ON THE VALUATION OF WARRANTS

JOHN C. HANDLEY

This article addresses a misconception in the literature concerning the valuation of warrants when a warrant is treated as an option on the stock of the underlying firm. The magnitude and timing of the impact of a warrant issue on the underlying stock price and on the wealth of the firm's shareholders is examined within a continuous-time arbitrage-free economy. In particular, it is shown that the stock price of the underlying firm conditionally reflects dilution at all times following the announcement of a warrant issue and notwithstanding that the warrants might not even have been issued yet. Valuing a warrant or convertible security as an option on the post-announcement underlying stock price means there is no need for any *explicit* adjustment for dilution to be made to the chosen option pricing model. © 2002 Wiley Periodicals, Inc. Jrl Fut Mark 22:765–782, 2002

INTRODUCTION

A warrant is a call option written by a corporation on its own common stock. When a warrant is exercised, new shares are issued and the exercise proceeds, if any, are paid into the firm. As a result, the wealth of

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the firm's nonparticipating shareholders may be diluted.[1] The source of this dilution is the game that is played between the warrantholders and the shareholders. Specifically, the rational exercise of a warrant represents a wealth gain to its holder and an equivalent loss to its writer, or, more precisely, there is a wealth transfer from the shareholders to the warrantholders.

The concept of dilution is well known and traditionally has been examined in the context of valuing a warrant compared to valuing an otherwise equivalent call option. Three consistent and equally valid approaches have been developed. The first, originally suggested by Black and Scholes (1973), and subsequently developed by Smith (1977) and Lauterbach and Schultz (1990), is to price the warrant as an option on the equity of the underlying firm, being the aggregate value of its shares and the warrants. The second, due to Galai and Schneller (1978), is to price the warrant as an option on the stock of an otherwise identical firm without warrants. The third, again due to Galai and Schneller (1978), is to price simply the warrant as an option on the stock of the underlying firm. All three approaches are general in the sense that they are not restricted to any particular option-pricing model such as Black–Scholes.

Unfortunately, a misconception concerning the dilution effect and the third approach continues to appear in the literature. Specifically, a number of authors inappropriately have suggested that dilution is ignored if a warrant is priced simply as a call option on the stock of the underlying firm. For example, Schulz and Trautmann (1994, p. 850) stated that, "Several studies on warrants . . . have ignored the dilution effect and valued the warrant as an otherwise identical call option on common stock of a firm without warrants." According to Hauser and Lauterbach (1996, p. 72), valuing warrants with the Black–Scholes option-pricing model "ignores the dilution effect of warrant exercise." Brealey and Myers (1996, p. 626) stated, "In practice investors often ignore dilution and calculate conversion value as the share price times the number of shares into which the bonds can be converted." Gemmill and Thomas (1997, p. 34) stated that valuing a warrant using the underlying stock price "ignores the dilution of V [the value of the all equity firm]."[2] Others

[1]If the firm is financed by other than stock and warrants, then the holders of the other corporate securities also may be subject to dilution.

[2]Note that the problem here is limited to this one aspect and certainly not with these articles in their entirety. For example, Schulz and Trautmann (1994) and Hauser and Lauterbach (1996) correctly took dilution into account in pricing a warrant as a call on the equity of the underlying firm, that is, in using the first of the three approaches mentioned above.

have gone even further and incorrectly made an explicit adjustment for dilution.[3]

An alternative view, and the one that is consistent with the Efficient Market Hypothesis, is that pricing a warrant as a call option on the stock of the underlying firm does not actually ignore dilution because dilution already should be reflected in the underlying stock price before expiration of the warrant.[4]

The contribution of this article is to dispel this misconception by presenting a rigorous theoretical proof of the proposition that the stock price of the underlying firm conditionally reflects dilution at all times following announcement of a warrant issue and notwithstanding that the warrants might not even have been issued yet. The assumed framework is a continuous-time arbitrage-free economy that allows the underlying stock price to follow one of a very wide class of stochastic processes. The key implication of this result is that using the (post-announcement) stock price of the underlying firm as the state variable to value a warrant in fact does not ignore dilution, but rather it simply obviates the need for any explicit adjustment for dilution to be made to the chosen option-pricing model.

The outline of the article is as follows. The next section describes the setup for the model. In the following section, the impact of a free issue of warrants on the underlying stock price and shareholder wealth is examined. This is followed by an extension of the model to allow for proceeds to be raised at the time of issue. A number of implications then are discussed. A conclusion is set out in the final section. All proofs are set out in the appendix.

