



The Comparative Statics of Self-Protection

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ABSTRACT

This article considers the effects of changes in initial wealth and loss size on optimal self-protection (loss prevention) activity in a simple model of self-protection choice. The effect of an initial wealth increase is shown to depend on both the probability of a loss and characteristics of the agent's absolute risk-aversion function, while increases in the size of the potential loss are shown to lead to increased self-protection under a variety of plausible restrictions on risk preference. A set of sufficient conditions for signing comparative static results are obtained.

The expenditure of resources to modify the probabilities of suffering losses is called "self-protection."¹ As noted by Briys and Schlesinger (1990), expenditures on self-protection do not merely trade income in one state of the world for income in another as market insurance contracts do. Rather, self-protection reduces income in all states, shifting the support of the wealth distribution to the left. Hence, increases in self-protection spending do not in general lead to less risky income prospects in the sense of stochastic dominance.² Consequently, the willingness of agents to engage in self-protection fails to parallel agents' willingness to buy market insurance in several important respects. Both risk-loving and risk-averse agents may buy self-protection, and more risk-averse agents will not generally spend more on self-protection than less risk-averse agents.³ This characteristic of self-protection makes its analysis considerably more

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¹ Self-protection is also referred to as "care" in the legal liability literature, and as "loss prevention" in the insurance context.

² For a discussion of the dominance concept applied to self-protection, see Briys and Schlesinger (1990).

³ See Sweeney and Beard (1991), Briys and Schlesinger (1990), Dionne and Eeckhoudt (1985), and Ehrlich and Becker (1972) for various observations on the relationship between self-protection and risk-aversion. Dionne and Eeckhoudt (1985) offer some interesting examples.

difficult than that of insurance. In addition, analysis of self-protection choice must account for the fact that any agent's optimal level of self-protection spending depends on an exogenous, "technical" relationship between spending and loss probability. This relationship summarizes the agent's opportunities to reduce loss probabilities and is a critical component in determining optimal self-protection choice.

The purpose of this article is to present a set of comparative static results for the simplest and most widely-used model of self-protection choice. Interest focuses on the effects of changes in initial wealth and potential losses on optimal self-protection expenditures.⁴ The goal is to seek results of a general character which do not depend on the particular form of the technical relationship between spending and loss probability, nor on restrictive characterizations of agents' von Neumann—Morgenstern utility functions for wealth. Further, the main findings will be stated in terms of the characteristics of agents' attitudes towards risk in a relatively straightforward fashion, and a set of results useful for evaluating economic problems that involve self-protection will be derived.

Optimal Self-Protection Choice

Consider the simplest model of self-protection choice in which an agent faces the potential of losing a fixed sum of money L . Letting y_0 be the agent's initial wealth, s her spending on self-protection, $p(s)$ the probability of suffering the loss L given self-protection expenditure $s \geq 0$, and $u(\cdot)$, the agent's von-Neumann Morgenstern utility function for money, write the agent's expected utility given self-protection spending level s as $v(s, L, y_0)$, where

$$v(s, L, y_0) \equiv p(s)u(y_0 - s - L) + [1 - p(s)]u(y_0 - s). \quad (1)$$

Letting $y \equiv y_0 - s$, and assuming the required differentiability of $p(\cdot)$ and $u(\cdot)$, the optimal level of self-protection spending s^* is given by the first-order condition

$$-p'(s^*)[u(y) - u(y-L)] - [1 - p(s^*)]u'(y) - p(s^*)u'(y-L) \leq 0, \quad (2)$$

with equality holding if $s^* > 0$. The second-order necessary condition for an interior solution requires

⁴ This analysis both differs from, and extends, the work of Ehrlich and Becker (1972) on this problem. Unlike Ehrlich and Becker, this paper analyzes changes in initial wealth rather than an "equal proportional increase in endowments," which surprisingly is quite a different problem. Further, the present analysis extends Ehrlich and Becker's result on loss sizes in several directions, producing a set of static conditions which are not apparent in their analysis of this issue. In fact, the condition for self-protection to be increasing in the loss given by Ehrlich and Becker conveys the erroneous impression that the curvature of the function $p(s)$ is important, while the results presented here establish that it is only the value of $p(s)$ at s^* that is significant for the static results.

$$-p''(s^*)[u(y) - u(y-L)] + 2p'(s^*)[u'(y) - u'(y-L)] + [1 - p(s^*)]u''(y) + p(s^*)u''(y-L) < 0. \quad (3)$$

The first-order condition given by (2) has an immediate conventional interpretation in terms of costs and benefits: the (expected) marginal benefit of self-protection spending $(-p'(s^*)[u(y) - u(y-L)])$ should be equal to the (expected) marginal "cost" of such spending $([1-p(s^*)]u'(y) + p(s^*)u'(y-L))$ at the optimal choice s^* .

