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Increased Risk Aversion and Risky Investment

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ABSTRACT

This article investigates the relationship between the size of investment in a risky asset and the degree of risk aversion. The necessary and sufficient conditions are established that permit the prediction of whether agents with differing degrees of risk aversion will increase or decrease investment in the risky asset. It shows, in particular, that when the marginal return to investment decreases (increases) with an improvement in the state of nature, greater risk aversion will induce higher (lower) investment.

Introduction

The relationship between the size of an investment and the degree of risk aversion has been studied under the Arrow-Pratt theory of risk aversion (Pratt, 1964; Arrow, 1971). This theory establishes a classical result for agents who maximize the expected utility of final wealth and who must divide their wealth between a safe asset and a single risky asset: the more risk averse the agent, the smaller the proportion of wealth he or she will invest in the risky asset.

There are two restrictive assumptions inherent in the Arrow-Pratt framework (for examples of models following the Arrow-Pratt tradition, see the references in Schlee, 1990). First, the payoff functions take the form of $W = K + ks$, where K is the initial wealth, k is investment, and s is a random net return which may take either negative or positive values. Once the value of s is determined, the payoffs must be linear in k. However, in practice, payoff functions assume a much more general form, namely, $W = W(k,s; K)$. With life insurance, for example, the benefit of the indemnity is the payoff to the insured. The benefit increases with the size of the premium, but the benefit is not necessarily a linear function of the premium. (See the Appendix for an example of an actual life insurance policy.)

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Second, the Arrow-Pratt framework implies that the marginal return to investment increases with risk; that is, a larger investment in a risky asset always entails a higher risk. In practice, however, the marginal return to investment may either increase or decrease as the state of nature improves. In farming, for example, investment can substitute for weather in the sense that larger investment may reduce risks associated with bad weather.

This article examines the effect of increased risk aversion on investment, with the general form of the payoff function assumed to be $W(k,s; K) = f(K-k)+y(k,s)$ (f and y are defined below). How an increase in the agent's risk aversion affects investment depends crucially on how the marginal returns of the risky asset respond to an improvement in the state of nature. Using a model analogous to Arrow and Pratt's, this article shows that, if the marginal return to investment decreases (increases) with an improvement in the state of nature, greater risk aversion will induce higher (lower) investment. That is, a more risk averse agent may actually invest more in the risky asset. On the other hand, the Arrow-Pratt result survives without the linearity assumption of the payoff function, as long as the marginal return increases as the state of nature improves. In fact, the Arrow-Pratt problem is a special case of the one discussed in this article.

The model is described in the next section. I then identify the necessary and sufficient conditions that determine whether a more risk averse agent will increase or decrease investment in a risky asset and discuss the implications of the results.

The Model

Consider an agent who is endowed with K units of capital goods (initial wealth) and faces two projects which would transform capital goods into consumption goods. Assume that the capital goods cannot be consumed directly and that the agent can invest in both projects. Of the two projects, one is safe and the other is risky. The safe project is represented by a production function $f(k)$ with $f(0) = 0$ and $f' > 0$, where $0 < k < K$. The risky project, on the other hand, is represented by a production function $y(k,s)$, with $y(0,s) = 0$ and $y_k > 0$ (where $y_k = \partial y(k,s)/\partial k$); s is a state variable which takes values randomly between 0 and 1. The distribution of s is governed by a density function $g(s)$ that is known to the agent. It is assumed that $y_s(k,s) > 0$ for any given $k \in (0,K]$; that is, s_1 is a better state than s_2 whenever $s_1 > s_2$. In addition, it is assumed that the returns to each project may exhibit increasing, decreasing, or constant returns to scale; however, the total return from investment exhibits decreasing or constant returns to scale. In other words, the rate of total returns is a decreasing or constant function of investment.

Moreover, the agent is endowed with a utility function $U(c)$, where U is increasing and strictly concave over the consumption variable c. In order to maximize utility, the agent must decide what to invest and how much. Let k be the amount of capital goods the agent invests in the risky project. The agent chooses k by solving the following problem:

$$\max E[U(f(K-k)+y(k,s))] \quad \text{s.t.} \quad 0 < k < K, \tag{1}$$

where the maximization is taken on k and the expectation is taken with respect to the random variable s.

