardless of whether an accident occurred. This is called a prudent behavior. Following Kimball (1990), "prudence is meant to suggest the propensity to prepare and forearm oneself in the face of uncertainty." When future incomes are uncertain, prudent consumers will save more. Leland (1968), Sandmo

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(1970), and Drèze and Modigliani (1972) showed that prudent behaviors are generated by the nonnegativity of the third derivative of the utility function. Prudence ($u''' \geq 0$) and risk aversion ($u'' \leq 0$) have different nature.

Most articles in the insurance literature provide static models, thereby excluding precautionary capital accumulation. Conversely, it is generally assumed in the literature on precautionary saving that the exogenous risk under consideration is not transferable, thereby excluding insurance strategies. The goal of this article is to determine the optimal strategy when both capital accumulation (deposits) and insurance are allowed.

Consider an agent facing at every instant a risk of liability generating some loss in wealth due to an "accident." In the short run, the agent is expected to transfer most of the risk to an insurer because he has no financial reserve. If he is sufficiently lucky and if his rate of consumption is not too large, he will be able to accumulate reserves in the form of deposits that will allow him to retain a larger proportion of the risk in the future. This is desirable for two reasons: First, financial reserves generate a positive return, and, second, reducing insurance coverage is profitable because insurance is costly. Two important problems should be solved with respect to this strategy: First, what is the optimal level of reserves and the demand for insurance in the long run? Second, we have to reconcile two conflicting objectives in the short run—protecting the agent against large losses and raising reserves to reduce the cost of the risk in the future (by reducing insurance coverage). These objectives conflict because the first is attained through spending enough money for insurance and the second saves on insurance costs today. As already observed by Dionne and Eeckhoudt (1984), for nonactuarial premia, it is not true that insurance is always more efficient than capital accumulation for the purpose of protection.

Using stochastic continuous-time control techniques, a necessary and sufficient condition is derived that guarantees a positive demand for insurance in the long run. The effect of transitory and permanent changes in the price of insurance is also examined either in the short run or in the long run. Contrary to the well-known result obtained in the standard static model, insurance may not be a Giffen good in the sense that a transitory increase in the loading factor of insurance always reduces the demand for insurance in the short run, under risk aversion alone.[1] The same result holds for the effect of a permanent change on the long-term demand for insurance. Ambiguity is found for the *transitory* effect of a *permanent* increase on the price of insurance, with the wealth and substitution effects being contradictory. A side result of this analysis sheds light on the effect of imposing compulsory insurance on the optimal rate of consumption by policyholders.

This article is connected to the literature on capital accumulation and growth initiated by Ramsey (1928). Bourguignon (1974) and Merton (1975) have extended the theory to incorporate uncertainty. Optimal consumption and

[1] For more details on insurance as a Giffen good, see Briys, Dionne, and Eeckhoudt (1989).

saving under uncertainty have been analyzed by Samuelson (1969) and Merton (1969, 1971) among others. Except for Briys (1986, 1988, 1990), these researchers assume that risks cannot be insured, that is, that markets are incomplete. Risky asset values are assumed to follow a Wiener process, which is not a relevant specification for the applications intended here.

In this article, I follow Briys (1986, 1988, 1990), who considers a Poisson process for the insurable risky asset value. This article differs from Briys's contribution in that I assume that the risk borne by the economic agent is exogenous and independent of the level of reserve. Briys considers the opposite problem in which it is the reserve itself that is subject to full loss. Briys also considers a portfolio allocation problem for the reserve. The two approaches are complementary and are a first step toward a unified theory of precautionary saving and insurance.

The Stochastic Dynamic Program Formulation

Consider a continuous-time model in which an economic agent chooses consumption and insurance rules in order to maximize the expected discounted utility of his consumption plan C(t) over his lifetime horizon $[t_0,T]$. Assuming an intertemporally separable utility function with discount factor (rate of impatience) $\delta \geq 0$, the problem is written as

$$I(W_0,t_0) = \max E\int_{t_0}^{T} u(C(t))e^{-\delta(t-t_0)}dt + e^{-\delta(T-t_0)}B(W(T)), \tag{1}$$

subject to $W(t_0) = W_0$—the initial wealth at t_0—and a budget constraint which will be defined below. We assume that the instantaneous utility function u(C) and the bequest function B(W(T)) are increasing and concave. The concavity of u(C) characterizes the limited degree of intertemporal substitutability and the individual's attitude toward risk.

