A NOTE ON STOP LOSS PREMIUMS

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ABSTRACT

This note uses a covariance model of insurance pricing to examine stop loss premiums. The premium is equal to the expected loss plus a term proportional to the covariance between the loss and another random variable Y which represents an index. Under this model premiums are additive. Three further properties of the stop loss premium are proposed and the conditions for the stop loss premium to have these properties are derived.

Introduction

In recent years several authors have developed models of insurance pricing within the framework of competitive markets. Many of the pricing paradigms developed by financial economists have been used as the basis of these models. Biger and Kahane (1978), Fairley (1979) and Urrutia (1986) base their approaches on the Capital Asset Pricing Model. In these models, the equilibrium insurance price depends on the covariance of the risk with the asset market portfolio. Borch (1984) developed a similar model based on the covariance of the risk with the insurance market portfolio. Ang and Lai (1987) constructed a model where insurance premiums depend both on the covariance of the risk insured with the asset market portfolio and the insurance market portfolio.

Under these models insurance premiums are additive. This additivity is a consequence of the no arbitrage principle, which provides the cornerstone for most of the asset pricing models in modern finance. The no arbitrage principle states that in a perfect frictionless market it is impossible to make a riskless profit with a zero net investment. While the insurance market is unlikely to

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conform exactly to a perfect frictionless market, it is nevertheless interesting to explore the implications for insurance pricing of this assumption. It implies, for example, that an insurer cannot make guaranteed profits from reinsurance operations. To illustrate, suppose that the insurer receives a premium, P, for a risk in the primary market. It should not be possible for the direct writer to reinsure the entire risk and receive a premium (premiums) greater than P.[1] Within this framework if a risk is carved up into several pieces the premium for the risk as a whole must equal the sum of the premiums for the individual pieces.

Approaches to premium determination in the actuarial literature suggest that the premium be set equal to the expected value of the claim plus a risk loading. The risk loading is often derived from solvency considerations. In recent years it has become fashionable in some actuarial journals to postulate so called " premiums principles" to determine the magnitude of this loading. These principles often reflect elegant mathematical analysis but in many cases their linkages to market based considerations are tenuous. For example, the standard deviation principle stipulates that the risk loading should be proportional to the standard deviation of the claim distribution. As noted by Borch (1974), this principle will not give rise to additive insurance premiums. In fact, in only a few cases will premium principles generate additive insurance premiums. Consequently many of these premium principles lead to models that are inconsistent with the no arbitrage principle.

The purpose of the present article is to examine the premiums for stop loss contracts, and in particular, to explore characteristics such as additivity and the nature of the loading, based on the models developed by Ang and Lai (1987) and Borch (1984). Within these models, the premium is set equal to the discounted value of the sum of two terms. The first term is equal to the expected claim amount and the second is a multiple of the covariance between the claim in question and the appropriate index. Given our assumptions, these premiums are additive thus precluding arbitrage.

In addition to additivity what other properties should "reasonable" stop loss premium have? It seems fundamental that the premium should exceed the expected value of the claim or the net premium. If the covariance between the claim amount under the stop loss contract and the market index is positive this will be assured - provided that the constant of proportionality is positive.

The expected claim under a stop loss contract decreases as the retention increases. It is suggested that the additional loading should also decrease as the retention limit increases. In other words, the premium minus the net premium should decrease as the retention limit increases. Finally, we are interested in situations where the ratio of the premium to the net premium

[1]Exceptions to the no arbitrage principle occasionally arise in practice. For example, in the presence of high commissions, reinsurance of 100 percent of a risk or a portfolio may produce a riskless profit if the direct underwriter receives a premium (net of expenses) greater than the reinsurance premium (net of commission) that must be paid. The authors are grateful to an anonymous referee for pointing out this exception.

increases as the retention limit increases. This last property seems to be a desirable feature of stop loss premiums.

Covariance Premium Principles

The equilibrium premium, PR[X], for a risk X in the Ang-Lai or Borch models is:

$$PR(X) = \frac{(E[X] + \lambda \, COV \, [x,y])}{(1+r)} \tag{1}$$

where y is a random variable corresponding to a "market" index. In the Borch model y represents the insurance market index, while in the Ang-Lai model it corresponds to the difference between the insurance market portfolio and the asset market portfolio. The parameter λ corresponds to the market price of risk. It is assumed $\lambda > o$. The riskless interest rate is denoted by "r". For simplicity it is assumed r is zero. The premium given by equation (1) is an increasing function of the covariance between X and y. In this note, assume that the risk X is positively correlated with y. Also assume that both X and y are positive.

Note that if the risk, X, is decomposed into two components, X_1 and X_2, where $X = X_1 + X_2$ then

$$PR[X] = PR[X_1] + PR[X_2] \tag{2}$$

This is what is meant by premium additivity and it follows because both the expectation operator and the covariance operator are linear.