THE SETUP

The setup for the analysis is based on the standard security-market model with continuous trading of Harrison and Pliska (1981). The time horizon is fixed at $T < \infty$ and the set of trading dates is $t \in [0, T]$. Uncertainty in the economy is modeled by a filtered probability space $(\Omega, \Im, \mathbf{F}, P)$. The filtration $\mathbf{F} = \{\Im_t \in [0, T]\}$ represents how information is revealed over time where $\Im_t$ is the accumulated information commonly available to investors at time t and $\mathbf{F}$ is assumed to satisfy the "usual conditions."

[3]See, for example, Kremer and Roenfeldt (1992).

[4]See, for example, Leonard and Solt (1990), Crouhy and Galai (1991), and Bensoussan, Crouhy, and Galai (1995).

The following assumptions are made:

i. There are two securities available for trade. First, let $V = \{V_t; t \in [0, T]\}$ be the exogenously determined asset-price process for a firm that is assumed to be a semimartingale and strictly positive, where V_t represents the asset value of the firm at time t. Then V also is right continuous with left limits ("RCLL") and adapted to the filtration F meaning V_t is $\Im_t$ – measurable for each time t.[5] To allow for the possibility of interim payouts on V during $t \in [0, T]$, let $v = \{v_t; t \in [0, T]\}$ be the associated cumulative dividend process, assumed to be RCLL, adapted and of finite variation where v_t represents the cumulative dividend on V up to and including time t. Given the possibility of jumps, V is defined unambiguously on an ex-dividend basis such that V_t represents the asset value of the firm at time t but excluding (after) any dividends paid at that time. Let $G^V = \{G_t^V; t \in [0, T]\}$ be the gain process for the asset price-dividend pair where $G_t^V = V_t + v_t$. Because V is a semimartingale and v is of finite variation, then G^V is also a semimartingale. Second, let $\beta = \{\beta_t; t \in [0, T]\}$ be the price process for a bond that is assumed to be continuous and deterministic such that the bond price at time t is $\beta_t = \exp(\int_0^t r_s\, ds)$ where r_t is the riskless interest rate at that time.

ii. There exists a probability measure Q equivalent to P such that the discounted gain process for the asset price-dividend pair is a martingale under Q. Then the arbitrage-free discounted asset price process, at any time $t \in [0, T)$ is given by:

$$\beta_t^{-1} V_t = E\left[\beta_T^{-1} V_T + \int_{(t,T]} \beta_u^{-1} dv_u \,\middle|\, \Im_t\right] \tag{1}$$

where E is the expectation operator under the martingale measure Q and dv_u is the "incremental" dividend paid at time u.[6]

iii. At time 0, the firm has on issue n shares of stock and no debt. Let $S = \{S_t; t \in [0, T]\}$ be the stock-price process defined on an

[5]Unlike a continuous process, an RCLL process may exhibit jump discontinuities. Specifically, let Z be an RCLL process where Z_t is the value of Z at time t and Z_{t-} is the left limit of Z at time t. Then the jump in Z at time t is $\Delta Z_t = Z_t - Z_{t-}$.

[6]Equation (1) similarly applies to any attainable $\Im_T$-measurable contingent claim with price-dividend pair (F, f) where f is adapted, RCLL, and of finite variation. See Chapter 6 of Duffie (1996) for further details.

ex-dividend basis, with associated cumulative dividend process, s, and associated gain process for the stock-price-dividend pair, G^S. In the absence of dividends, each share of stock contractually entitles its holder to an equal proportion of the asset value of the firm at time T. Accordingly, the stock initially may be treated as an $\Im_T$ − measurable contingent claim on the assets of the firm with payoff:

$$S_T = \frac{V_T}{n} \tag{2}$$

iv. At time $\alpha \in (0, T]$, the firm makes an announcement that it will for certain, at some time $t \in [\alpha, T]$, issue m European warrants, each of which will entitle its holder to buy γ shares of stock at an exercise price of X per share at time T. None of the firm's shareholders participate in the issue.