The budget constraint may be described as follows: financial reserves take the form of deposits. Return r(t) on deposits is risk-free. The instantaneous flow of noncapital income is denoted w(t). These two exogenous variables may vary through time. Finally, the agent bears an exogenous risk. It is assumed that this risk may be specified as a Poisson process q(t) with parameter $\pi > 0$. The random variable q(t) expresses the number of accidents incurred in the interval $[t_0,t]$. Formally,

$$P[q(t)-q(s)=k] = \frac{[\pi(t-s)]^k}{k!}e^{-\pi(t-s)}$$

for any $t_0 \leq s \leq t \leq T$. It implies that the event "incurring one or more losses" in a time period of duration dt is $\pi dt + o(dt)$. The severity of the loss in case of an accident is $\tilde{x}(t)$, which is a random variable whose cumulative distribution function F_t is allowed to depend upon time. In the absence of insurance, the budget constraint would be defined by the following stochastic differential equation:

$$dW(t) = [w(t)+r(t)W(t)-C(t)]dt - \tilde{x}(t)dq(t).$$

In addition to the accumulation strategy to protect against the adverse consequences of loss, I introduce the possibility of buying partial insurance. More specifically, I assume that the agent has the opportunity to purchase proportional coinsurance contracts. The insurance pricing is linear in that a contract with a retention rate (or percentage-of-loss deductible) D at time t can be purchased at price (1-D)P(t). P(t) is the full insurance premium at time t. Such a linear pricing rule is sustained by the classical assumption that the insurance premium is proportional to the actuarial value of the policy. The selection of the percentage-of-loss deductible D(t) at any time is left to the discretion of the individual. The case D = 1 corresponds to self-insurance; the case D = 0 corresponds to full insurance. Because insurance is costly, self-insurance is the less expensive strategy. The actual budget constraint is, then,

$$dW(t)=[w(t)+r(t)W(t)-C(t)-(1-D(t))P(t)]dt-D(t)\tilde{x}(t)dq(t). \tag{2}$$

The problem is to select consumption and insurance rules—C(W,t) and D(W,t)—which would maximize $I(W_0,t_0)$ subject to equation (2). Function I(W,t) is the value function—formally, the discounted value of expected utility along the optimal path. The optimal path of wealth is denoted W(t,q(t)). Assuming sufficient differentiability for the value function, we may use Bellman's Principle of Optimality to write

$$I(W,t) = \max_{C,D} E[u(C)dt+e^{-\delta dt}I(W(t+dt),q(t+dt))],$$

or, after taking the limit for dt → 0 and using the Poisson differential operator, assuming that I is twice continuously differentiable,[2]

$$0 = \max_{C,D} u(C)-\delta I+I_t+I_W\frac{\partial W}{\partial t}-\pi[I(W,t)-EI(W-D\tilde{x}(t),t)], \tag{3}$$

with

$$\frac{\partial W}{\partial t} = w+rW-C-(1-D)P.$$

The first four terms on the right-hand side of equation (3) correspond to the increment in I conditional to q(t+dt) = q(t) (no loss), and the fifth term is the expected decrease in I due to the likely occurrence of a Poisson event with probability π. This equation is called the Hamilton-Bellman-Jacobi equation in stochastic dynamic calculus. Associated to it is the terminal condition

$$I(W,T) = B(W). \tag{4}$$

Conditions (3) and (4) determine the optimal strategy of the policyholder. Suppose for the moment that value function I(W,t) is known. Then, observe that the stochastic aspect of the problem has dropped out. The problem of the

[2] For a discussion of Poisson differential equations and control, see Kushner (1967) and Malliaris and Brock (1985). Merton (1975) applies this technique in portfolio selection.

individual at any time is reduced to the two-dimensional problem of selecting a level of current consumption and a deductible rate, and by difference, a level of saving. The first-order conditions for C and D are written as follows:

$$u'(C(W,t)) = I_W(W,t), \tag{5}$$

and

$$P(t)I_W(W,t) = \pi E\tilde{x}(t)I_W(W - D(W,t)\tilde{x}(t),t). \tag{6}$$

Condition (5) states that the marginal utility of current consumption along the optimal path must equal the marginal value of capital stock, that is, the marginal expected utility of future consumption. Otherwise, it is easy to show that a small reallocation along the optimal wealth would increase I(W,t). This condition is well known in the literature of lifetime consumption and capital accumulation. Condition (6) may be interpreted as balancing the effects of increasing the percentage-of-loss deductible. The right-hand side of condition (6) is the expected instantaneous reduction in the value function in the next dt period of time, since the probability of an accident is $\pi dt + o(dt)$. The left-hand side is the increase in the value function due to the reduction in the premium flow by P(t). Along the optimal path, this small reallocation should not change the value function, so condition (6) should hold. These first-order conditions are similar in spirit to those obtained by Briys (1986, 1988, 1990).