First, look briefly at proportional risk sharing. Let X_1 be a constant fraction "a" of X and X_2 equal bX, where $a+b = 1$ and both a and b are positive. In this case the agent retains X_1 and cedes X_2 to an insurer. The premium for the contract is

$$PR[X_2] = bPR[X] \tag{3}$$

Splitting up the risk in this way implies that the premium is also split up in exactly the same proportion. Note that the premium for this type of contract is an increasing function of the covariance between X and y.

Consider now a stop loss contract with stop loss limit L. Let X_1 denote the distribution of the retained portion of the risk and X_2 represent the distribution of the insured portion.

$$X_1 = \text{Minimum } [X,L] \tag{4a}$$

$$X_2 = \text{Maximum } [X-L, O] \tag{4b}$$

The premium for the stop loss contract is

$$PR[X_2] = E[X_2] + \lambda COV[X_2,Y] \tag{5}$$

The three properties suggested earlier for a stop loss premium can be summarized as follows:

(i) Positive Loading: this implies $PR(X_2)$-$E(X_2)$ is positive,

(ii) Loading Decreases as Stop Loss Limit Increases:
this means that $PR(X_2)$ - $E(X_2)$ decreases as L increases,

(iii) Relative Loading Increases as Stop Loss Limit Increases: this means that the ratio

$\frac{PR(X_2)}{E(X_2)}$ increases as L increases.

The conditions on the distribution of X and Y for these three properties to hold are given in three separate theorems.

Theorem One

The premium given by equation (5) possesses Property (i) if and only if $COV[X-L)_+, Y]$ is positive

Proof: Since λ is assumed positive, the proof follows immediately from the conditions of the theorem and equation (5). The symbol $(X-L)_+$ indicates that when X exceeds L its value is $(X-L)$ and that its value is zero elsewhere.

The conditions under which Property (ii) holds are given in the next theorem.

Theorem Two

Property (ii) holds provided that:

$$E\,(Y \mid X > L) - E(Y) > 0 \tag{6}$$

and

$$\text{Prob}\,[X > L] > 0 \tag{7}$$

Proof: We note that

$$PR(X_2) - E(X_2) = (COV[X-L)_+, Y])$$

The left hand side will be a decreasing function of L if the right-hand side is positive. The derivative (with respect to L) of the right hand side is:

$$\frac{d}{dL}\{E[(X-L)_+Y]\} - \frac{d}{dL}\,d\{E[X-L)_+]E(y)\}$$

Now

$$\frac{d}{dL}\,E[(X-L)_+] = -PROB[X > L]$$

Furthermore

$$\frac{d}{dL}\,E[(X-L)_+Y] = -\text{Prob}[X > L]E(Y|X > L)$$

Hence, combining terms the derivative is

$$-\,\mathrm{PROB}[X > L]\ \{E(Y|X > L) - E(Y)\}$$

The result follows in view of equations (6) and (7)

The intuition behind equation (6) is that the expected value of Y when there is a claim $[X > L]$ is greater than the unconditional expected value of Y.

The conditions for Property (iii) are given in Theorem Three.

Theorem Three

Property (iii) holds provided that

$$\mathrm{COV}[X,Y|X > L] > 0 \tag{8}$$

and

$$\mathrm{Prob}[X > L] > 0 \tag{9}$$

First note that equation (8) is equivalent to the condition

$$\mathrm{COV}\ [(X-L)_+,\ Y|X > L] > 0 \tag{10}$$

Proof: The relative loading will be an increasing function of L provided that

$$\frac{d}{dL}\ \frac{PR[(X-L)_+]}{E[X-L)_+]}\ \text{is positive}$$

Using the results derived in proving Theorem Two it can be shown that

$$\frac{d}{dL}\ \frac{PR[(X-L)_+]}{E[(X-L)_+]} = \frac{\mathrm{PROB}\ [X > L]}{\{E[X-L)_+]\}^2}\ \mathrm{COV}\ [(x-L)_+,\ Y\,|\,X > L]$$

$$= \frac{\mathrm{PROB}[X > L]}{\{E[(X-L)_+]\}^2}\ \mathrm{COV}[(X,Y)\,|\,X > L]$$

From equations (8) and (9) the last expression is positive, and this completes the proof.

Equation (8) has an intuitive interpretation. It states that there is a positive correlation between Y and those claims that exceed the stop loss limit, L.

Concluding Remarks

This note explores some features of stop loss premiums based on recent models that have appeared in the literature. The covariance premium principle results in premiums that are additive. It was suggested that stop loss premiums should satisfy three additional properties. Some results were derived indicating the relationship between properties and the characteristics of the distributions of the random claim and the market index. It was assumed that $\lambda > 0$ and $Y > 0$. Similar results would be obtained if it had been assumed that $Y < 0$, $\lambda < 0$.

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