On the Fundamental Law of Active Portfolio Management: *How to Make Conditional Investments Unconditionally Optimal*

GUOFU ZHOU

GUOFU ZHOU
is a professor of finance at Olin Business School, Washington University, in St. Louis, MO.
zhou@wustl.edu

The key insight of the fundamental law of active portfolio management, pioneered by Grinold [1989], is to show how an active manager can transform his alpha forecasts into valued-added for his active portfolio. In particular, it states that the information ratio (IR) is proportional to the information coefficient (IC) and the square root of the market breadth (BR). Because it addresses the critical issue of value creation, the law has received enormous attention, and continues to be developed and explored as a central research topic for portfolio performance and evaluation. For example, Clarke, de Silva, and Sapra [2004, 2006] analyzed the law under various constraints, Qian and Hua [2004] studied the impact of variability in IC, and Chincarini and Kim [2007] made a distinction between the conditional and unconditional IR. Quantitative portfolio management books, such as Grinold and Kahn [2000], Chincarini and Kim [2006], and Qian, Hua, and Sorensen [2007], all devote major chapters to the law and its applications in portfolio management.

The law provides a linear strategy with active positions proportional to the forecasts. This strategy is only conditionally optimal because it is optimal each period, conditional on the information variables at that time. In practice, there can be too many conditional variables so that it is impossible for clients and third-party performance evaluators to know all of them. In addition, forecasting models are proprietary, and some of the conditional variables may come and go. Therefore, it is very difficult to conditionally assess a manager's performance. Historical returns on an active portfolio are, however, readily available, and their sample average return and variance are commonly used to assess the long-term return and risk of the active portfolio. Although returns can be affected by conditional strategies, the long-term return and risk measures are unconditional, because, intuitively, they are not impacted too strongly by any particular realization of the conditional variables. Put another way, we have only a single number for the expected return and a single number for the risk, and we do not have multiple expected returns and risks conditional on some variables being X, Y, and Z. Because the unconditional performance is often used by clients, it is the unconditional value-added—the value-added over the long haul or over multiple periods—that the manager usually strives earnestly for.

Because the law employs a trading strategy that is optimal only conditionally, the resulting value-added is not the unconditional value-added. The question we address here is that if our objective is the unconditional value-added, not the conditional value-added implicitly assumed by the law, can we find a new strategy for maximizing the unconditional objective? The next question of practical importance is whether the linear strategy of the law has good performance in terms of maximizing the unconditional value-added even though it is designed to maximize the conditional value-added.

As it turns out, we can provide a simple analytical solution to the optimal investment strategy as well as a simple formula for computing the maximum unconditional value-added. Our results draw heavily on important insights from Hansen and Richard [1987], and especially from Ferson and Siegel [2001] as well as Chiang [2008]. While Hansen and Richard were the first to analyze the problem of maximizing the unconditional objective with conditional information, Ferson and Siegel elegantly solved such a problem in terms of a conditionally efficient mean-variance frontier, and Chiang made a significant extension of Ferson and Siegel's work by adding benchmarks and by allowing various constraints. In contrast to their studies, we focus on the value-added, that is, the risk-adjusted returns or the expected utilities, and analyze the impact of the new strategy in the context of the fundamental law of active portfolio management.

We find that our new strategy smoothes the linear trading strategy over time, and in so doing, maximizes the unconditional value-added. This strategy makes substantial economic differences in obtaining a higher unconditional valued-added than what can be achieved based on the linear strategy. In fact, the effectiveness of the linear strategy depends crucially on the volatility—or more precisely, kurtosis—of the signals. If they are too volatile, say t-distributed with a high kurtosis, the unconditional valued-added based on the linear strategy can be close to zero or even negative, while the new smoothed strategy remains effective. For simplicity and for better illustration of these ideas, we will first analyze an unconstrained problem, and then discuss the usual problem, which has both a portfolio and a zero-beta restriction on the active portfolio.

UNCONSTRAINED STRATEGY

Following Grinold and Kahn [2000], we consider a managed portfolio with return R_p in excess of the risk-free rate, which is assumed to be zero without loss of generality because we are working with excess returns. Let R_B be the excess return on the benchmark portfolio, so that we have

Great plans start with great planning.