Program (3) is concave in C and D if I is concave in wealth, a condition that will be verified below. Given condition (5), the concavity of I implies that C is an increasing function of wealth. Due to nonlinearities, system (5) and (6) has in general no analytical solution. But, hopefully, there are two exceptions to this observation. First, when the instantaneous utility function exhibits constant relative risk aversion, the system may be solved analytically. The next section fully describes the optimal rules in that case. The other exception is when insurance is provided at a fair price, that is, $P(t) = \pi E\tilde{x}(t)$. Then, all risk-averse individuals will purchase full insurance, since condition (6) is obviously satisfied with $D(W,t) = 0, \forall W,t$. With full insurance, we are back to the certainty case, and the problem simplifies to the well-known optimal lifetime consumption problem under certainty. In sum, when insurance prices are fair, consumption decisions and insurance decisions are separate, a simple result already observed by Dionne and Eeckhoudt (1984) and Briys (1986, 1988, 1990).

A weaker separation property immediately implied by first-order conditions (5) and (6) is that the decision to buy insurance at any time is separate from the consumption decision. This is due to the fact that C does not appear in condition (6). It means that, given his current wealth and the time remaining, the agent selects the instantaneous level of insurance coverage independently of his decision on current consumption.

A well-known feature of the static demand for insurance is that insurance can be a Giffen good if relative risk aversion is larger than unity. Because insurance is an inferior good under decreasing absolute risk aversion, an increase in the cost of insurance generates a wealth effect that tends to increase the demand for insurance. If the wealth effect exceeds the substitution effect, insurance is a Giffen good. It is noteworthy that insurance cannot be a Giffen good in a continuous-time model. This is stated in the following Proposition.

Proposition 1

An increase in the instantaneous price of insurance reduces the instantaneous demand for insurance.

Proof. Suppose that P(t) is increased by dP in interval [t-dt,t]. This change has no effect on the characteristics of the value function I(.,t). Fully differentiating equality (6) implies that

$$\frac{dD(W,t)}{dP(t)} = -\frac{I_W(W,t)}{\pi E(\tilde{x}(t))^2 I_{WW}(W-D(W,t)\tilde{x}(t),t)},$$

which is obviously positive, since I is concave in wealth.■

The intuition is that there is only a substitution effect at play in the continuous-time model. An increase in the full insurance premium in an infinitesimal period of time does not modify wealth at a given insurance coverage, thereby making insurance less desirable. Notice that the outcome is ambiguous when we consider a permanent change in the cost of insurance, since function I is affected by the permanent change. This problem will be examined later in this article.

Closed-Form Solution with Power Utility Functions

For the remainder of this article, it is assumed that both the instantaneous utility function and the bequest function exhibit constant relative risk aversion:

$$u(C) = \frac{C^\gamma}{\gamma} \qquad \gamma \le 1,$$

and

$$B(W) = b\frac{W^\gamma}{\gamma} \qquad b \ge 0,$$

where $1-\gamma$ is the Arrow-Pratt index of relative risk aversion, which is assumed to be nonnegative. These are not innocuous assumptions. Isoelastic utility functions exhibit positive prudence, an assumption that plays a central role in determining precautionary saving. Isoelastic utility functions also exhibit decreasing absolute risk aversion. Decreasing absolute risk aversion is expected to generate a demand for insurance which is decreasing in wealth, as it does in static models.

Proposition 2

Define the agent's "adjusted wealth" at t to be Y(t) = W(t) + Ŵ(t), where Ŵ(t) is the discounted value at time t of future noncapital incomes net of future full insurance premia:

$$\hat{W}(t) = \int_t^T (w(s)-P(s))e^{-\int_t^s r(\tau)d\tau} ds.$$

If functions u and B exhibit constant relative risk aversion $1 - \gamma \geq 0$, so does value function I:

$$I(W,t) = j(t)\frac{(W+\hat{W}(t))^\gamma}{\gamma}. \tag{7}$$

It sustains an optimal strategy in which both consumption and the percentage-of-loss deductible are proportional to the adjusted wealth:

$$C(W,t) = c(t)(W+\hat{W}(t)). \tag{8}$$

$$D(W,t) = d(t)(W+\hat{W}(t)). \tag{9}$$

The per-dollar-of-wealth consumption c(t) and the per-dollar-of-wealth retention rate d(t) at time t satisfy the following conditions (together with $c(T)^{\gamma-1} = b$):

$$(1-\gamma)\frac{c'(t)}{c(t)} = (1-\gamma)c(t)-\delta+\gamma(r(t)+d(t)P(t))-\pi(1-E[1-d(T)\tilde{x}(t)]^\gamma). \tag{10}$$

$$P(t) = \pi E[\tilde{x}(t)(1-d(t)\tilde{x}(t))^{\gamma-1}]. \tag{11}$$

This proposition is proven in the Appendix.