Pratt (1964) suggests a formula for comparing two agents' risk aversion: Agent A is more risk averse than agent B if there exists an increasing, strictly concave, and twice differentiable function Φ such that $U^A = \Phi(U^B)$, where U^A and U^B are the respective utility functions.

The Effect of an Increase in Risk Aversion on Investment

The Sufficient Conditions

The effect of an increase in risk aversion on investment is summarized by the following proposition:

Proposition 1. Let k_0 be the original optimal investment level. If $\frac{\partial^2 y}{\partial k \partial s} < 0$, then the level of risky investment increases as the investor becomes more risk averse in the sense of Pratt, and if $\frac{\partial^2 y}{\partial k \partial s} > 0$, then the level of risky investment decreases as the investor becomes more risk averse. However, if $\frac{\partial^2 y}{\partial k \partial s} = 0$, then the investment level is independent of the investor's risk attitude. Proof: Suppose k_0 is the original optimal investment level. Then k_0 satisfies the first-order condition

$$E[U'(f(K-k_0)+y(k_0,s))(y_k(k_0,s)-f'(K-k_0))] = 0. \tag{2}$$

Suppose that the investor becomes more risk averse. There exists a strictly concave function Φ, so that the investor's utility function is now $\Phi(U(\cdot))$. Let $V(k) = E[\Phi(U(f(K-k)+y(k,s)))]$. Then

$$V'(k_0) = E[\Phi'(U)U'(f(K-k_0)+y(k_0,s))(y_k(k_0,s)-f'(K-k_0))].$$

Define $\hat{s}$ by $y_k(k_0,\hat{s}) = f'(K-k_0)$ and denote $U(f(K-k_0)+y(k_0,s))$ by $U(s)$. Because both U and Φ are increasing and concave,

$$U(s) \begin{cases} < U(\hat{s}) & \text{if } s < \hat{s} \\ > U(\hat{s}) & \text{if } s > \hat{s} \end{cases}$$

$$\phi'(U(s)) \begin{cases} > \Phi'(U(\hat{s})) & \text{if } s < \hat{s} \\ < \Phi'(U(\hat{s})) & \text{if } s > \hat{s} \end{cases} \tag{3}$$

Consider $Y = E\{[\Phi'(U(s))-\Phi'(U(\hat{s}))]U'(f(K-k_0)+y(k_0,s))(y_k(k_0,s)-f'(K-k_0))\}$. Suppose that $\frac{\partial^2 y}{\partial k \partial s} < 0$ (the cases where $\frac{\partial^2 y}{\partial k \partial s} > 0$ or $\frac{\partial^2 y}{\partial k \partial s} = 0$ can be proved similarly). Then,

$$y_k(k_0,s) - f'(K-k_0) \begin{cases} > 0 & \text{if } s < \hat{s} \\ < 0 & \text{if } s > \hat{s} \end{cases} \tag{4}$$

Using expressions (3) and (4) together with $U' > 0$, we arrive at $Y > 0$. It follows that

$$V'(k_0) > \Phi'(U(\hat{s}))E[U'(f(K\text{-}k_0)+y(k_0,s))(y_k(k_0,s)\text{-}f'(K\text{-}k_0))].$$

Applying equation (2) to the right-hand side of the above inequality, we find that

$$V'(k_0) > 0. \tag{5}$$

Finally, we need to check the second-order condition. If we denote $\Phi(U(\cdot))$ by $\psi(\cdot)$, the second-order derivative of $V(k)$ becomes

$$V''(k) = E[\psi''(\cdot)(y_k\text{-} f')^2 + \psi'(\cdot)(y_{kk}+ f'')].$$

Note that Φ and U are both concave, so ψ is concave, and therefore $\psi'' < 0$. On the other hand, the total return to investment exhibits decreasing or constant returns to scale, so $y_{kk} + f'' \leq 0$. The second-order condition is thus satisfied. In particular,

$$V''(k_0) \leq 0. \tag{6}$$

Together, inequalities (5) and (6) imply that

$$\text{when } \frac{\partial^2 y}{\partial k \partial s} < 0, \text{ then } k^* \equiv \text{argmax}V(k) > k_0. \blacksquare$$ [1]

Proposition 1 states that, whenever the states of nature affect the marginal returns of the risky project, this relationship determines the effects of increased risk aversion on investment. If the marginal return is an increasing function of s at k_0 ($\frac{\partial^2 y}{\partial k \partial s} > 0$), then increased risk aversion will discourage the risky investment. Conversely, if the marginal return is a decreasing function at k_0 ($\frac{\partial^2 y}{\partial k \partial s} < 0$), then increased risk aversion will stimulate the risky investment. Whenever the marginal return is not affected by the states, then investment level is independent of the investor's risk attitude.

