THE ACCURACY AND EFFICIENCY OF ALTERNATIVE OPTION PRICING APPROACHES RELATIVE TO A LOG-TRANSFORMED TRINOMIAL MODEL

HSUAN-CHI CHEN*
DAVID M. CHEN
SAN-LIN CHUNG

This article presents a log-transformed trinomial approach to option pricing and finds that various numerical procedures in the option pricing literature are embedded in this approach with choices of different parameters. The unified view also facilitates comparisons of computational efficiency among numerous lattice approaches and explicit finite difference methods. We use the root-mean-squared relative error and the minimum

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*Correspondence author, Department of Finance, Yuan Ze University, 320 ChungLi, Taiwan, R.O.C.; e-mail: chenh@saturn.yzu.edu.tw

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- *Hsuan-Chi Chen is an Associate Professor in the Department of Finance at Yuan Ze University in Taiwan, R.O.C.*
- *David M. Chen is a Professor in the Department of Accounting at Fu-Jen Catholic University in Taiwan, R.O.C.*
- *San-Lin Chung is an Associate Professor in the Department of Finance at National Central University in Taiwan, R.O.C.*

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convergence step to evaluate the accuracy and efficiency for alternative option pricing approaches. The numerical results show that the equal-probability trinomial specification of He (1990) and Tian (1993) and the sharpened trinomial specification of Omberg (1988) outperform other lattice approaches and explicit finite difference methods. Jrl Fut Mark 22:557–577, 2002

1. INTRODUCTION

Two well-known numerical procedures for valuing derivative securities are the lattice approach and the finite difference approach. The binomial lattice approach developed by Cox, Ross, and Rubinstein (CRR; 1979) and Rendleman and Bartter (RB; 1979) has been extended by various authors. With different parameter choices,[1] CRR and RB showed that the binomial model converges to the Black–Scholes (1973) model. Hull and White (1988) provided a slightly modified version of the binomial lattice approach. Trigeorgis (1991) designed a log-transformed version of the binomial method for valuing complex investments with multiple interacting options. More recently, Jabbour, Kramin, and Young (2001) revisited the binomial models developed by CRR, RB, and Trigeorgis and proposed two alternative binomial models based on the continuous-time and discrete-time geometric Brownian motion processes.

A natural extension of the binomial lattice approach in the option pricing literature is the trinomial lattice approach. Parkinson (1977) employed a trinomial pricing method as a numerical integration procedure to value American puts. Boyle (1988) developed a trinomial lattice approach with different parameter choices and extended it to value options on two underlying assets. Cho and Lee (1995) extended Boyle's trinomial lattice model by incorporating the information contained in the skewness as well as the mean and the variance of the log-normal distribution to uniquely determine the parameters within the trinomial model. Applying Gauss–Hermite quadrature to the recursive integration problem proposed by a compound option model, Omberg (1988) derived a family of discrete-time jump processes in option pricing. Tian (1993) introduced a modified approach to the selection of lattice parameters including probabilities and jumps in the binomial and trinomial models.

Comparing the trinomial process to the binomial process, the binomial process enables the hedging argument to be made in discrete time, whereas the trinomial process is otherwise much more natural. It has

[1]See Ombert (1987) for more details.

considerable pedagogical value to assume that over a small period of time, an asset price can go up, remain the same, or go down. The hedging argument aside, it is quite unnatural to assume that an asset price only can go up or go down, as a binomial process does.[2]

The finite difference approach was examined by Brennan and Schwartz (1978) and extended by Courtadon (1982) and Hull and White (1990). Brennan and Schwartz showed that the explicit finite difference method with logarithmic transformation is equivalent to a trinomial lattice approach. The trinomial lattice approach may also be regarded as a form of numerical integration where the probabilities are taken from a three-jump process. However, the variance of their approximating trinomial process is a downward biased estimate of the variance of the transformed diffusion process.

Geske and Shastri (1985) made an interesting comparison of several finite difference methods and the binomial pricing method and concluded that the explicit finite difference method with logarithmic transformation is more efficient than the implicit method, since it does not require the inversion of matrices. The binomial technique has a strong intuitive appeal and is more readily implemented than the finite difference methods. Rubinstein (2000) explored the connection between the binomial lattice approach and the explicit finite difference method. He showed that both methods are equivalent if two steps of the binomial tree are examined each time, which in turn generate a trinomial tree.