The announcement of the warrant issue implies a possible jump in both the number of shares on issue, $N = \{N_t; t \in [0, T]\}$ and in the asset value of the firm, V, at time T. For all times strictly before maturity, the number of shares is fixed at $N_t = n$, whereas the number of shares at maturity depends upon whether the warrants are exercised:

$$N_T = N_{T-} + m\gamma I_{\{\cdot\}} = n + m\gamma I_{\{\cdot\}} \tag{3}$$

where $I_{\{\cdot\}}$ is the indicator function for the event "the warrants are exercised."[7] Similarly, and in the absence of any new information at time T, the asset value of the firm at maturity is:

$$V_T = V_{T-} + m\gamma X I_{\{\cdot\}} \tag{4}$$

Equation (4) provides for the possible payment of cash into the firm on exercise of the warrants.[8]

Because each share of stock entitles its holder to an equal proportion of the asset value of the firm at time T, then equivalently each warrant entitles its holder to buy a $\gamma/(n + m\gamma)$ proportion of the asset value of the firm at time T for an exercise price of γX. Let $W = \{W_t; t \in [0, T]\}$ be the warrant price process defined on an ex-dividend basis with associated

[7] Because the warrants are European, then the rational exercise of one warrant implies the rational exercise of all warrants simultaneously.

[8] The exercise proceeds may be treated as a (possible) negative dividend paid by the firm with the cumulative dividend process for the firm equal to $v_t = -m\gamma X I_{\{\cdot\}}$ for $t = T$, and zero otherwise.

cumulative dividend process, w, and associated gain process for the warrant price-dividend pair, G^W. Each warrant is an $\Im_T$ − measurable contingent claim on the assets of the firm with payoff:

$$w_t = \max\left[0, \gamma\left(\frac{V_T}{n + m\gamma} - X\right)\right] \tag{5}$$

for $t = T$ and zero otherwise, and:

$$W_T = 0 \tag{6}$$

Equation (6) reflects the fact that the warrants cease to exist at maturity, that is, either they lapse or they are exchanged for stock. In the absence of arbitrage, Eqs. (5) and (6) may alternatively be expressed as:

$$W_{T-} = \frac{\gamma(V_{T-} - nX)}{n + m\gamma} I_{\{\cdot\}} \tag{7}$$

Equation (7) shows the condition for optimal exercise is $V_{T-}/n > X$, which is an $\Im_{T-}$ − measurable event.[9] Because V is exogenous, then the exercise decision is independent of the number of warrants exercised. For the purposes of Eq. (3), Eqs. (4) and (7) interpret $I_{\{\cdot\}}$ as the indicator function for the event $V_{T-}/n > X$.

Importantly, the announcement of the warrant issue causes the boundary condition for the stock to change, nontrivially, from (2) to:

$$S_{T-} = \frac{V_{T-} - mW_{T-}}{n} \tag{8}$$

and

$$S_T = \frac{V_T}{N_T} \tag{9}$$

The only securities with a claim on the assets of the firm immediately before maturity are the stock and the warrants which leads to Eq. (8). Because the warrants cease to exist at maturity, the only security with

[9]By Eq. (4), V_T is then $\Im_{T-}$-measurable rather than $\Im_T$-measurable. Because V is the sole source of uncertainty in the economy, this is equivalent to assuming the filtration F is left continuous at maturity, that is, $\Im_{T-} = \Im_T$.

a claim on the assets of the firm at time T is the stock which leads to Eq. (9).

WHEN THE WARRANTS ARE ISSUED FOR FREE

In order to isolate the dilutionary impact of the warrant issue on the underlying stock price, assume the firm receives no proceeds from the issue, there is no asymmetric information concerning the underlying asset value of the firm, and, for simplicity, the announcement and issue dates coincide. In this case, the asset-price process is continuous at the time of announcement. The assumption that the warrants are issued for nil consideration temporarily avoids complications that arise from having to decide what is to be done with the issue proceeds. Working backwards, the first point in time to be considered is the maturity date of the warrants.

Proposition 1. *In the absence of arbitrage, the stock-price process S is continuous at the warrant maturity date T.*

Wilmott, Dewynne, and Howison (1993) and Sidenius (1996), amongst others, previously have noted this result. The important implication is that the stock price does not *wait* until maturity to adjust for dilution. So, when does it (first) adjust?

Proposition 2. *Let time α be the first announcement date of the warrant issue. Let S_α and W_α be the price of the stock and warrant at time α and $S_{\alpha-}$ be the left limit of S at time α. Then, in the absence of arbitrage, there is a jump in the stock price at the time of announcement such that:*

$$S_\alpha = S_{\alpha-} - \frac{m}{n} W_\alpha \tag{10}$$

A number of remarks are in order. First, the above result is consistent with Huang (1985a, 1985b) and the Efficient Markets Hypothesis.[10] The announcement of the warrant issue represents new information or an *information surprise* in relation to the contingent claim S, and therefore leads to a jump discontinuity in S. However, not only does Eq. (10) identify the *timing* of the jump, but it also specifies the

[10]Motivated by empirical tests of the Efficient Markets Hypothesis, Huang (1985a, 1985b) mathematically formalized the relationship between information flow and equilibrium asset prices.