Figure 1 illustrates this proposition for the special case where there are only two states of the world. The horizontal axis represents levels of investment and the vertical axis represents outputs of the risky project. For simplicity, we consider a special case in which s takes only two values: 0 or 1. In each panel, the upper curve represents the production function of the risky project in state $s = 1$, and the lower curve represents the production function of the risky project in state $s = 0$. Since the slopes of the curves at k_0 in each panel are $y_k(k_0,0)$ and $y_k(k_0,1)$, respectively, panel A is associated with the case $y_k(k_0,0)$

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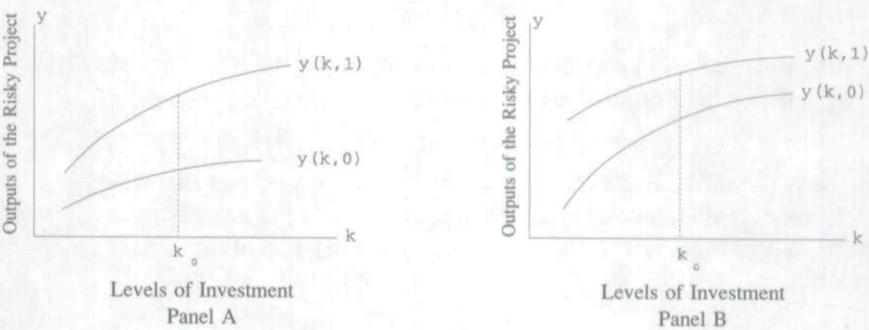
[1] I gratefully acknowledge an anonymous referee's suggestion on the proof of this proposition.

$< y_k(k_0,1)$,[2] and panel B is associated with the case $y_k(k_0,0) > y_k(k_0,1)$. In panel A, the difference between y(k,1) and y(k,0) becomes larger as k increases. That is, as k increases, the risky project becomes riskier, and a more risk averse agent will choose a smaller k. In contrast, in panel B, the difference between y(k,1) and y(k,0) becomes smaller as k increases. As k increases, the risky project becomes less risky, and a more risk averse agent will choose a larger k.

Figure 1
Marginal Return and Risky Investment



Comparison with Arrow-Pratt

If it had been assumed that the marginal return is an increasing function of the states of nature, then the Arrow-Pratt result would have followed. According to Arrow and Pratt, when part of the assets x are to be held as cash and the rest, say a, invested in a risky project with random rate $\tilde{\imath}$, then the payoff will be $x + a\cdot\tilde{\imath}$ (see Pratt, 1964, Section 13). Rewriting this result in the notation of this article, $W = K + k\cdot s = K-k + k(1+s)$ with $f(K-k) = K-k$ and $y(k,s) = k(1+s)$, where $x = K$, $a = k$, and $\tilde{\imath} = s$. Notice that $y_k = 1+s$, $y_{kk} = 0$, $y_{ks} = 1$, $f' = 1$, and $f'' = 0$. Since $y_{kk} + f'' = 0$, constant returns to scale, Proposition 1 can be applied. Applying the proposition, we find that an increase in the degree of risk aversion reduces k since $\frac{\partial^2 y}{\partial k \partial s} = 1 > 0$. Thus, the Arrow-Pratt result is a special case of Proposition 1 with $\frac{\partial^2 y}{\partial k \partial s} = 1 > 0$.

In order to obtain the Arrow-Pratt result, the payoffs need not be linear; any concave payoff function of investment suffices. More importantly, Proposition 1 states that an increase in risk aversion does not necessarily lead to a smaller investment in a risky asset, even when there is only one such asset. The investment level can either increase or remain constant when agents become more risk averse, results not predicted by the Arrow-Pratt theory. These results

[2] This is an expression of $\partial^2 y/\partial k \partial s > 0$ when s takes only two values.