In the isoelastic case, the instantaneous consumption and retention rate are proportional to the adjusted wealth Y(t), which is the sum of current wealth W(t) and $\hat{W}(t)$, where $\hat{W}(t)$ is the discounted value of the flow of noncapital incomes net of full insurance premia. It implies that full insurance is optimal when current wealth goes down to $-\hat{W}(t)$. When $W = -\hat{W}(t)$ at time t, the individual is in a situation of having just enough instantaneous capital/wage earnings to pay for full insurance continuously until T. Since his or her marginal utility of wealth tends to infinity when W tends to $-\hat{W}$, the priority is to escape any risk to reduce further his or her level of wealth, therefore fully insuring the exogenous risk and reducing consumption to zero. This is a dead-end strategy in which the consumer has no other alternative than to insure the risk at full. One can indeed verify that, if adjusted wealth vanishes at time t, adjusted wealth will optimally be locked at zero until T, since we have

$$\frac{d(W+\hat{W}(t))}{dt}\Big|_{W=-\hat{W}(t)} = w(t)-r\hat{W}(t)-P(t)+\hat{W}'(t) = 0,$$

since $C(W,t) = D(W,t) = 0$ at $W = -\hat{W}(t)$. Larger consumption today implies with probability one that consumption will become negative at some future date if this excess consumption is financed through reducing capital reserves or through reducing expenses on insurance. Hereafter, it is assumed that W_0 is larger than $-\hat{W}(t_0)$. Otherwise, it is clear from the above discussion that the value function would be trivial ("death" with probability one), with an infinite number of solutions.

Observe that the retention rate per dollar of adjusted wealth, d(t), is a function of the instantaneous price of insurance and the instantaneous distribution of severity in case of an accident. Changes in the future price of insurance or in the future distribution of severity do not affect the instantaneous demand for insurance per dollar of adjusted wealth. Insurance demand is also independent of the time remaining to T. In a sense, myopia is optimal. Notice also that d is unaffected by changes in the discount factor or in the interest rate.

In fact, condition (11) is similar to the first-order condition obtained in the standard static model of proportional insurance (Mossin, 1968). Considering the usual linear pricing formula $P(t) = (1+\theta(t))\pi E\tilde{x}(t)$, where $\theta(t)$ is the loading factor at time t and $\pi E\tilde{x}(t)$ is the actuarial value of the full insurance policy, condition (11) simplifies to

$$(1+\theta(t))E\tilde{x}(t) = E[\tilde{x}(t)(1-d(t)\tilde{x}(t))^{\gamma-1}].$$

The following corollary summarizes all results on the instantaneous demand for insurance.

Corollary 1

If the utility function and the bequest function exhibit the same constant relative risk aversion, then the rate of retention per dollar of adjusted wealth satisfies the following properties:

1. It depends only upon the current risk and the current price of insurance.
2. It vanishes when insurance is provided at a fair price ($\theta = 0$), and it is positive when insurance is provided at an unfair price.
3. An increase in the loading factor increases the per-dollar rate of retention.
4. A more risk-averse agent will select a smaller retention rate per dollar of adjusted wealth.

The proof of this corollary is similar to that for results in the standard static model and is therefore left to the reader.

The last property is a simple extension of a well-known property of the static demand for insurance. In a word, the decision on the retention rate per dollar of adjusted wealth is mainly static. This is not the case for the consumption decision, which is the solution of the first-degree ordinary differential equation (10) with terminal condition $c(T) = b^{\frac{1}{\gamma-1}}$. The next section analyzes the optimal path of wealth, consumption, and insurance coverage when the terminal time is "far away," that is, in the infinite horizon case.

Long-Term Convergence with a Nonrandom Severity

Let us consider the case where $T \to \infty$, with constant parameter values for θ, r, and w, and with a degenerate severity distribution $\tilde{x}(t) = 1$ with probability one.

The constancy of parameters allows long-term convergence analysis, whereas the assumption on the distribution of severity allows for expositional simplicity. The severity is normalized to unity, so d can be interpreted either as the retention rate or as the straight deductible of the insurance contract. The full insurance premium flow P(t) is $(1+\theta)\pi$.