magnitude of the jump. Second, the decline of $\frac{m}{n}W_\alpha$ in the underlying stock price at the time of announcement may be interpreted as an instantaneous adjustment for dilution on a per-share basis. To see this note that the price of a warrant W_α represents the ($\Im_\alpha-$ conditional) expected discounted payoff on the warrant at maturity, calculated under the martingale measure Q. Because the payoff on the warrant represents the transfer of wealth from the shareholders to the warrantholders that arises from the exercise of the warrant, then W_α may be interpreted as the expected discounted value of this dilution. The key points are that stock-price dilution arises *from* rather than *upon* exercise and that the stock price *first* adjusts for this at the time the issue is announced. Third, proposition 2 represents a generalization of a result originally derived by Galai and Schneller (1978) within a simple one-period model framework. Fourth, if the warrants are issued on a date strictly after announcement, then Eq. (10) still holds. In this case, it readily can be shown that the stock price first adjusts for dilution at the time of announcement notwithstanding that the warrants are not issued until a later date. In the absence of any further information surprises, the stock-price process then will be continuous on the issue date.

Proposition 3. *At all times following the announcement of the warrant issue and before maturity, the stock price conditionally reflects dilution at that time.*

Proposition 3 represents a generalization of a result of Crouhy and Galai (1991) and may be expressed as, for all $t \in [\alpha, T)$, the arbitrage free price of the stock is:

$$S_t = \frac{1}{n} V_t - \frac{m}{n} W_t \tag{11}$$

Clearly Eq. (11) is *very* well known—the obvious interpretation being the commonly adopted balance-sheet-like definition that at any point in time the asset value of the firm equals the aggregate value of all of the firm's corporate securities; however, its importance in the context of this article is more subtle. Specifically, because W_t represents the expected discounted value of dilution, then the relevant interpretation of Eq. (11) here, is that the post-announcement stock price of the firm reflects the underlying asset value conditionally adjusted for dilution, on a per-share basis. Combining the above three propositions, the stock price first adjusts for dilution at the time the issue is first announced. Thereafter, and in the absence of any further information surprises, the stock price is continuous, meaning that as new information is revealed continuously, the stock price is updated conditionally for dilution on a continuous basis over time.[11,12]

WHEN THE WARRANTS ARE ISSUED FOR CASH

Consider the case where each warrant is issued at a price of I, thereby providing the firm with total proceeds on issue of mI. This now raises the complication of having to decide what the firm does with these proceeds. A number of different approaches have been adopted in the literature, all with the common objective of ensuring that the warrant issue does not effect the probability distribution of the rate of return on the asset value of the firm. Alternatively, a firm may use the issue proceeds for a variety of other purposes, including investment in risky and/or riskless assets with the resultant change to the probability distribution of V depending upon how the proceeds actually are used. To eliminate these secondary effects, it is assumed here that part of the issue proceeds are distributed immediately to shareholders as a cash dividend of D per share, while the balance of the proceeds $(mI - nD)$ are retained and invested in a scale expansion of the firm.

To give this effect, the issue proceeds of I per warrant may be treated as a negative dividend on V and W, and the cash dividend of D per share may be treated as a positive dividend on V and S. Assuming no other payouts over the period $(0, T)$, then the cumulative dividend process for the warrant is defined by:

$$\Delta w_t = -I \quad \text{if} \quad t = \alpha; \quad \frac{\gamma(V_{T-} - nX)}{n + m\gamma} I_{\{\cdot\}}$$
$$\text{if} \quad t = T; \quad 0 \quad \text{otherwise} \tag{12}$$

[11]The assumption of no further information surprises may be relaxed. If the filtration were discontinuous, then the conditional adjustment for dilution over time also would be discontinuous.

[12]For completeness, the impact of the warrant issue on shareholder wealth readily follows. In summary, shareholder wealth first adjusts for dilution at the time the issue is announced, and then is conditionally updated for subsequent changes in stock price over time such that if—and by—the time the warrants are exercised, shareholders already will have borne fully the dilutionary impact on their wealth, being an amount equal to the ex-post payoff on the warrant at that time. Alternatively, shareholders could hedge the initial impact of dilution at announcement by subsequently investing in a self-financing trading strategy that replicates the payoff on the warrant. Note that these results are based on the assumption that none of the firm's shareholders participate in the issue. At the other extreme, if all shareholders of the firm participate in the issue on a pro-rata basis, then the underlying stock price will adjust for dilution on the ex-date rather than on the announcement date. In this case, the shareholders and warrantholders are one and the same, and hence there is no transfer of wealth before the ex-date and no adjustment for dilution to the stock price before the ex-date.