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obtain because the riskiness of any investment depends in part upon how the marginal return responds to changes of the states of nature. Whenever the marginal return is a decreasing function of s, larger investment can actually reduce risk. Under this circumstance, a more risk averse agent will choose to invest more. The Arrow-Pratt theory does not address this situation.

The Necessary Conditions

Thus far, we have identified the conditions that are sufficient to induce an agent to invest more (or less) in the risky asset as he or she becomes more risk averse. Proposition 2 identifies the necessary conditions.

Proposition 2. Suppose $y_k(k_0,s)$ is monotonic in s. When the investor becomes more risk averse, a necessary condition for a higher (lower or constant) level of risky investment is $\frac{\partial^2 y}{\partial k \partial s}|_{k=k_0} < 0 \left(\frac{\partial^2 y}{\partial k \partial s}|_{k=k_0} \geq 0\right)$. Proof: The notations are the same as those used in proving Proposition 1. Suppose $k^* > k_0$. Then, $V'(k_0) > 0$ implies $Y > 0$. Therefore,

$$\int_0^{\hat{s}} [\Phi'(U(s) - \Phi'(U(\hat{s}))]U'(s)(y_k(k_0,s) - f'(K-k_0))g(s)ds$$

$$+ \int_{\hat{s}}^1 [\Phi'(U(s)) - \Phi'(U(\hat{s}))]U'(s)(y_k(k_0,s) - f'(K-k_0))g(s)ds > 0$$

Recall that $\Phi'(U(s)) > \Phi'(U(\hat{s}))$ when $s < \hat{s}$ and vice-versa. Since $y_k(k_0,s)$ is monotonic in s and $U' > 0$, it must be true that $y_k(k_0,s)-f'(K-k_0) > 0$ when $s < \hat{s}$ and that $y_k(k_0,s)-f'(K-k_0) < 0$ when $s > \hat{s}$. Recall also that $y_k(k_0,\hat{s}) = f'(K-k_0)$. It follows that

$$y_k(k_0,s) \begin{cases} > y_k(k_0,\hat{s}) & \text{if } s < \hat{s} \\ < y_k(k_0,\hat{s}) & \text{if } s > \hat{s} \end{cases}$$

Therefore, $\frac{\partial^2 y}{\partial k \partial s} < 0$ at k_0. The other two cases can be proved similarly.

Together, Propositions 1 and 2 provide the necessary and sufficient conditions that permit the prediction of whether agents with differing degrees of risk aversion will increase or decrease investment in a risky asset. By recognizing that the effect of an increase in risk aversion on investment depends on how the marginal returns to investment respond to changes in the states of nature, it is possible to establish a more general result concerning the relationship between risk aversion and risky investments. As shown above, this result includes the Arrow-Pratt result as a special case.

Conclusion

The prevailing Arrow-Pratt theory of risk aversion makes very specific assumptions about the nature of the risky asset (how the marginal returns of the asset respond to changes of the states) in analyzing how risk aversion affects investment. This article demonstrates the importance of the nature of an asset in determining the effect of changes in risk aversion. The riskiness of any investment depends on both the risk attitude of the investor and the nature of the asset. Agents may raise or lower their level of risky investment, depending not only upon differences in their risk attitude but also on the way in which the marginal returns of the asset respond to changes of the states.

Appendix
A Sample Life Insurance Policy

The following are selected statistics of a life insurance policy offered by Provident Mutual Life Insurance Company of Philadelphia in 1985, as reported by the National Underwriting Company (1985):

	Premium Rates			
Death Benefit	Age 0	Age 10	Age 20	Age 30
$ 5,000- 49,999	5.10	6.62	9.00	12.79
50,000- 99,999	4.70	6.16	8.45	12.25
100,000-249,999	4.58	6.03	8.30	12.11
250,000 and over	4.51	5.95	8.21	12.02

The table shows the premium rates for an ordinary male at ages 0, 10, 20, and 30. The premium rate reflects the premium per $1,000 of death benefit. Let G(b) and R(b) respectively denote the premium and premium rate for a policy of the amount of b. It has been established that G(b) = bR(b) (see Bowers et al., 1986). Notice that R(b) varies with b over different ranges of b as shown in the above table. It is easy to verify that the relationship between G(b) and b is not linear.

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