What makes a retirement plan great? A sound asset allocation strategy. Strong investment options. Ongoing monitoring to keep the plan on track. Morningstar investment research tools combine powerful analytics with comprehensive institutional data to help you make sure your plan is built on solid ground.

MORNINGSTAR®

©2008 Morningstar, Inc. All rights reserved.

$$R_p = R_B + R_A \tag{1}$$

where R_A is the return on the active portfolio. Suppose that the benchmark is a portfolio of N risky assets. We can always decompose the excess return on the i-th asset into

$$R_i = \alpha_i(z) + \beta_i R_B + \epsilon_i(z) \tag{2}$$

where $\alpha_i(z)$ is the i-th asset's alpha conditional on information or on a vector of signals z; β_i is the i-th asset's unconditional beta; and $\varepsilon_i(z)$ is the residual with zero mean conditional on z. The unconditional alphas are assumed to be zero as in Grinold and Kahn.

Denote by $\alpha(z)$ the N-vector formed by the conditional alphas of all the assets and by Σ the conditional covariance matrix of all the residuals. As in Grinold and Kahn, we assume, for notational simplicity, that Σ is constant. For any given risk-aversion parameter γ, the law, as derived by Grinold and Kahn, is to find an optimal investment strategy with portfolio weights $w(z)$ to maximize the conditional valued-added, such that

$$\max V_c(z) = \mu_A(z) - \frac{\gamma}{2}\sigma_A(z)^2 \tag{3}$$

where

$$\mu_A(z) = w(z)'\alpha(z) \quad \sigma_A(z)^2 = w(z)'\Sigma w(z) \tag{4}$$

are the conditional active portfolio expected return and variance, respectively, because $R_A = w(z)'R$ with R an N-vector of the asset returns. For the preceding maximization problem, Grinold and Kahn impose the standard constraint that the weights are orthogonal to the betas (i.e., no beta risk). For simplicity, we will analyze the unconstrained problem first, leaving the constrained problem to the next section. It is well known that Σ is, in general, a matrix of rank $N-1$, because $w_B'\Sigma = 0$ with w_B as the benchmark portfolio weights. For illustration in this section, however, we assume that Σ is nonsingular. This scenario can arise if an active manager is interested in only part of the residual assets or if the benchmark weights change over time. Nevertheless, the singularity case will not impose any fundamental difficulties for our analysis, as shown in the next section.

Under the above-simplifying assumptions, the optimal active portfolio weights, based on the first-order condition of Equation (3), are well known as

$$w_c(z) = \frac{1}{\gamma}\Sigma^{-1}\alpha(z) \tag{5}$$

and the conditional information ratio (IR), defined as the ratio of the conditional expected active return to its conditional standard deviation, is then given by

$$\text{IR}\ (z) = \sqrt{\alpha(z)'\Sigma^{-1}\alpha(z)} \tag{6}$$

The law, pioneered by Grinold [1989], states that the value-added (i.e., the risk-adjusted extra return relative to the benchmark) is, apart from a scalar of risk aversion, the expected IR squared,

$$\text{VA}_0^* = \frac{1}{2\gamma}E[\alpha(z)'\Sigma^{-1}\alpha(z)] \tag{7}$$

where the expectation is taken over all possible realizations of z, which can be any variables or factors a manager uses to forecast the alphas in practice. Grinold [1989] also intuitively explained this formula in terms of IR by relating it to the ability to perform the alpha forecasts, and provided a simple formula,

$$\text{IR} = \text{IC}\sqrt{N} \tag{8}$$

where IC is the information coefficient (skill) measured in terms of the correlation between the forecasts with the future returns, which are assumed constant across assets for purposes of the formula. The formula specifies that IR is proportional to IC and the square root of N, which here is the market breadth (BR). If the manager doubles his forecasting accuracy (IC), then IR doubles. Applying the same level of skill to a portfolio of 500 stocks will generate 10 times more value than applying it to 5 stocks. As a result, one should play well and play often (increase IC and increase BR), which is what derives the valued-added.