the cumulative dividend process for the firm is defined by:

$$\Delta v_t = nD - mI \quad \text{if} \quad t = \alpha; \quad -m\gamma X I_{\{\cdot\}} \quad \text{if} \quad t = T; \quad 0 \quad \text{otherwise} \tag{13}$$

and the cumulative dividend process for the stock is defined by:

$$\Delta s_t = D \quad \text{if} \quad t = \alpha; \quad 0 \quad \text{otherwise} \tag{14}$$

This leads to the following natural extension to Eq. (10) for the behavior of the underlying stock price at the time of announcement.

Proposition 4. *In the absence of arbitrage, there is a jump in the stock price on announcement such that:*

$$S_\alpha = S_{\alpha-} - \frac{m}{n}W_\alpha + \frac{m}{n}I - D \tag{15}$$

Equation (15) indicates that not only dilution, but also the issue price of the warrants and the expected use of the issue proceeds, determines whether the stock price in fact falls on announcement. This necessitates a finer usage of the term dilution. Let *gross* stock-price dilution refer to the reduction in the firm's stock price, which arises from warrant exercise, and let *net* stock-price dilution refer to gross dilution net of any issue proceeds received by the firm. When the warrants are issued for nil consideration, the two amounts coincide, but when cash proceeds are raised on issue, it is net rather than gross dilution that better reflects the effect of the warrant issue on the underlying stock price and on the wealth of the shareholders. In this case, the jump of $\frac{m}{n}(I - W_\alpha)$ in Eq. (15) may be interpreted as an instantaneous adjustment for net dilution on a per-share basis.[13]

IMPLICATIONS FOR THE VALUATION OF WARRANTS

As mentioned earlier, there are three alternative, yet consistent, approaches to valuing a European warrant on the stock of an unlevered firm:

i. as an option on the equity of the underlying firm;

[13]In particular, if the warrants are issued for fair value ($I = W_\alpha$), then there is no net dilution in shareholder wealth on announcement, as shareholders may be considered to have been conditionally compensated in full for the warrant issue at that time and therefore conditionally compensated in full for expected gross dilution. Similarly, there is no net stock price dilution on announcement and no jump in stock price if, in addition, the issue proceeds are wholly retained.

ii. as an option on the stock of an otherwise identical firm without warrants; and

iii. as an option on the stock of the underlying firm.

In the first case, the arbitrage-free value of the warrant is equal to the value of an otherwise equivalent call option on the equity of the firm multiplied by a dilution factor based on the number of new shares issued in the event the warrant is exercised. In the second case, the arbitrage-free value of the warrant is equal to the value of an otherwise equivalent call option on the stock of the otherwise identical firm without warrants multiplied by the same dilution factor. In the third case, the arbitrage-free value of the warrant is simply equal to the value of an otherwise equivalent call option on the stock of the firm. Note that only in the first two cases is the resultant option value adjusted by a dilution factor.

The results in the previous two sections have the following implications. First, using the post-announcement stock price of the firm as the underlying state variable to value the warrant, case iii, does not in fact ignore dilution, but rather it obviates the need for an *explicit* adjustment for dilution to be made to the chosen option-pricing model. Because the stock price conditionally reflects dilution at all times post announcement, then, in effect, dilution is *implicitly* incorporated into the underlying stock price and hence, implicitly incorporated into the pricing model. Clearly, an explicit adjustment for dilution would be required if the warrant were valued as a function of the underlying stock price before announcement of the issue.[14] Second, using the post-announcement equity per share of the firm as the underlying state variable to value the warrant, case i, essentially involves making *two* adjustments for dilution. First, the theoretical adjustment for dilution (i.e., the value of the warrant) is added back to the *observed* conditionally diluted stock price in order to arrive at an estimate of the *notional* undiluted stock price.[15] The boundary condition for the warrant, expressed as a function of this notional undiluted stock price, then is adjusted specifically for dilution. Third, historic volatility estimates based on observed stock prices post announcement already reflect dilution while estimates based on observed stock prices pre-announcement do not. Therefore, a time series

[14]For empirical evidence that stock prices incorporate expected dilution effects, see for example Schulz and Trautmann (1994), Bensoussan et al. (1995), and Huson, Scott, and Wier (2001).

[15]In the case of traded warrants, it is an empirical issue as to whether the market price of the warrant is equal to its theoretical price.

of observed stock prices across the announcement date is not strictly comparable.