Comparative Static Results for Changes in Initial Wealth

The initial goal is evaluating the effect of changes in the agent's initial wealth y_0 on optimal self-protection spending s^* . If the purchase of self-protection is identified with the purchase of an economic good such as a burglar alarm, security guard service, or smoke detector, then the issue of how changes in (initial) wealth y_0 influence self-protection choice amounts to an evaluation of the income sensitivity of demand for the relevant good or service. Without confusion, then, one could say that, when optimal self-protection spending s^* increases as y_0 increases, self-protection is a "normal good," while decreased spending induced by increased wealth implies that self-protection is "inferior."

Performing the necessary differentiation to condition (2), and letting v_{ss} denote the expression in (3), the desired term is

$$\partial s^*/\partial y_0 = \{-p'(s^*)[u'(y) - u'(y-L)] - [1 - p(s^*)] u''(y) - p(s^*)u''(y-L)\}/(-v_{ss}). \quad (4)$$

To begin interpretation of expression (4), note first that for any value of spending $\hat{s}$ that is strictly between 0 and L , and any strictly positive number K , one can always find some loss probability function $p(s)$ such that $\hat{s}$ is the optimal level of expenditure and $p(\hat{s}) < K$.⁵ Further, for some (but not all) values of $\hat{s}$, one can also find another loss probability function $\tilde{p}(\hat{s})$ such that $\hat{s}$ is the optimal level of expenditure and $\tilde{p}(\hat{s})$ is arbitrarily close to (but less than) one.⁶ These facts are important because they imply that, for any utility function $u(\cdot)$, the agent's "optimal probability" of suffering the loss $p(s^*)$ could equal any value between zero and one, depending on the nature of the loss probability function.⁷

Extensive algebraic manipulation of expression (4) and first-order condition (2) produces the following sign condition:

⁵ A constructive proof of this claim, while quite lengthy and omitted here, is available from the authors on request.

⁶ Optimal spending s^* cannot be arbitrarily close to L when $p(s^*)$ is close to one because $s = 0$ would then be preferable. An extensive examination of this issue is available from the authors on request.

⁷ Hence, one cannot rule out any value of $p(s^*)$ such that $0 < p(s^*) < 1$ *a priori*.

$$\text{sgn}[\partial s^*/\partial y_0] = \text{sgn}[\bar{r}_{\text{end}} - \bar{r}_{\text{full}}] \quad (5)$$

where

$$\bar{r}_{\text{end}} \equiv \{r(y)(1-p(s^*))\exp[-I(y)] + r(y-L)p(s^*)\exp[-I(y-L)]\} / \{(1-p(s^*))\exp[-I(y)] + p(s^*)\exp[-I(y-L)]\} \quad (6)$$

$$\bar{r}_{\text{full}} \equiv \left\{ \int_{y-L}^y r(x)\exp[-I(x)]dx \right\} / \left\{ \int_{y-L}^y \exp[-I(x)]dx \right\} \quad (7)$$

$$I(x) \equiv \int_{y-L}^x r(s)ds \quad (8)$$

and $r(s) \equiv -u''(s)/u'(s)$ is the Arrow-Pratt measure of absolute risk-aversion. Hence, condition (5) states that the normality or inferiority of self-protection spending depends on a comparison between $\bar{r}_{\text{end}}$, a weighted average of the measures of absolute risk-aversion $r(\cdot)$ at the wealth "endpoints" y and $y-L$, and $\bar{r}_{\text{full}}$, a weighted average of $r(\cdot)$ over the entire interval $[y, y-L]$.