Let us make two remarks about the fundamental law for our purposes in this article. First, the trading strategy prescribed by Equation (5) is a linear function of the forecasts, $\alpha(z)$. If we double our forecasts across the board for the alphas, we double our position. This is consistent with our intuition that the better the odds, the more we bet. However, the strategy does not handle outliers well. Imagine that if we have 10-times larger forecasts than the usual in 1% of the time, then the linear strategy will suggest that we invest 10 times more than the usual amount in the 1% scenario. While this is reasonable given the risk at that time, the resulting position (win or lose) will look like an outlier when one assesses the long-term risk and return. As a result, from a long-term perspective, the larger bet would be worse had we kept our usual position. There is, fortunately, an optimal way to increase our position, but not as high as 10 times. These points will be clear from the analysis that follows.

The second remark is more subtle. It is about what VA_0^* really means. First, we need to distinguish between conditional and unconditional risks. It is well known that the U.S. stock market has a long-term average standard deviation of about 20%. This is the unconditional risk of the stock market, and we can measure it by using realizations of stock returns without conditioning on anything else. Now suppose that one manager can use a variable, say GDP, to predict 90% of the stock market move. Then, to this manager, the risk of the market is only 10% conditional on his use of the GDP information. This is the conditional risk. Its measurement obviously also requires the realization of GDP. For each realization of GDP, the manager has a corresponding value for risk, and hence must have multiple values of risk for varying values of GDP. In contrast, without information on GDP, another manager can only assess the unconditional risk—with a single value of risk measured perhaps using the historical sample variance—and cannot estimate VA_0^*.

Is VA_0^* the unconditional value-added because it is an unconditional number that is independent of z? The answer is no. It is the result of integrating out z from the conditional return adjusted by the conditional risk, as shown explicitly in Equation (4). If the linear investment strategy is played over and over again, VA_0^* is the average payoff of all those conditional bets. In contrast, the unconditional value-added is evaluated based on the unconditional active portfolio expected return and variance, which, for an arbitrary portfolio strategy $w(z)$, are given by

Build a strong plan foundation.

Develop a sound asset allocation strategy with Morningstar® EnCorr®

Optimal asset allocations rise out of reliable inputs and rigorous testing. The recently enhanced Morningstar EnCorr helps you construct asset allocation policies from the ground up. Financial models like Black-Litterman and Ibbotson's building blocks present options for developing and testing assumption inputs. Resampling strengthens efficient portfolios by accounting for the uncertainties of asset class expectations. And Morningstar EnCorr is one of only a few products that feature liability-relative optimization so you can consider both assets and liabilities when assessing portfolio risk. New features include risk budgeting that identifies the true risk each asset class brings a portfolio.

New version now available for trial

- +1 866 910-0840
- EnCorr@morningstar.com
- http://global.morningstar.com/EnCorrJPM

MORNINGSTAR®

©2008 Morningstar, Inc. All rights reserved.

$$\mu_A = E[R_A] = E[w(z)'\alpha(z)] \tag{9}$$

and

$$\begin{aligned}\sigma_A^2 = \mathrm{Var}[R_A] &= E[w(z)'RR'w(z)] - (E[w(z)'\alpha(z)])^2 \\ &= E(w(z)'[\alpha(z)\alpha(z)' + \Sigma]w(z)) \\ &\quad - (E[w(z)'\alpha(z)])^2\end{aligned} \tag{10}$$

respectively, where the law of the iterated expectations and the standard zero-correlation assumption between the alphas and residuals are used to obtain the second expression for $\mathbf{s}_A^2$, which will be useful later.

Another way to understand the difference between the conditional and unconditional value-added is to assess VA_0^* from the viewpoint of estimation. Suppose we have time-series observations on returns of the active portfolio, $R_{A1}, R_{A2}, \ldots, R_{AT}$, where T is the sample size. The sample mean and variance are then easily computed, as are the sample Sharpe ratio and the risk-adjusted return. The key point is that the sample mean and variance will approximate μ_A and $\mathbf{s}_A^2$, respectively. Hence, the risk-adjusted return should approximate the unconditional value-added function,

$$V = \mu_A - \frac{\gamma}{2}\sigma_A^2 \tag{11}$$

This differs from the conditional objective, Equation (3), in that it is the unconditional mean and variance that are used to compute the risk-adjusted return.