While the model presented in this article is necessarily complicated to allow for jumps in the underlying price processes, the central idea of the article can be illustrated by way of a simple numerical example within a one-period binomial framework. Consider an all equity firm with five shares outstanding, a current stock price of \$20 per share, a risk-free rate of 10%, an end of period stock price of either \$26 or \$18 and an equal (risk neutral) probability of an up state or down state occurring. By Eq. (2), the value of the underlying assets at the end of the period is either \$130 or \$90. Suppose the firm announces that it has just issued (for free) one European warrant that will entitle its holder to buy one share of stock at the end of the period at an exercise price of \$20. By Eq. (4), the value of the underlying assets at the end of the period will now be either \$150 or \$90, reflecting the fact that the warrant will be exercised in the up state but not in the down state. Similarly by Eq. (9), the stock price at the end of the period will now be either \$25 (\$150/6) or \$18 (\$90/5), which implies a new current stock price of $\frac{0.5(25) + 0.5(18)}{1.1}$ or \$19.545, that is, the stock price adjusts for dilution on announcement of the warrant issue. Because the stock price already reflects dilution, then the warrant may be valued as an option on the stock of the underlying firm without any need for an explicit adjustment for dilution and is equal to $\frac{0.5(5) + 0.5(0)}{1.1}$ or \$2.273. If the warrant is valued as an option on the stock of an otherwise identical firm without warrants (the firm prior to announcing the warrant issue), then an explicit adjustment for dilution is required and the same value would result, that is, the value of the otherwise equivalent call option is $\frac{0.5(6) + 0.5(0)}{1.1}$ or \$2.727, which, after multiplying by the dilution factor of $\frac{1}{1 + 1/5}$ or 0.833, becomes \$2.273.

Finally, if the warrant is to be priced as an option on the stock of the underlying firm, then the key difficulty is deciding how to deal with the stochastic volatility of the underlying stock price.[16] Crouhy and Galai (1991) and Bensoussan, Crouhy, and Galai (1995) suggested that by issuing warrants, part of the total asset risk of an unlevered firm is shifted from the stockholders to the warrantholders, and the degree of this risk sharing changes continuously as a function of the asset value and with the passage of time.[17] This difficulty, however, does *not* mean that a

[16]This follows from the standard assumption that the underlying equity value of the firm follows a geometric Brownian motion.

[17]Geske (1979) considered a similar problem in relation to valuing a call option on the stock of a levered firm. Geske (1979) pointed out that financial leverage causes stock-price volatility to be stochastic, which, in turn, motivated the development of his compound option model.

warrant cannot be priced as an option on the underlying stock. Rather, it simply means that the chosen option-pricing model must allow for a stochastic stock-price volatility.[18] This is a topic for further research.

CONCLUSION

In this article, the impact of a warrant issue on the stock price of the underlying firm and on the wealth of the firm's shareholders is examined within a framework based on Harrison and Pliska's (1981) standard security market model with continuous trading. Extending previous research, it has been shown that, at all times following announcement of the issue and before maturity, both the stock price and shareholder wealth conditionally reflect dilution at that time notwithstanding that the warrants may not have been issued yet. This result has a number of important implications concerning the valuation of warrants and other convertible securities. In particular, no explicit adjustment for dilution is required in the case where the warrant is treated as a contingent claim on the underlying post-announcement stock price of the firm.

APPENDIX

Proofs

Proof of Proposition 1

The stock price immediately before maturity is given by Eq. (8). Substituting from Eq. (7):

$$S_{T-} = \frac{V_{T-}}{n} - \frac{m}{n}\frac{\gamma(V_{T-} - mX)}{n + m\gamma}I_{\{\cdot\}} \tag{16}$$

The stock price at maturity is given by Eq. (9). Substituting from Eqs. (3) and (4):

$$S_T = \frac{V_T}{N_T} = \frac{V_{T-} + m\gamma X I_{\{\cdot\}}}{n + m\gamma I_{\{\cdot\}}} \tag{17}$$

[18]Alternatively, if the warrant is to be priced as an option of the equity of the underlying firm, Schulz and Trautmann (1994) presented a methodology for simultaneously arriving at the value of the warrant, the value of the firm, and the value of its volatility using observed stock prices and estimates of unobserved stock price volatility.