Optimal Strategy

These assumptions imply that functions I, C, and D depend only upon W. They depend upon time only through time variations of wealth. It means that $\hat{W}(t)$, j(t), c(t), and d(t) are time independent. In particular, $\hat{W}(t)$ equals $\hat{W} = \frac{w-(1+\theta)\pi}{r}$. After some manipulations, we obtain

$$d = 1-(1+\theta)^{\frac{1}{\gamma-1}}, \tag{12}$$

and

$$c = \frac{\delta-\gamma r-\pi\theta}{1-\gamma}+d(1+\theta)\pi. \tag{13}$$

For the sake of illustration, consider the logarithmic case ($\gamma = 0$). Then d equals $\frac{\theta}{1+\theta}$, and c equals δ. This is the same consumption strategy as in the standard Ramsey problem with no uncertainty. Therefore, one can state that the consumption rule is not affected by uncertainty under logarithmic utility functions.

It is noteworthy to examine the optimal behavior in case of an accident, with net loss D. Two strategies are available in such a circumstance: First, the agent could reduce consumption by $\Delta C = D$ during one period of time to compensate for the loss. The loss is fully amortized into a loss of short-term consumption, with no effect in the long run. Alternatively, the agent could amortize the shock into a small, long-run reduction in consumption. The second strategy is optimal. Indeed, in the logarithmic case, for example, we obtain that

$$C(W)-C(W-D(W)) = cD(W) = \delta D(W).$$

If an accident occurs, consumption is reduced by an amount that is a small fraction δ of the net loss. Consumers reduce their financial reserve at the occurrence of a loss rather than drastically reduce current consumption to compensate for the loss. The existence of financial reserves implies that the incidence of a loss is much less dramatic than in static models in which consumers are obliged to adapt current consumption to current losses. This allows for increasing the deductible. Because I is a linear transformation of u, the optimal retention rate per dollar of adjusted *wealth* in the dynamic model is the same as the optimal retention rate per dollar of *consumption* in the static model.

The Path of Wealth

We now solve the stochastic differential equation (2) along the optimal path. After substituting for $Y(t) = W(t)+\hat{W}_0$, it yields

$$\frac{dY}{Y} = \frac{rW+w-P-C+DP}{Y}dt-\frac{D}{Y}dq$$

$$= [r-c+d(1+\theta)\pi]dt-d\ dq$$

$$= \left[\frac{r-\delta+\pi\theta}{1-\gamma}\right]dt-d\ dq(t).$$

Adjusted wealth changes smoothly at a constant rate across time as long as no accident occurs. If an accident occurs ($dq = 1$), adjusted wealth is reduced by deductible D which is proportional to wealth. Adjusted wealth instantaneously drops by a given percentage $d = \frac{D}{Y}$, yielding

$$Y(t,q+1) = (1-d)Y(t,q).$$

It follows that $Y(t,q) = (1-d)^q\ Y(t,0)$. Since $Y(t,t_0) = Y_0 \exp\frac{r-\delta+\pi\theta}{1-\gamma}$, we obtain the following expression (normalizing $t_0 = 0$):

$$Y(t,q(t)) = Y_0(1-d)^{q(t)}\exp\left[\frac{r-\delta+\pi\theta}{1-\gamma}\right]t, \tag{14}$$

where $Y_0 = W_0 + \hat{W}_0$. Or, equivalently,

$$W(t,q(t)) = (W_0+\hat{W}_0)(1-d)^{q(t)}\exp\left[\frac{r-\delta+\pi\theta}{1-\gamma}t\right]-\hat{W}_0. \tag{15}$$

This equation determines wealth at time t as a function of the number of accidents already incurred. Observe that there is no feasible solution when W_0 is less than $-\hat{W}_0$. Indeed, that case would lead to negative Y all along the path (14), generating negative consumption and deductible. This is unfeasible because the isoelastic utility function is defined on R^+. Condition (14) also implies that if W_0 is larger than $-\hat{W}_0$, then capital stock W will remain larger than this lower bound all along the optimal path. When capital stock tends toward this lower bound, the consumer becomes infinitely risk averse and reduces the deductible in order to avoid zero consumption.

In the absence of loss, the growth rate of capital stock is $(r-\delta+\pi\theta)/(1-\gamma)$, which is larger than the rate of accumulation $(r-\delta)/(1-\gamma)$ obtained in models without uncertainty (which corresponds to the case $\theta = 0$, as observed above). The rate of growth of the so-called precautionary saving is $\pi\theta/(1-\gamma) > 0$, which is increasing in the unit cost of insurance and decreasing in risk aversion. When insurance becomes more expensive, consumers increase the rate of capital accumulation in order to forearm themselves in the face of the increased retained risk.