In practice, the signals z and predictive models are usually well guarded by managers and are likely unknown to their clients and outsiders. The performance of managers is often, and perhaps can only be, solely assessed based on the returns of their portfolio(s). This is because without z or $\mathbf{a}(z)$, it is impossible for the clients and third-party performance evaluators to assess the conditional returns and conditional risks of the active portfolio. As a result, the tradeoff between the unconditional mean return and the unconditional risk is a decision that most, if not all, active portfolio managers must make. The strategy of the law, however, does not optimize this unconditional tradeoff, but optimizes the value-added conditional on z by solving the associated maximization problem, Equation (3). Therefore, the value-added delivered by the law is a performance measure of the conditional objective, not the unconditional objective, which is the focus of this article.

An active manager can use information for both conditional and unconditional objectives. A general trading strategy is a function of the signals z, so the task of the manager is to choose an investment strategy, such as portfolio weights $w(z)$, in order to maximize the unconditional value-added given by Equation (11). Mathematically, the maximization problem is variational because the manager needs to find a function of the signals z rather than a set of parameters. This is in general a challenging task.

Fortunately, Ferson and Siegel [2001] provided an analytical solution to the conditional mean-variance efficient frontier. The utility-maximizing solution we seek must be on that frontier. Based on Ferson and Siegel's analysis, we guess a solution

$$w_u(z) = \frac{k}{\gamma}[\alpha(z)\alpha(z)' + \Sigma]^{-1}\alpha(z) \tag{12}$$

where k is a constant. Plugging this into Equation (11), we immediately obtain the optimal k, that is,

$$k^{-1} = E\frac{1}{1 + \alpha(z)'\Sigma^{-1}\alpha(z)} \tag{13}$$

The solution can alternatively be derived with Chiang's [2008] general approach based on the calculus of variations that can easily solve such an unconstrained problem as well as problems with multiple constraints.[1] Note that k is constant regardless of the signals and that we need only the distributional assumption for z to compute it. Once this is done, the same k can be applied each time for all possible realizations of z. In contrast with the earlier strategy $w_c(z)$, and apart from k, we now add the $\mathbf{a}(z)\mathbf{a}(z)'$ term into the Σ matrix, making the strategy highly nonlinear in the alpha forecasts. But computationally $\mathbf{a}(z)\mathbf{a}(z)' + \Sigma$ is as easy to invert as the previous Σ. Hence, to maximize the unconditional risk-adjusted return, $w_u(z)$ can be easily implemented each time an active manager has signals z.

To gain further economic understanding of the new solution, we rewrite $w_u(z)$, using a standard inverse formula for patterned matrices as

$$w_u(z) = \Pi(z)\frac{1}{\gamma}\Sigma^{-1}\alpha(z) \tag{14}$$

where

$$\Pi(z) = \frac{1/(1+\alpha(z)'\Sigma^{-1}\alpha(z))}{E[1/(1+\alpha(z)'\Sigma^{-1}\alpha(z))]} \tag{15}$$

This says that $w_u(z)$ is as diversified as $w_c(z)$ because it is proportional to $w_c(z)$. It differs, however, from $w_c(z)$ in an important way by the scale function $\Pi(z)$. There are two important and intuitive explanations. First, the expectation of $\Pi(z)$ is one, which means that the new strategy, on average, takes approximately the same positions as before. The second explanation, which drives the real difference, is that the z-dependent scalar is relatively small when $\mathsf{a}(z)$ is large and relatively large when $\mathsf{a}(z)$ is small. If it is larger than one, we bet more aggressively than the traditional linear strategy, and if it is less than one, we bet less aggressively. By definition, the denominator of $\Pi(z)$, or k, is a measure of the variability in the alpha forecasts. Hence, the new strategy suggests that we should bet more aggressively if the current alpha forecasts are less variable than their long-term average. In contrast, the linear strategy does not consider the current alpha values relative to its long-term values, and simply bets linearly with them. In short, $w_u(z)$ essentially suggests that we smooth our bets over time to have better risk and return profiles in the long term.

The valued-added with the new optimal investment strategy can also be written as a function of $\Pi(z)$ such that

$$\mathrm{VA}^* = \frac{1}{2\gamma}E[\Pi(z)\alpha(z)'\Sigma^{-1}\alpha(z)] \tag{16}$$

This explicit formula for VA^* has an interesting economic interpretation. Previously, the value-added was a sum of the conditional value-added across the states of nature or the realization of the signals, but now, is a weighted sum of the conditional value-added with weighting function $\Pi(z)$. The weighting comes into play because the strategy itself is now weighted. Because VA^* is equal to the unconditional IR squared (apart from the risk-aversion con-

Construct a plan with the right investments.