The jump in the stock price at maturity is defined as $S_T - S_{T-}$. Substituting from (16) and (17):

$$S_T - S_{T-} = \frac{V_{T-} + m\gamma X I_{\{\cdot\}}}{n + m\gamma I_{\{\cdot\}}} - \frac{V_T}{n} + \frac{m}{n}\frac{\gamma(V_{T-} - nX)}{n + m\gamma} I_{\{\cdot\}} \qquad (18)$$

Working through the algebra and recognizing that $I^2_{\{\cdot\}} = I_{\{\cdot\}}$ yields $S_T - S_{T-} = 0$, that is, the stock-price process is left continuous at time T, which is then sufficient to establish the continuity of S at time T.[19] ❑

Proof of Proposition 2

Let the warrant issue be first announced at time α such that the marginal increase in information commonly available to investors at the announcement date, $\Im_{\alpha-} \subset \Im_\alpha$ consists of notification of the detailed terms of the warrant issue. Immediately before announcement (time $\alpha-$) the stock may be treated as an $\Im_T$ – measurable contingent claim on the assets of the firm with a payoff given by Eq. (2). By Harrison and Pliska (1981), the discounted price of the stock at that time is:

$$\begin{aligned}
\beta_{\alpha-}^{-1} S_{\alpha-} &= E\left[\beta_T^{-1} S_T + \int_{(\alpha-,T]} \beta_u^{-1}\, ds_u \,\middle|\, \Im_{\alpha-}\right] \\
&= E[\beta_T^{-1} S_T \mid \Im_{\alpha-}] \\
&= E\left[\beta_T^{-1}\frac{V_T}{n} \,\middle|\, \Im_{\alpha-}\right] \\
&= \frac{1}{n}E[\beta_T^{-1} V_T \mid \Im_{\alpha-}] = \frac{1}{n}\beta_{\alpha-}^{-1} V_{\alpha-}
\end{aligned} \qquad (19)$$

The second step comes from the assumption that $s_t = 0$ for $t \in [0, T]$, the third step comes directly from Eq. (2) and the last follows from Eq. (1) after taking into account the assumption that, *pre-announcement*, $v_t = 0$ for $t \in [0, T]$. On announcement, the stock may be treated as

[19]In this study, any process that is a martingale with respect to (Ω, $\Im$, F, P) will be taken to be an RCLL process [Theorem 4.6 of Elliott (1982)]. Let (F, f) be the price-dividend pair for any attainable $\Im_T$-measurable contingent claim on the assets of the firm where f is adapted, RCLL, and of finite variation. It readily can be shown that if F is left continuous at time t, then F also is continuous at that time.

an $\Im_{T-}$ – measurable contingent claim on the assets of the firm with a payoff given by Eq. (8). Again by Harrison and Pliska (1981), the price of the stock at that time is:

$$\begin{aligned} \beta_\alpha^{-1} S_\alpha &= E\left[\beta_{T-}^{-1} S_{T-} + \int_{(\alpha,T)} \beta_u^{-1} ds_u \,\middle|\, \Im_\alpha\right] \\ &= E[\beta_{T-}^{-1} S_{T-} \mid \Im_\alpha] \\ &= E\left[\beta_{T-}^{-1}\left(\frac{V_{T-}}{n} - \frac{m}{n} W_{T-}\right) \,\middle|\, \Im_\alpha\right] \\ &= \frac{1}{n} E[\beta_{T-}^{-1} V_{T-} \mid \Im_\alpha] - \frac{m}{n} E[\beta_{T-}^{-1} W_{T-} \mid \Im_\alpha] \end{aligned} \tag{20}$$

where W_α is the price of the warrant at time α. The second step follows from the assumption that $s_t = 0$ for $t \in [0, T]$, and the third step follows from Eq. (8). Applying Eq. (1) to the price-dividend pair for the firm:

$$\beta_\alpha^{-1} V_\alpha = E\left[\beta_{T-}^{-1} V_{T-} + \int_{(\alpha,T)} \beta_u^{-1} dv_u \,\middle|\, \Im_\alpha\right] = E[\beta_{T-}^{-1} V_{T-} \mid \Im_\alpha] \tag{21}$$

since following announcement, $dv_u = 0$ for $u \neq T$. Similarly, applying Eq. (1) to the price-dividend pair for the warrant:

$$\beta_\alpha^{-1} W_\alpha = E\left[\beta_{T-}^{-1} W_{T-} + \int_{(\alpha,T)} \beta_u^{-1} dw_u \,\middle|\, \Im_\alpha\right] = E[\beta_{T-}^{-1} W_{T-} \mid \Im_\alpha] \tag{22}$$

since following announcement, $dw_u = 0$ for $u \neq T$. Substituting Eqs. (21) and (22) into Eq. (20):

$$\beta_\alpha^{-1} S_\alpha = \frac{1}{n} \beta_\alpha^{-1} V_\alpha - \frac{m}{n} \beta_\alpha^{-1} W_\alpha \tag{23}$$