The purpose of this article is to present a log-transformed trinomial approach to option pricing and provide various numerical procedures in the large literature with this unified view.[3] We also recognize the conditions for this approach to converge to the Black–Scholes partial differential equation. Finally, we conduct numerical simulation in this unified framework to investigate the computational accuracy and efficiency for several specifications in the literature. The equal-probability trinomial specification of He (1990) and Tian (1993), and the sharpened trinomial specification of Omberg (1988), are shown to outperform other lattice approaches and explicit finite difference methods. Though the downward-biased estimate of the variance arising from the

[2]As pointed out by the referee, comparing the trinomial process to higher-order processes, making higher-order processes recombining may restrict the parameters more. For example, Omberg (1988) has proved that for higher-order processes ($n > 3$), the nodes constructed from the Gauss–Hermite method are not recombined. This fact highlights an advantage of the trinomial process.

[3]Brennan and Schwartz (1978) and Trigeorgis (1991) also apply logarithmic transformations to the finite difference method and the binomial model, respectively.

explicit finite difference method is corrected in our framework, pricing errors are slightly improved. Overall, we find that trinomial methods are superior to the explicit finite difference methods and the binomial models in terms of the accuracy and efficiency. Corrections to higher moments do not necessarily improve the pricing errors for each category of methodology.

The remainder of the paper is organized as follows: Section 2 presents the log-transformed trinomial approach and discusses the convergence of this approach. Section 3 contains several examples that appear in the literature. Section 4 compares the numerical accuracy and the convergence property among numerous lattice approaches and the explicit finite difference method using the log-transformed trinomial framework. Section 5 concludes the paper.

2. DESIGN OF THE LOG-TRANSFORMED TRINOMIAL APPROACH

We assume that the risk-neutral valuation argument is valid and regard the option pricing problem as a problem in numerical analysis. Under risk neutrality, the stock price S can be assumed to follow a stochastic process

$$\frac{dS}{S} = r\,dt + \sigma\,dz$$

where r is the risk-free interest rate, σ is the constant instantaneous standard deviation of the stock return, and dz is a standard Wiener process.

Define $G = \ln S$. From Ito's lemma, the process followed by G is

$$dG = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\,dz$$

where dG is normally distributed with mean $(r - \frac{1}{2}\sigma^2)\,dt$ and variance $\sigma^2 dt$. For convenience of notation, we let $Var(dG) = u$ and $E(dG) = eu$, where $u \equiv \sigma^2\,dt$ and $e \equiv \frac{r}{\sigma^2} - \frac{1}{2}$.

We can develop a log-transformed trinomial process G_i^n to approximate the above continuous stochastic process G as follows. The time to maturity, T, of the stock option is divided into n equal time steps with each duration $h = T/n$. Thus, $\nu = \sigma^2 h$ can be used to approximate u. Within each small time interval, G_i^n follows a trinomial process (a three-jump process). It can move upward by an amount H_1 with risk-neutral probability P_1, downward by an amount H_3 with probability P_3, or take a middle jump by an amount H_2 with probability P_2. That is,

$$G_{i+1}^n = \begin{cases} G_i^n + H_1 & \text{(up state with probability } P_1) \\ G_i^n + H_2 & \text{(middle state with probability } P_2) \\ G_i^n + H_3 & \text{(down state with probability } P_3) \end{cases}$$

where $H_1 > H_2 > H_3$.

Because the trinomial process G_i^n is used to approximate the continuous stochastic process G, some restrictions on the first two moments are imposed. First, the sum of risk-neutral probabilities equals to 1. Second, the mean of the proposed trinomial distribution is equal to the mean of the normal distribution. Furthermore, the variance of the proposed trinomial distribution is equal to the variance of the normal distribution. These restrictions will be

$$P_1 + P_2 + P_3 = 1 \tag{1}$$

$$P_1H_1 + P_2H_2 + P_3H_3 = e\nu \tag{2}$$

$$P_1H_1^2 + P_2H_2^2 + P_3H_3^2 = \nu + e^2\nu^2 \tag{3}$$

These restrictions are also enough to guarantee the convergence to the Black–Scholes partial differential equation. To see this, the risk-neutral valuation approach gives us the value of an option at time t, $V(S_t, t)$, and we have

$$\begin{aligned} V(S_t, t) = e^{-rh}[&P_1V(S_te^{H_1}, t + h) + P_2V(S_te^{H_2}, t + h) \\ &+ P_3V(S_te^{H_3}, t + h)] \end{aligned} \tag{4}$$