When δ is less than $r+\pi\theta$, deposits are accumulated in the absence of an accident: the agent consumes less than the instantaneous income net of the insurance premium paid. Otherwise, if δ is larger than $r+\pi\theta$, the agent consumes more than the income net of the insurance premium. In this latter case, wealth decreases through time even in the absence of an accident. This is optimal because of the large rate of impatience. Notice that $\pi\theta$ is the net expected gain of increasing the deductible by one dollar. Indeed, it reduces the premium by $(1+\theta)\pi$, whereas it increases the expected loss net of the indemnity paid by π. In other words, $\pi\theta$ is the expected return of investing in retention. This allows us to state that *capital is accumulated in the absence of loss if and only if the rate of impatience is less than the sum of the risk-free rate and the expected return on risk retention.*

However, it is possible that adjusted wealth along the optimal path decreases in expectation even when δ is less than $r+\pi\theta$ because the agent periodically incurs losses. If the capital accumulation in the absence of loss is not large enough to compensate for expected losses, the growth rate of expected adjusted wealth will be negative. In order to compute this growth rate, notice that if $q(t)$ is a Poisson process with parameter π, then for any scalar k

$$E[k^{q(t)}] = \exp[-\pi t(1-k)].$$

Applied to equation (15) with $k = (1-d)$, it yields

$$E[W(t)] = (W_0+\hat{W}_0)\exp\left[\frac{r-\delta+\pi\theta}{1-\gamma}-\pi d\right]t - \hat{W}_0. \tag{16}$$

Obviously, the expected rate of growth of capital stock is the rate of growth in case of no loss minus the expected loss expressed as a fraction of Y. This equation is rewritten as

$$E[W(t)] = (W_0+\hat{W}_0)\exp\left[\frac{\beta-\delta}{1-\gamma}t\right] - \hat{W}_0, \tag{17}$$

with

$$\beta = r + \pi\theta - (1-\gamma)\pi d. \tag{18}$$

We can now address the problem of convergence and stability of the absorbing state $W = -\hat{W}_0$. Since $d \geq 0$, it is clear from equation (15) that the process will converge to the absorbing state with probability one if $\delta > r+\pi\theta$. Otherwise, convergence is ambiguous, as stated above. Equation (17) shows that expected wealth is decreasing through time if and only if the rate of impatience is larger than β, which is strictly less than $r+\pi\theta$.

We now have a clear picture of optimal paths depending upon the value of the relevant parameters δ, r, π, θ, and γ. Three cases are possible.

- $\delta \geq r + \pi\theta$. In this case $\partial W/\partial t < 0$ and $dEW/dt < 0$. Capital stock monotonically converges to $-\hat{W}_0$, the full insurance state.
- $\beta < \delta < r+\pi\theta$. In this case $\partial W/\partial t > 0$, but $dEW/dt < 0$. Despite the fact that consumers accumulate capital during no-loss periods, the likelihood of loss events eventually leads W to $-\hat{W}_0$ with probability one, given the optimal deductible per dollar of adjusted wealth.

- $\delta < \beta$. In this case $\partial W/\partial t > 0$, and $dEW/dt > 0$. The rate of growth of capital stock in the absence of loss is sufficiently large to increase the consumer's expected wealth. This will eventually put the consumer in a position to become his or her own insurer.

The main result of this section is summarized in the following Proposition.

Proposition 3

The optimal path W(t,q(t)) converges to the full insurance state $-\hat{W}_0$ with probability 1 if and only if $\delta > \beta = r+\pi\theta - (1-\gamma)\pi d > r + \pi\theta$.

Proof. From equation (17), we know that W(t,q(t)) converges in expectation toward $-\hat{W}_0$. By equation (15), W(t,q(t)) is bounded below by $-\hat{W}_0$; the optimal path of wealth converges to this lower bound with probability 1.■

Considering long-term behavior, everything relies on the sign of δ-β. If it is positive, the preference for consumption over wealth is so strong that insurance is preferred to capital accumulation. If it is negative, capital accumulation is preferred to insurance in the long run. In the short run, however, insurance is required to preserve capital accumulation. Insurance in this case is a transitory strategy.

Long-Run Effects of Permanent Changes

In the previous section, the effect of an increase in the loading factor on the instantaneous demand for insurance was examined. It was shown that the loading factor unambiguously reduces the demand for insurance, because there is no wealth effect at play. In the first part of this section, the effect of a permanent increase in the loading factor is analyzed. A permanent increase in the cost of insurance reduces $\hat{W}_0 = \frac{w-(1+\theta)\pi}{r}$, the discounted value of future noncapital income net of the flow of insurance premiums. Under constant relative risk aversion, it tends to increase the demand for insurance. As we know, the substitution effect decreases the demand for insurance. Indeed, fully differentiating equation (9) yields

$$\frac{dD}{d\theta} = (W+\hat{W}_0)\frac{dd}{d\theta} + d\frac{d\hat{W}_0}{d\theta}.$$

The first term in the right-hand side (the substitution effect) is positive as stated in Corollary 1, whereas the second term (the wealth effect) is negative. One cannot conclude that a permanent increase in the price of insurance reduces short-run demand.