Canceling the discount factors from each of Eqs. (19) and (23) and then taking the first from the second:

$$S_\alpha = S_{\alpha-} + \frac{1}{n}(V_\alpha - V_{\alpha-}) - \frac{m}{n} W_\alpha = S_{\alpha-} - \frac{m}{n} W_\alpha \tag{24}$$

The last step follows from the assumed continuity of the asset price process at the announcement date. ❑

Proof of Proposition 3

By proposition 2, the stock price first adjusts for dilution at the time the issue is announced ($t = \alpha$). Now consider *any* time $t \in (\alpha, T)$. By Eq. (1), the discounted price of the stock at that time is:

$$\beta_t^{-1} S_t = E\left[\beta_{T-}^{-1} S_{T-} + \int_{(t,T)} \beta_u^{-1} ds_u \,\middle|\, \Im_t \right] \tag{25}$$

which, using similar arguments to those used to derive Eq (23), simplifies to Eq. (11), where W_t is the price of the warrant at time t. ❑

Proof of Proposition 4

Immediately before announcement (time $\alpha-$) the stock may be considered to be an $\Im_T$ − measurable contingent claim on the assets of the firm with a payoff [Eq. (2)] and therefore the discounted price of the stock at that time is given by Eq. (19):

$$\beta_{\alpha-}^{-1} S_{\alpha-} = \frac{1}{n} \beta_{\alpha-}^{-1} V_{\alpha-} \tag{26}$$

On announcement (time α) the stock may be considered to be an $\Im_{T-}$ − measurable contingent claim on the assets of the firm with a terminal payoff [Eq. (8)] and a cumulative payout process [Eq. (14)]. By Eq. (1), the price of the stock at that time is:

$$\begin{aligned}
\beta_\alpha^{-1} S_\alpha &= E\left[\beta_{T-}^{-1} S_{T-} + \int_{(\alpha,T)} \beta_u^{-1} ds_u \,\middle|\, \Im_\alpha \right] \\
&= E[\beta_{T-}^{-1} S_{T-} | \Im_\alpha] + E\left[\int_{(\alpha,T)} \beta_u^{-1} ds_u \,\middle|\, \Im_\alpha \right] \\
&= E[\beta_{T-}^{-1} S_{T-} \mid \Im_\alpha] \\
&= \frac{1}{n} E[\beta_{T-}^{-1} V_{T-} | \Im_\alpha] - \frac{m}{n} E[\beta_{T-}^{-1} W_{T-} | \Im_\alpha]
\end{aligned} \tag{27}$$

The third step follows from Eq. (14), that is, $ds_u = 0$ for $u \neq \alpha$ while the last step follows after substitution from Eq. (8). Applying Eq. (1) to the price-dividend pair for the firm (V,v):

$$\beta_\alpha^{-1} V_\alpha = E\left[\beta_{T-}^{-1} V_{T-} + \int_{(\alpha,T)} \beta_u^{-1} dv_u \,\middle|\, \Im_\alpha \right] = E[\beta_{T-}^{-1} V_{T-} | \Im_\alpha] \tag{28}$$

where the second step follows from Eq. (13), that is, $dv_u = 0$ for $u \in (\alpha, T)$. Similarly, applying Eq. (1) to the price-dividend pair for the warrant (W,w):

$$\beta_\alpha^{-1} W_\alpha = E\left[\beta_{T-}^{-1} W_{T-} + \int_{(\alpha,T)} \beta_u^{-1} dw_u \,\middle|\, \Im_\alpha\right] = E[\beta_{T-}^{-1} W_{T-} \mid \Im_\alpha] \quad (29)$$

where the second step follows from Eq. (12), that is, $dw_u = 0$ for $u \in (\alpha, T)$. Substituting Eqs (28) and (29) into Eq (27):

$$\beta_\alpha^{-1} S_\alpha = \frac{1}{n}\beta_\alpha^{-1} V_\alpha - \frac{m}{n}\beta_\alpha^{-1} W_\alpha \quad (30)$$

Canceling the discount factors from each of Eqs. (26) and (30) and then taking the first from the second gives the desired result:

$$S_\alpha = S_{\alpha-} - \frac{m}{n} W_\alpha + \frac{1}{n}(V_\alpha - V_{\alpha-}) = S_{\alpha-} - \frac{m}{n} W_\alpha + \frac{m}{n} I - D \quad (31)$$

The last step comes from the jump discontinuity of the asset price process on announcement. ❑

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