On the other hand, the Taylor series expansion gives

$$\begin{aligned} &V(S_te^{H_i}, t + h) \\ &= V(S_t, t) + \frac{\partial V}{\partial t}h + \frac{\partial V}{\partial S_t}(e^{H_i} - 1)S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}(e^{H_i} - 1)^2S_t^2 + \cdots \\ &= V(S_t, t) + \frac{\partial V}{\partial t}h + \frac{\partial V}{\partial S_t}\left(H_i + \frac{H_i^2}{2}\right)S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}H_i^2S_t^2 + \cdots \end{aligned}$$

Therefore,

$$
\begin{aligned}
e^{rh}V(S_t, t) &= \sum_{i=1}^{3} P_i V(S_t e^{H_i}, t + h) \\
&= V(S_t, t) + \sum_{i=1}^{3}\left(P_i \frac{\partial V}{\partial t} h\right) + \frac{\partial V}{\partial S_t}\left[\sum_{i=1}^{3} P_i\left(H_i + \frac{H_i^2}{2}\right)\right] S_t \\
&\quad + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}\left(\sum_{i=1}^{3} P_i H_i^2\right) S_t^2 + \cdots
\end{aligned}
$$

Substituting equations (1) to (3) into the above equation gives

$$
\begin{aligned}
rV(S_t, t) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t}\left[r + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)^2 h\right] S_t \\
&\quad + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}\left[\sigma^2 + \left(r - \frac{1}{2}\sigma^2\right)^2 h\right] S_t^2 + \cdots
\end{aligned}
$$

As $h \to 0$

$$
rV(S_t, t) = \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}
$$

This is the well-known Black–Scholes partial differential equation (PDE).[4] Hence we could interpret the log-transformed trinomial model as a numerical method to solve the Black–Scholes PDE given the validity of the risk-neutral valuation approach.

In addition, from Lindeberg's Central Limit Theorem,[5] the discrete distribution converges to the appropriate normal distribution. Under the assumption of the risk-neutral valuation approach, the value of a European option is the discounted value of the expected terminal payoff. He (1990) and Nelson and Ramaswamy (1990) have addressed the convergence of the contingent claim price.

3. CASES OF THE LOG-TRANSFORMED TRINOMIAL MODEL

Note that we have six parameters P_1, P_2, P_3, H_1, H_2, and H_3, but restrictions (1) to (3) have determined the convergence of the discrete valuation equation (4). This implies that we have infinitely many choices of six parameters

[4] See Cox et al. (1979, p. 254).
[5] See Feller (1971, p. 262) or Billingsley (1986, p. 369).

to obtain the convergent value of an option using the log-transformed trinomial model. It is possible to make a unique choice from six parameters if we are willing to add more interesting restrictions. Under this point of view, many numerical methods correspond to the proper choice of the six parameters in the log-transformed trinomial model, while there seems to be a lot of different discrete-time approaches at first glance.

For efficiency of computation, one has to reduce the number of nodes in the valuation algorithm. A simple method for reducing the number of nodes is to impose the recombined condition, that is,

$$H_1 + H_3 = 2H_2 \tag{5}$$

The recombined condition means that an up state followed by a down state is the same as a down state followed by an up state. Also, the location of G_i^n after up-down jumps or down-up jumps is the same as that after middle-middle jumps.

If two more proper conditions are imposed, the choice of six parameters can be determined uniquely. To illustrate the unified view for discrete-time numerical methods, we present several examples discussed in the literature.

Case 1

$$H_2 = 0 \tag{6}$$

$$P_2 = 0 \tag{7}$$

The choice of these parameters corresponds to the binomial model appearing in Trigeorgis (1991). Solutions to equations (1)–(3) and (5)–(7) are

$$P_1 = \frac{1}{2} + \frac{e\nu}{2H_1}, \quad P_2 = 0, \quad P_3 = \frac{1}{2} - \frac{e\nu}{2H_1},$$

$$H_1 = \sqrt{\nu + e^2\nu^2}, \quad H_2 = 0, \quad H_3 = -\sqrt{\nu + e^2\nu^2}$$

Without considering the logarithmic transformation, Hull and White (1988) provided one binomial specification which corresponds to this case. As $n \to 0$ we have

$$H_1 \to \sqrt{\nu} = \sigma\sqrt{h}$$

$$P_1 = \frac{1}{2} + \frac{e\nu}{2H_1} \to \frac{1}{2} + \frac{1}{2\sigma}\left(r - \frac{1}{2}\sigma^2\right)\sqrt{h}$$

and the limits of H_1 and P_1 are exactly the parameters $\log u$ and q that show up in Cox et al. (1979).[6]