We now turn to the analysis of the long-run effect of a permanent change in the cost of insurance. As shown above, full insurance is demanded in the long run with probability 1 if and only if $\delta \geq \beta$; otherwise, the insurance demand vanishes as time goes to infinity. Parameter β, as expressed by equation (18), is a function of θ, with

$$\frac{\partial \beta}{\partial \theta} = \pi\left[1-(1+\theta)^{\frac{2-\gamma}{\gamma-1}}\right].$$

This result is positive since $\gamma \leq 1$ and $\theta \geq 0$, so the rate of growth in the expected adjusted wealth $\frac{\beta-\delta}{1-\gamma}$ is an increasing function of the loading factor. Therefore, there exists a bound on θ under which full insurance is bought in the long run, and above which no insurance is bought in the long run, with certainty. The upper bound $\bar{\theta}$ on the loading factor for insurers to be competitive in the long run is implicitly defined by the following equation:

$$\frac{\delta - r}{\pi} = \bar{\theta} - (1-\gamma)\left[1-(1+\bar{\theta})^{\frac{1}{\gamma-1}}\right]$$

It is easily shown that there exists a unique root to this equation. When the interest rate equals the rate of impatience of the policyholder, this bound is zero; it is positive if the rate of impatience exceeds the interest rate. This bound is an increasing function of $\delta - r$ and a decreasing function of π. As a numerical example, consider an individual with a probability of loss equal to the difference of his or her discount factor and the risk-free rate. Assume also a relative risk aversion of $1 - \gamma = 2$, a standard estimation. Then, solving the above equation yields $\bar{\theta} = 1.806$.

Proposition 4

If the utility function of policyholders exhibits constant relative risk aversion and if the severity of the loss in case of an accident is certain, then there exists a critical value $\bar{\theta}$ to the loading factor such that all policyholders purchase full insurance in the long run with probability 1 if the actual loading factor is less than $\bar{\theta}$. This bound is positive if the rate of impatience of the policyholder exceeds the interest rate.

Another interesting comparative statics property is on the effect of imposing permanent compulsory full insurance. In this case, we are back to the standard consumption-saving problem under certainty, with a flow of noncapital income net of insurance payment w - P. The standard result in the infinite horizon model under constant relative risk aversion $1 - \gamma$ is that consumption per dollar of adjusted wealth equals[3]

$$c^0 = \frac{\delta - \gamma r}{1-\gamma}.$$

Below, the optimal consumption strategy under permanent compulsory insurance is compared with the optimal consumption strategy with a dynamically optimal insurance coverage.[4] Compulsory insurance has two effects. The first is a wealth effect: Because insurance is costly, compulsory insurance reduces expected future wealth, everything else unchanged. It tends to reduce consumption. The other effect is a pure risk effect: Eliminating partial retention

[3] This can be seen in the model presented here by taking $\pi = 0$ and by replacing noncapital income w by w - P.

[4] For another analysis of compulsory insurance, see Briys (1988).

of the risk eliminates the need for precautionary saving. This tends to increase consumption. The global effect of compulsory insurance on the rate of consumption is therefore undetermined. Indeed, comparing c^0 with the optimal rate of consumption in the first-best case, as expressed by equation (10), yields

$$c - c^0 = -\frac{\pi\theta}{1-\gamma} + (1 - (1+\theta)^{\frac{1}{\gamma-1}})(1+\theta)\pi .$$

The first term in the right-hand side measures the wealth effect, which is negative; the second term measures the positive risk effect. The sign of the sum of these terms can easily be derived as a function of γ. In particular, if γ equals zero, the right-hand side equals zero: With a logarithmic utility function, imposing full insurance has no effect on the rate of consumption. The proof of the first part of the following proposition is a straightforward consequence of the above equality.

Proposition 5

In the infinite horizon model with an isoelastic utility function, imposing full insurance generates the following effects:

1. It reduces the optimal rate of consumption per dollar of adjusted wealth if relative risk aversion is larger than unity, and it increases the optimal rate of consumption per dollar of adjusted wealth if relative risk aversion is less than unity. With a logarithmic utility function, imposing full insurance has no effect on consumption.
2. It reduces the rate of growth of capital accumulation in the absence of an accident by $\pi\theta/(1-\gamma) \geq 0$.
3. It reduces the rate of growth of the expected adjusted wealth.