Select and monitor investment managers with Morningstar Direct℠

Your search begins with access to continually updated data covering more than 260,000 global investments. Morningstar Direct delivers data on separate accounts, collective investment trusts, hedge funds, mutual funds, and other securities. Its sophisticated tools and depth of data help you conduct style and peer analysis to identify top managers in each asset class. After selecting the best managers, you can monitor their performance over time to make sure all investments remain consistent with your plan's goals. Morningstar tools help you create a well-constructed retirement plan—one that's always as sound as the day it was built.

Take a free trial

- +1 866 910-0840
- MorningstarDirect@morningstar.com
- http://global.morningstar.com/DirectJPM

MORNINGSTAR®

©2008 Morningstar, Inc. All rights reserved.

stant), the same interpretation for V_0^* also applies to the maximum of the unconditional IR.

For easy computation, we can write

$$\mathrm{VA}^* = \frac{1}{2\gamma}(k-1) \tag{17}$$

so the key is to determine k. In practice, our forecasts are usually linear functions of signals or risk factors. In this case, as in Grinold and Kahn [2000], we can assume $\mathbf{a}(z) = Az$, where A is an $N \times m$ matrix of loadings on z, and z is an m-vector of signals standardized to have zero mean and unit standard deviation. Using the common normality assumption on z, one can show that VA* is an increasing function of BR. Moreover, using an asymptotic expansion, it is easy to see that the earlier intuition "play well and play often" also holds true. The exact value of k can be computed based on the following integral:

$$k^{-1} = \int_0^\infty \frac{e^{-t}}{\Pi_{i=1}^m (1+2t\lambda_i)^{\frac{1}{2}}} \mathrm{dt} \tag{18}$$

where $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_m > 0$ are the eigenvalues of $A' \Sigma^{-1} A$. Alternatively, k can be computed from its definition via Monte Carlo simulation. Once the value of k is determined, the strategy can be implemented each period, and then the unconditional value-added is computed from Equation (17). It is of interest to note that the maximum value-added depends only on the eigenvalues of $A' \Sigma^{-1} \mathrm{A}$. In other words, no matter how different the factor loadings and risk matrices are in two different applications, they yield the same value-added as long as the eigenvalues of their respective quadratic products are the same.

The important economic question of practical interest is whether the new VA* is substantially higher than what can be achieved based on the traditional linear strategy $w_c(z)$.[2] To answer this question, consider a special case in which we have a single signal z each period ($m = 1$). In this case, letting $q^2 = A'\Sigma^{-1}A$, we can write

$$k^{-1} = E\frac{1}{1+z^2q^2} \tag{19}$$

which can be analytically evaluated as $\Phi(-1/q)(q\mathrm{f}(1/q))$ with use of the distribution function, Φ, and density function, f, of the standard normal random variable.[3] Thus, it is clear that VA* is a function of q alone. The greater the q, the greater the k, and so the greater the VA*.

To determine the value-added of the traditional strategy, $w_c(z)$, we need to plug it into the unconditional utility function, Equation (11), and then we obtain

$$\mathrm{VA}_c^* = \frac{1}{2\gamma}(q^2 - 2q^4) \tag{20}$$

which is the unconditional valued-added achieved by using $w_c(z)$. It is evident that VA_c^* is a quadratic function of q^2 and obtains its maximum at $q^2 = 25\%$. This value seems high in practice since it implies a valued-added of $25/(2 * \mathrm{g}) = 4.17\%$ when g takes a typical value of 3. We are mainly interested in comparing VA* and VA_c^* in a smaller range of values for q^2, say $q^2 = 6\%$ to 14%, which are more likely to occur. Such a comparison is difficult to do analytically, but easy numerically.

Ignoring the risk-aversion dependent scalar, $1/2\mathrm{g}$, Panel A of Exhibit 1 reports both VA* and VA_c^*, as well as their percentage differences, when q^2 = 6%–14% at 2% increments. When q^2 = 6%, VA* and VA_c^* are 5.4% and 4.6%, respectively, with a percentage difference of 19.2%. As q^2 increases, so does the difference. When q^2 = 14%, VA* and VA_c^* are 11.5% and 6.2%, respectively. Their difference is more than 5%, which is clearly significant. The percentage difference now is also impressive with a value of 87.1%. Note that VA* and VA_c^* are unconditional expected risk-adjusted returns per period. If the strategies are played 12 times a year, the effects will be compounded 12 times. Therefore, the differences will have an even greater impact when considered on an annual basis.