Case 2

$$P_2 = 0 \tag{8}$$

$$P_1H_1^3 + P_2H_2^3 + P_3H_3^3 = 3ev^2 + e^3v^3 \tag{9}$$

The choice of the parameters generates an alternative binomial model where the third moment of the discrete time process is the same as that of the corresponding continuous time process.[7] The third moment condition is imposed by equation (9). Solving equations (1)–(3), (5), (8), and (9) gives

$$P_1 = P_3 = \frac{1}{2}, \quad P_2 = 0,$$

$$H_1 = ev + \sqrt{v}, \quad H_2 = ev, \quad H_3 = ev - \sqrt{v}$$

Rendleman and Bartter (1979), Jarrow and Rudd (1983), and Omberg (1988) discussed this binomial model. For the binomial specification without taking the logarithmic transformation into account, see the result in Tian (1993). Using spread-sheet simulation, we find that the up and down probabilities in the BIN model of Tian (1993) are close to 0.5 when the number of time steps is large.

Case 3

$$H_2 = 0 \tag{10}$$

$$P_1H_1^3 + P_2H_2^3 + P_3H_3^3 = 3ev^2 + e^3v^3 \tag{11}$$

This case considers that stock prices are able to remain unchanged and the third moment condition is imposed. Solutions to equations (1)–(3), (5), (10), and (11) are

$$P_1 = \frac{1}{2} + \frac{ev}{2H_1} - \frac{v}{H_1^2}, \quad P_2 = \frac{2v}{H_1^2}, \quad P_3 = \frac{1}{2} - \frac{ev}{2H_1} - \frac{v}{H_1^2},$$

$$H_1 = \sqrt{3v + e^2v^2}, \quad H_2 = 0, \quad H_3 = -\sqrt{3v + e^2v^2}$$

[6]See Cox et al. (1979, p. 249).

[7]If the random variable X is normally distributed with $N(\mu, \sigma^2)$, then $E(X - \mu)^3 = 0$ and $E(X - \mu)^4 = 3\sigma^4$. The arrangement of these equations shows that $E(X)^3 = \mu^3 + 3\mu\sigma^2$ and $E(X)^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$.

This case is similar to that of Boyle (1988). Cho and Lee (1995) proposed one equivalent trinomial specification without making the logarithmic transformation.

Case 4

$$\sqrt{P_1} + \sqrt{P_3} = 1 \tag{12}$$

$$H_2 = 0 \tag{13}$$

If we treat $\sqrt{P_1}$ ($\sqrt{P_3}$) in equation (12) as the probability of up (down) movement in the binomial process and examine equations (1) and (12), it is natural to interpret the trinomial tree as the combination of a two-step binomial tree. Equation (13) means that the stock price remains unchanged in the middle state. From equations (1), (2), (3), (5), (12), and (13), we obtain

$$P_1 = \left(\frac{1}{2} + \frac{e\nu}{2H_1}\right)^2, \quad P_2 = \frac{\nu}{H_1^2}, \quad P_3 = \left(\frac{1}{2} - \frac{e\nu}{2H_1}\right)^2,$$

$$H_1 = \sqrt{2\nu + e^2\nu^2}, \quad H_2 = 0, \quad H_3 = -\sqrt{2\nu + e^2\nu^2}$$

Note that Case 1 is a degenerated trinomial tree or a binomial tree, and the previous interpretation links the relationship between probabilities in this case and those of Case 1. The choice of these parameters is shown in Parkinson (1977, p. 28).

Case 5

$$P_1 = P_3 = p \tag{14}$$

$$P_1H_1^3 + P_2H_2^3 + P_3H_3^3 = 3e\nu^2 + e^3\nu^3 \tag{15}$$

From equation (14), it follows that the risk-neutral probability of an up movement is the same as that of a down movement. Also, the third moment is matched using equation (15). Equations (1)–(3), (5), (14), and (15) imply

$$P_1 = P_3 = p, \quad P_2 = 1 - 2p$$

$$H_1 = e\nu + \sqrt{\frac{\nu}{2p}}, \quad H_2 = e\nu, \quad H_3 = e\nu - \sqrt{\frac{\nu}{2p}}$$

There are several cases with the symmetric choice of risk-neutral probabilities and the match of the third moment.

(a) If $p = \frac{1}{3}$, the equal-probability case shows up in He (1990). Tian (1993) also developed an equal-probability trinomial specification without making the logarithmic transformation.