The summary nature of condition (5) becomes apparent when it is recalled that, in general, $p(s^*)$ may take on any value in the interval $(0,1)$, implying that $\bar{r}_{\text{end}}$ could assume any value between $r(y)$ and $r(y-L)$. Further, $\bar{r}_{\text{full}}$ lies strictly between the minimum and maximum values of $r(\cdot)$ on the interval $[y, y-L]$, provided that $r(\cdot)$ is not constant over the entire interval. This reasoning yields the following result:

- Result A:*
1. If $r(\cdot)$ is constant throughout the interval $[y-L, y]$, then $\partial s^*/\partial y_0 = 0$.
 2. If $r(\cdot)$ is monotonically decreasing (DARA) at all levels of wealth in $[y-L, y]$, then there exists a number p^c , $0 < p^c < 1$, such that:

$$\begin{aligned} \partial s^*/\partial y_0 &< 0 & \text{if } p(s^*) < p^c \\ \partial s^*/\partial y_0 &> 0 & \text{if } p(s^*) > p^c \end{aligned}$$
 3. If $r(\cdot)$ is monotonically increasing (IARA) at all levels of wealth in $[y-L, y]$, then there exists a number p^c , $0 < p^c < 1$, such that:

$$\begin{aligned} \partial s^*/\partial y_0 &> 0 & \text{if } p(s^*) < p^c \\ \partial s^*/\partial y_0 &< 0 & \text{if } p(s^*) > p^c \end{aligned}$$

Result A illustrates that, in general, the effect of a change in initial wealth y_0 on self-protection choice depends fundamentally on the magnitude of the probability of suffering the loss $p(s^*)$, and on the behavior of the absolute risk-aversion function $r(\cdot)$.

Given any monotone behavior of $r(\cdot)$ over the interval $[y-L, y]$, the critical probability p^c of Result A can be calculated explicitly from knowledge of $r(\cdot)$ by solving for the probability p^c that equates $\bar{r}_{\text{end}}$ and $\bar{r}_{\text{full}}$. Since both $\bar{r}_{\text{end}}$ and $\bar{r}_{\text{full}}$ are (somewhat complex) functions of the values that the risk-aversion function assumes over the interval $[y, y-L]$, any change in the shape of $r(\cdot)$ affects the level of the critical probability p^c . This can be illustrated by comparing the four risk-aversion functions sketched in Figure 1, labelled r_1 , r_2 , r_3 , and r_4 . Note that all four functions are decreasing in wealth (so DARA obtains), and that all four functions obtain equal values at the "endpoints" $y-L$ and y . Let p_i^c denote the critical probability corresponding to the risk-aversion function r_i , $i = 1, 2, 3, 4$.

Let $r_1(\cdot)$ represent any monotonically decreasing risk-aversion function. The function r_2 is derived from r_1 by adding a "perturbation" h to r_1 that reduces risk-aversion at "lower" levels of wealth and raises it at "higher" levels of wealth while leaving unchanged the simple (uniformly weighted) average level of risk-aversion over $[y, y-L]$.⁸ That is:

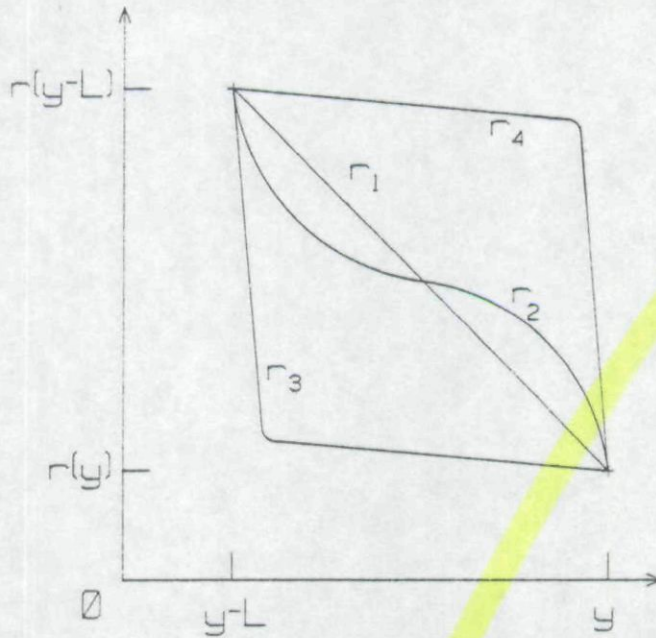
$$r_2(x) = r_1(x) + h(x), y-L \leq x \leq y, \text{ such that: (i) } h(y) = h(y-L) = 0; \text{ (ii) } \int_{y-L}^x h(s)ds < 0 \text{ for } y-L < x < y; \text{ and (iii) } \int_{y-L}^y h(s)ds = 0.$$

Such a perturbation leaves the value of $\bar{r}_{\text{end}}$ unchanged, (see equation (7)), but reduces $\bar{r}_{\text{full}}$. Hence, $p_2^c < p_1^c$. That is, any mean preserving perturbation of the risk-aversion function that lowers risk-aversion at smaller levels of wealth while raising it at larger levels (and leaving risk-aversion unchanged at the endpoints) causes the critical probability p^c to fall, increasing the range of loss probabilities over which self-protection is a normal good.