The next comparison is more striking and examines the impact of kurtosis. Intuitively, the greater the smoothing, the greater the gains of the unconditional optimal strategy over the conditional strategy. Moreover, the more outliers of the signals, the more important the smoothing. This intuition can be further explained analytically. Suppose that z has an arbitrary distribution with excess kurtosis κ. Plugging the linear strategy $w_c(z)$ into the unconditional utility function, we obtain

$$\mathrm{VA}_c^* = \frac{1}{2\gamma}\left[q^2 - (\kappa+2)q^4\right] \tag{21}$$

To be specific, assume the signal z is t-distributed with six degrees of freedom (df), which fits many returns. If $\kappa = 3$, it implies that z has many outliers relative to the normally distributed signal and so does the associated linear strategy $w_c(z)$. Panel B of Exhibit 1 provides the same comparison as Panel A except the signal distribution is replaced by t. When q^2 is 6% or 8%, the results are not much different. However, when q^2 = 12%, the percentage gain jumps to 91.4%, and when q^2 = 14%, it rises to 149.4%. Quantitatively, the higher the q^2 the greater the decrease in VA_c^*, as indicated by Equation (21). In fact, when q^2 goes higher than 20%, the problem is severe. For example, at a value of q^2 = 20%, VA_c^* is exactly zero, and the percentage gain is infinity; that is, there is no value whatsoever in using the conditional strategy when the unconditional risk-adjusted return is concerned. When $q^2 > 20\%$, VA_c^* is negative, implying that there is unconditional value destruction in using VA_c^*. In other words, the manager is better off holding the risk-free asset. This finding is quite robust as VA_c^* depends only on the level of kurtosis, regardless of any distributional assumptions. Note that VA^*, computed by the same formula used in the preceding example, is also affected by κ via its impact on the valuation of k (computed now by simulation), but the effects are small.

Clearly, the greater the kurtosis, the smaller the q^2 that is needed to make VA_c^* zero. In practice, many, perhaps most, useful factors do have high kurtosis. For example, Tu and Zhou [2004] showed that many small size portfolios and even well-diversified portfolios can possess an excess kurtosis of 3 or greater, and some factors have kurtosis close to 50! Hence, the implication of the kurtosis problem is highly relevant and is of significant practical importance. In summary, our results suggest that extreme care must be taken when using volatile alpha forecasts or highly kurtotic signals.

In short, we find that the linear strategy does not maximize the unconditional value-added. When the forecasts or signals have a high kurtosis, the problem worsens significantly. In performance evaluation, it is theoretically debatable whether the conditional value-added or unconditional value-added should be used to judge managers. However, due to its easy computation, the unconditional value-added is widely used in practice. Given the unconditional measure, we show that there can be significant underperformance with the use of the conditionally optimal strategy. Conditional information can, however, be used to maximize the unconditional value-added, and we provide the optimal strategy to do so.

CONSTRAINED STRATEGY

After understanding the strategy in the simpler, unconstrained problem, we are now ready to analyze the exact problem posed by Grinold and Kahn [2000] with two extra constraints. As mentioned previously, the first constraint is that all the assets in the benchmark index are used to form the active portfolio, which implies both $w_B'\alpha(z) = 0$ and $w_B'\Sigma = 0$ with w_B as the benchmark portfolio weights. The second constraint is $w(z)'\mathbf{b} = 0$ so that the active portfolio does not have any benchmark risk. Interestingly, following Appendix 2A of Chincarini and Kim [2006], we can transform the constrained problem into an unconstrained problem in this context, so that both the solution and insights from the unconstrained problem can be readily applied to the constrained problem.