The second property is due to

$$\left.\frac{dY}{Y}\right|_{d=0} - \left(\frac{dY}{Y}\right)_{opt,dq=0} = (r - c^0) - (r - c + dP) = -\frac{\pi\theta}{1-\gamma} \leq 0$$

when there is no accident. The third property is derived from the following inequality:

$$\left.\frac{dY}{y}\right|_{d=0} - E\left[\left(\frac{dY}{Y}\right)_{opt}\right] = (r - c^0) - \left(\frac{r - \delta + dP}{1-\gamma} - \pi d\right)$$
$$= \frac{\pi}{1-\gamma}[(1-\gamma)d - \theta] \leq 0.$$

Conclusion

When insurance is costly, I have shown that, under some conditions, (partial) insurance coverage is dominated by precautionary capital accumulation and self-insurance in the long run. A model was used in which the decision-maker continuously solves the conflict between two opposite objectives: from one side, given the agent's attitude toward risk, it is valuable to insure risk in the short run; from the other side, given the cost of insurance, the agent's expected wealth would be increased by being its own insurer, which can be done only if financial reserves are accumulated. There are two ways to attain this objective: reduce current consumption or reduce the current level of coverage. The conflict between consumption, capital accumulation, and insurance appears clearly. This is the classic conflict between solvency and profitability. As stressed by Blanchard and Fischer (1989, p. 290), consumers who want to avoid negative consumption will want to exercise extreme prudence and build wealth as a precaution. When costly insurance is provided, I have shown that this result holds for a specific region of the parameters of the problem. I provided an upper bound to the loading factor for the insurer to survive in the long run, in a stable environment.

The optimal dynamic insurance strategy has some specific properties that differ from those of the classical static model. For example, a transitory increase in the cost of insurance unambiguously reduces the demand for insurance. A permanent increase in the cost of insurance has an ambiguous effect on the demand for insurance in the short run, but it unambiguously reduces demand in the long run. The effect of imposing full insurance on consumption was shown to depend upon whether relative risk aversion is larger or smaller than the classical critical value 1.

Appendix

Proof of Proposition 2

Let us consider a trial solution $I(W,t) = j(t)\frac{(w+h(t))^{\gamma}}{\gamma}$. It must be shown to satisfy the Hamilton-Bellman-Jacobi condition (3), for some specific functions j(t) and h(t). Terminal condition (4) implies that j(T) = b and h(T) = 0. Under this trial solution, the first-order condition (6) on D is rewritten as

$$P(t)(W+h(t))^{\gamma-1} = \pi E[\tilde{x}(t)(W-D\tilde{x}(t)+h(t))^{\gamma-1}],$$

which implies that D(W,t) = d(t)(W+h(t)) with d(t) defined by equation (11). The first-order condition (5) on consumption is rewritten as

$$[C(W,t)]^{\gamma-1} = j(t)(W+h(t))^{\gamma-1}.$$

It implies that C(w,t) = c(t)(W+h(t)), with

$$c(t) = [j(t)]^{\frac{1}{\gamma-1}}.$$

It still must be verified that the Hamilton-Bellman-Jacobi equation (3) holds for this trial solution. Replacing C and D by their optimal value yields

$$0 = \frac{[c(W+h)]^{\gamma}}{\gamma} - \delta j\frac{(W+h)^{\gamma}}{\gamma} + j'\frac{(W+h)^{\gamma}}{\gamma} + jh'(W+h)^{\gamma-1}$$

$$+ j(W+h)^{\gamma-1}[w-P-rh+(W+h)(r-c+Pd)]$$

$$- \pi\left[j\frac{(W+h)^{\gamma}}{\gamma} - j\frac{E(W-d(W+h)\tilde{x}+h)^{\gamma}}{\gamma}\right].$$

After some rearrangements, it simplifies to

$$0 = c^{\gamma} -\delta j+j'+j\gamma(r-c+Pd)-\pi j[1-E(1-d\tilde{x})^{\gamma}]$$
$$+\gamma(W+h)^{-1}[h'-rh+w-P].$$

Define h(t) in such a way that h′(t) = r(t)h(t)-w(t)+P(t), and h(T) = 0. It implies that h(t) = $\hat{W}(t)$ which is defined in Proposition 2. It also implies that the last term in the right-hand side of the above equation vanishes. Replacing j(t) by $c(t)^{\gamma-1}$, the above equation can be rewritten as

$$0 = c^{\gamma}-\delta c^{\gamma-1}+(\gamma-1)c^{\gamma-2}c'+c^{\gamma-1}\gamma(r-c+Pd)-\pi c^{\gamma-1}[1-E(1-d\tilde{x})^{\gamma}],$$

or, equivalently,

$$0 = c - \delta + (\gamma-1)c^{-1}c' + \gamma(r-c+Pd) - \pi[1-E(1-d\tilde{x})^{\gamma}],$$

which is equivalent to condition (10).■

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