The risk-aversion functions r_3 and r_4 represent extreme forms of decreasing absolute risk-aversion. The curve r_3 approximates a step-function for which $r(x)$ falls to the level of $r(y)$ immediately beyond wealth level $y-L$. Hence, by (6) and (7), the critical probability $p_3^c = 0$, so that self-protection is a normal good for almost all loss probabilities in this case. The opposite extreme is illustrated by r_4 , in which $r(x)$ remains near $r(y-L)$ until wealth almost equals y , at which point it falls rapidly to $r(y)$. By (6) and (7), the critical probability $p_4^c = 1$, so that self-protection is nearly always an inferior good in this case.

⁸ Figure 1 illustrates the case in which the perturbed function r_2 crosses r_1 only once in the interior of $[y-L, y]$, so that the meanings of "lower" and "higher" levels of wealth are unambiguous. The conditions on h , however, do not prevent the perturbed function r_2 from crossing r_1 any odd number of times. If r_2 crosses r_1 multiple times in the interior of $[y-L, y]$, one must interpret "lower" and "higher" as pertaining to levels of wealth within the subintervals bounded by the evenly numbered crossing points (taking $y-L$ as the first evenly numbered crossing, and y as the last).

Figure 1
Four Risk-Aversion Functions .



The examples of Figure 1 provide some insights into the role of the absolute risk-aversion function in determining the normality of self protection under DARA. The length of the interval of probability values over which self-protection is a normal good for a person depends in a complex fashion on the shape of that person's risk-aversion function over the entire interval of wealth between the two possible outcomes. Surprisingly, the length of the interval of probability values over which self-protection is normal can be arbitrarily close to 0 in the case of one set of preferences while being arbitrarily close to 1 with a second set of preferences that is qualitatively quite similar to the first set. One can imagine both risk-aversion functions r_3 and r_4 as characterizing a person whose aversion to risk is approximately constant at all levels of wealth except one (between $y-L$ and y), at which risk aversion falls from $r(y-L)$ to $r(y)$. The function r_3 then would represent the case that the drop in risk-aversion occurs near the level of wealth of the loss state, while the function r_4 would represent the case that the drop in risk-aversion occurs near the level of wealth of the no-loss state. Clearly, for a person with such preferences, the location of loss and no-loss levels of wealth, relative to that at which risk-aversion changes, is a critical determinant of normality or inferiority of self-protection. While this result may not seem odd because the average level of risk-aversion (over the relevant range) of a person characterized by r_3 is significantly lower than that of a person characterized by r_4 , examples r_1 and r_2 caution against using the simple average level of risk-aversion (or any other relatively simple summary characteristic) to predict the length of the interval of probabilities over which self-protection is normal. Conclusions can not

be reached from knowledge of general characteristics of a (nonconstant) risk-aversion function; an almost complete specification of the risk-aversion function is required.

Comparative Static Results for Changes in the Loss L

Consider next an analysis of the effect of a change in the size of the loss L on optimal self-protection choice s^* . Differentiation of (2) implies that

$$\partial s^*/\partial L = [-p'(s^*)/p(s^*) + u''(y-L)/u'(y-L)]p(s^*)u'(y-L)/(-v_{ss}) . \quad (11)$$

Employing (2), (3) and (11), algebraic manipulation yields the result that

$$\text{sgn}[\partial s^*/\partial L] = \text{sgn}\{\bar{J} - r(y-L)\} \quad (12)$$

where

$$\bar{J} \equiv \left\{ \int_{y-L}^y J(x) \exp[-I(x)] dx \right\} / \left\{ \int_{y-L}^y \exp[-I(x)] dx \right\} \quad (13)$$

and

$$J(x) \equiv r(x) + (p(s^*) \cdot L)^{-1} \exp[-I(y) - I(x)] . \quad (14)$$

Thus, the static result is seen to depend on comparison of the weighted average of the function $J(x)$ with the value of the risk-aversion function at the loss state wealth level. Noting that $J(x)$ equals $r(x)$ plus a positive term, one obtains the following:

Result B: $\partial s^*/\partial L > 0$ whenever $r'(x) \geq 0$.