EXHIBIT 1
Differences in Valued-Added

$q^2 = A'\Sigma^{-1}A$	0.06	0.08	0.10	0.12	0.14
	Panel A: Normal-Distributed Signal				
VA^*	0.054	0.071	0.086	0.101	0.115
VA_c^*	0.046	0.054	0.060	0.062	0.062
Diff (%)	0.192	0.297	0.434	0.617	0.871
	Panel B: t-distributed Signal with df = 6				
VA^*	0.051	0.065	0.079	0.092	0.105
VA_c^*	0.042	0.048	0.050	0.048	0.042
Diff (%)	0.211	0.358	0.580	0.914	1.494

Let w_{B1} and w_{B2} be the first $N-1$ and N-th components of w_B, respectively, and define $\mathbf{b}_1$ and $\mathbf{b}_2$ similarly. Without loss of generality, we assume both w_{B2} and $\mathbf{b}_2$ are nonzero, and then $c = w_{B1}/w_{B2}$ and $d = \mathbf{b}_1/\beta_2$ are well defined $N-1$ vectors. Then, we can show that

$$w(z)'[\alpha(z)\alpha(z)' + \Sigma]w(z) = w^*(z)'[\alpha_1(z)\alpha_1(z)' + \Sigma_{11}]w^*(z) \tag{22}$$

where

$$w^*(z) = (I_{N-1} + cd')w_1(z) \tag{23}$$

and both $\mathbf{a}_1$ and Σ_{11} are the first $N-1$ component of $\mathbf{a}$ and the first $N-1$ submatrix of Σ, respectively. We can also verify that $w(z)'\mathbf{a}(z) = w^*(z)'\mathbf{a}_1(z)$. Note that the active portfolio expected return and variance, Equations (9) and (10), are always true regardless of any constraints. Because of our simplifications, they are determined solely by the variables from the first $N-1$ assets.[4] Therefore, all of our formulas in the previous section follow except that now $\mathbf{a}_1(z)$ plays the role of the previous $\alpha(z)$, $\mathbf{a}_1(z)\mathbf{a}_1(z)' + \Sigma_{11}$ plays the role of the previous $\mathbf{a}(z)\mathbf{a}(z)' + \Sigma$, and so forth. In particular, we obtain the maximum unconditional valued-added

$$\text{VA}^* = \frac{1}{2\gamma}(k_1 - 1) \tag{24}$$

with

$$k_1^{-1} = E\frac{1}{1 + \alpha_1(z)'\Sigma_{11}^{-1}\alpha_1(z)} \tag{25}$$

Because there are no more redundant assets, the rank of Σ_{11} is $N-1$, and so it must be invertible. This avoids the earlier nonsingularity assumption. Moreover, the formula for the optimal new strategy, $w^*(z)$, still takes the same form, which is based on backing out $w_1(z)$ from Equation (23). For the traditional linear trading strategy, an expression similar to that for $w_c(z)$ holds because of Chincarini and Kim's [2006] result. Therefore, all of our conclusions in the previous section must carry through to the constrained problem.

CONCLUSION

The fundamental law of active portfolio management pioneered by Grinold [1989] tells an active manager how to transform his alpha forecasts into valued-added for his active portfolio by using a linear strategy with active positions proportional to the forecasts. This linear strategy is conditionally optimal because it is optimal each period, conditional on the forecasts at that time. However, the unconditional value-added (the valued-added over the long haul or over multiple periods) is usually what the manager strives earnestly for. Under this unconditional objective, the linear strategy can generate zero unconditional value-added or even a negative value if the forecasts or signals have a high kurtosis. To overcome this problem, we provide an investment strategy that maximizes the unconditional value-added with the optimal use of conditional information, with the help of important insights from Hansen and Richard [1987], and especially from Ferson and Siegel [2001], as well as Chiang [2008]. Our strategy is nonlinear in the forecasts, but has a simple economic interpretation. When the alpha forecasts are high, we invest less aggressively than the linear strategy, and when the forecasts are low, we invest more aggressively. In this way, we tend to smooth our value-added over time and, hence, on a risk-adjusted basis, our long-term unconditional value-added will in most cases be substantially higher than that based on the linear strategy, particularly when the alpha forecasts experience high kurtosis.[5]

ENDNOTES

The author is grateful to I-Hsuan (Ethan) Chiang, Ludwig Chincarini, Wayne Ferson, Campbell Harvey, Daehwan Kim, Hong Liu, Edward Qian, Farris Shuggi, Jack Strauss, and especially Raymond Kan for helpful discussions and comments.

[1]It should be noted that Ferson and Siegel [2001] also discuss utility maximization, but their utility contains only the second moments (not the variance), and hence their utility solution (in their appendix) differs from ours. But our solution is on their explicitly solved frontier, which is the focus of their article. One can verify that our solution does maximize the unconditional expected utility either mathematically by Ferson and Siegel's novel method or by Monte Carlo simulations.

[2]Chiang [2008] is the first to apply Ferson and Siegel's [2001] methodology to the benchmark case. In the context of tracking error, Chiang provides via simulation an interesting comparison

between conditionally efficient tracking error and unconditional.

[3]The author is grateful to Raymond Kan for this and Equation (18).

[4]This result can be extended to allow $K < N$ linear constraints and can be alternatively proven with the use of the generalized inverse of a matrix.

[5]The optimal use of conditional information for unconditional utility gains has a number of applications. In particular, the metric is useful for assessing the value of market timing strategies, predictive variables, and conditional models. Our studies here, however, like most others, ignore parameter estimation errors and the related robust portfolio constructions that are getting increasing attention (see Fabozzi, Kolm, Pachamanova, and Focardi [2007]). While the impact of estimation errors on the law is well understood for the linear strategy based on Zhou [2008], little is known about the more challenging nonlinear strategy. These are interesting issues for future studies and are beyond the scope of this article.

REFERENCES

Chiang, I-Hsuan E. "Modern Portfolio Management with Conditioning Information." Working paper, Boston University, 2008.

Chincarini, Ludwig B., and Daehwan Kim. *Quantitative Equity Portfolio Management.* New York: McGraw-Hill, 2006.

——. "Another Look at the Information Ratio." *Journal of Asset Management*, Vol. 8, No. 5 (2007), pp. 284–295.

Clark, Roger, Harindra de Silva, and Stephen Thorley. "Portfolio Constraints and the Fundamental Law of Active Management." *Financial Analysts Journal*, Vol. 58, No. 5 (2002), pp. 48–66.

——. "The Fundamental Law of Active Management." *Journal of Investment Management*, Vol. 4, No. 3 (2006), pp. 54–72.

Fabozzi, Frank J., Petter N. Kolm, Dessislava Pachamanova, and Sergio M. Focardi. *Robust Portfolio Optimization and Management.* New York: Wiley, 2007.

Ferson, Wayne E., and Andrew F. Siegel. "The Efficient Use of Conditioning Information in Portfolios." *Journal of Finance*, Vol. 56, No. 3 (2001), pp. 967–982.

Grinold, Richard C. "The Fundamental Law of Active Management." *The Journal of Portfolio Management*, Vol. 15, No. 3 (1989), pp. 30–37.

Grinold, Richard C., and Ronald N. Kahn. *Active Portfolio Management*, 2nd ed. New York: McGraw-Hill, 2000.

Hansen, Lars P., and Scott F. Richard. "The Role of Conditioning Information in Deducing Testable Restrictions Implied By Dynamic Asset Pricing Models." *Econometrica*, Vol. 55, No. 3 (1987), pp. 587–613.

Qian, Edward, and Ronald Hua. "Active Risk and Information Ratio." *Journal of Investment Management*, Vol. 2, No. 3 (2004), pp. 20–34.

Qian, Edward, Ronald Hua, and Eric Sorensen. *Quantitative Equity Portfolio Management: Modern Techniques and Applications.* New York: Chapman & Hall, 2007.

Sorensen, Eric, Edward Qian, Robert Schoen, and Ronald Hua. "Multiple Alpha Sources and Active Management." *Journal of Portfolio Management*, Vol. 30, No. 2 (2004), pp. 39–45.

Tu, Jun, and Guofu Zhou. "Data-Generating Process Uncertainty: What Difference Does It Make in Portfolio Decisions?" *Journal of Financial Economics*, Vol. 72, No. 2 (2004), pp. 385–421.

Zhou, Guofu. "On the Fundamental Law of Active Portfolio Management: What Happens if Our Estimates Are Wrong?" *Journal of Portfolio Management*, Summer 2008, pp. 26–33.

To order reprints of this article, please contact Dewey Palmieri at dpalmieri@iijournals.com or 212-224-3675.

©Euromoney Institutional Investor PLC. This material must be used for the customer's internal business use only and a maximum of ten (10) hard copy print-outs may be made. No further copying or transmission of this material is allowed without the express permission of Euromoney Institutional Investor PLC. Source: Journal of Portfolio Management and http://www.iijournals.com/JPM/Default.asp