Hence, when risk-aversion is not decreasing in income, increases in the size of the loss always increase self-protection effort.

The case of DARA is somewhat more complex. However, one can derive a set of useful sufficient conditions that insure that $\partial s^*/\partial L > 0$. Manipulation of (12), (13), and (14) allows one to conclude that $\partial s^*/\partial L > 0$ whenever $\exp[-I(y) - I(x)] > p(s^*)L[r(y-L) - r(x)]$ for all x between $y-L$ and y . Assuming DARA, this condition can be manipulated to produce Result C:

Result C: Under DARA, $\partial s^*/\partial L > 0$ whenever $\exp[-Lr(y-L)] > p(s^*)L[r(y-L) - r(y)]$.

Result C gives a useful sufficient condition for self-protection spending to be increasing in the size of the loss L . In general, $\partial s^*/\partial L > 0$ whenever the probability of the loss $p(s^*)$, the size of the loss L , or the expected loss

$p(s^*)L$ is sufficiently small, or when $r(y-L)$ and $r(y)$ are nearly equal. In terms of the expected loss $p(s^*)L$, $\partial s^*/\partial L > 0$ whenever $p(s^*)L < \exp[-L r(y-L)]/[r(y-L) - r(y)]$, a condition that can be evaluated for many practical problems given some assumptions about the degree of risk-aversion $r(\cdot)$.

When *relative* risk-aversion $xr(x)$ is increasing in wealth with DARA, a very precise condition can be derived. Letting $R(x) = xr(x)$ denote the measure of relative risk-aversion, with $R'(x) > 0$, we obtain the result that $\partial s^*/\partial L > 0$ whenever $(x/y)^{R(y)} > p(s^*)L/x$ for all levels of wealth x in the interval $[y-L, y]$, where y is y_0 -s as before, and $R(y)$ is the level of relative risk-aversion at y . Hence, if the ratio of any level of wealth x in $[y-L, y]$ to wealth y , raised to the power $R(y)$, exceeds the ratio of the expected loss $p(s^*)L$ to that wealth level s , then increases in the loss size L must increase self-protection effort. Since this requirement is most stringent at $x = y - L$, one obtains Result D:

Result D: $\partial s^*/\partial L > 0$ when $R'(x) > 0$ and $[(y-L)/y]^{R(y)} > p(s^*)L/(y-L)$.

Hence, with $R'(x) > 0$ and DARA, self-protection effort increases in the loss L whenever the expected loss is a sufficiently small proportion of after loss wealth so that $[(y-L)/y]^{R(y)}$ exceeds it. Assuming, for example, that $R(y) = 1$, this just requires that $(y-L)/y > p(s^*)L/(y-L)$, an easily verified condition in most practical applications.⁹

Conclusion

This article has investigated the effects of changes in initial wealth and loss sizes on optimal behavior in the simplest model of self-protection. Since optimal self-protection choice partially depends on an exogenously given "loss probability function," and since increases in self-protection spending do not decrease the riskiness of income prospects, the static results are unsurprisingly ambiguous. However, one can manage to both characterize self-protection choice in terms of risk-aversion, and derive a set of potentially usable results for analysis of economic problems involving self-protection.

Self-protection is a normal good under DARA when the probability of the loss is sufficiently large, but is an inferior good otherwise. (Related results are obtained when risk preferences are not characterized by DARA.) The critical size of this loss probability can be calculated solely from knowledge of the behavior of the absolute risk-aversion function $r(\cdot)$, although this critical probability may vary widely from case to case.

The effect of changes in the size of the potential loss L likewise can be characterized in terms of loss probabilities and risk-aversion. When absolute risk-aversion is increasing or constant in income, increases in L always

⁹ See Arrow (1970), chap. 3, for an argument that $R(x)$ may be close to 1.

induce increased self-protection efforts. Alternately under DARA, increases in L induce increased self-protection whenever the potential loss L , the probability of the loss, or the expected loss are sufficiently small. Again, "sufficiently small" can be stated solely in terms of the behavior of the absolute risk-aversion function $r(\cdot)$. If, under DARA, it is also true that relative risk-aversion is increasing in wealth, then self-protection effort will increase as the loss increases whenever the expected loss is a sufficiently small fraction of after loss wealth.

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