(b) If $p = \frac{1}{2}$, this case degenerates to the binomial case or Case 2.

(c) If $p = \frac{1}{6}$, the case appears in Omberg (1988) and Figlewski and Gao (1999). Omberg derived it as the trinomial process corresponding to Gauss–Hermite integration. Figlewski and Gao derived it directly by imposing the restriction of matching the fourth moment. To see this, we obtain

$$P_1H_1^4 + P_2H_2^4 + P_3H_3^4 = e^4\nu^4 + 6e^2\nu^3 + 3\nu^2$$

The right-hand side of the above equation is exactly the fourth moment of a normal random variable. In the same manner, Tian (1993) considered the fourth-moment-matching trinomial specification without using the logarithmic transformation.

(d) If $p = \frac{1}{\pi}$, it is the sharpened trinomial process proposed in Omberg (1988). It should be noted that the Omberg sharpened-trinomial process has close-to-equal, though not exactly equal, probabilities. Also, Boyle (1988) optimized a trinomial process by trial and error to determine the jump probabilities and jump amplitudes. Boyle found that best results are obtained when the probabilities are roughly equal, as is the case of the sharpened trinomial process. Omberg (1988) compared these two model specifications in more detail.

Case 6

We briefly discuss the explicit finite difference approximation of the log-transformation of the Black–Scholes equation:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV \tag{16}$$

Define $y = \ln S$. Equation (16) can be transformed to[8]

$$\frac{\partial V}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial y^2} = rV \tag{17}$$

[8]See Brennan and Schwartz (1978, pp. 462–465).

Before replacing the partial derivatives by finite differences, we consider

$$V(y, t) = V(ik, jh) = V_{i,j}$$

where k and h are the discrete increments in the stock price dimension and the time dimension, respectively.

To approximate equation (17), at node (i, j), define

$$\frac{\partial V}{\partial y} = \frac{V_{i+1,j+1} - V_{i-1,j+1}}{2k}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{V_{i+1,j+1} - 2V_{i,j+1} + V_{i-1,j+1}}{k^2}$$

$$\frac{\partial V}{\partial t} = \frac{V_{i,j+1} - V_{i,j}}{h}$$

Substituting the above equations into (17) and rearranging terms, we obtain

$$(1 + rh)V_{i,j} = P_1 V_{i+1,j+1} + P_2 V_{i,j+1} + P_3 V_{i-1,j+1} \tag{18}$$

where

$$P_1 = \frac{\nu}{2k^2} + \frac{e\nu}{2k} \tag{19}$$

$$P_2 = 1 - \frac{\nu}{k^2} \tag{20}$$

$$P_3 = \frac{\nu}{2k^2} - \frac{e\nu}{2k} \tag{21}$$

We have $P_1 + P_2 + P_3 = 1$; therefore equation (18) can be viewed as a risk-neutral valuation model. Moreover, the explicit finite difference approximation corresponds to a log-transformed trinomial process.

Case 6(a)

In the usual explicit finite difference method, the upward and downward jumps are k and $-k$, respectively. And the middle jump is 0. Under this choice of parameters, the first moment is

$$P_1 \cdot k + P_2 \cdot 0 + P_3 \cdot (-k) = e\nu$$

which satisfies equation (2). The second moment is

$$P_1 \cdot k^2 + P_2 \cdot 0 + P_3 \cdot (-k)^2 = \nu$$

which is less than the right-hand side of equation (3). Consequently, the variance of the corresponding trinomial process is downward biased.[9] Furthermore, when we introduce a stretch parameter λ so that $k = \lambda\sqrt{\nu}$, we naturally identify the "trinomial" specification of Kamrad and Ritchken (1991, pp. 1642–1643) as Rubinstein (2000) did.

Case 6(b)

To correct the variance deficiency, we select P_1, P_2 and P_3 in equation (18), and impose conditions in equations (2), (3), and (5). These restrictions simultaneously give

$$H_1 = e\nu\left(1 - \sqrt{\frac{1}{1 - e^2\nu}}\right) + \sqrt{\frac{1}{1 - e^2\nu}}k \tag{22}$$

$$H_2 = e\nu\left(1 - \sqrt{\frac{1}{1 - e^2\nu}}\right) \tag{23}$$

$$H_3 = e\nu\left(1 - \sqrt{\frac{1}{1 - e^2\nu}}\right) - \sqrt{\frac{1}{1 - e^2\nu}}k \tag{24}$$

Therefore, if we make some adjustments in the magnitude of jumps, the problem of downward biased variance is eliminated. This log-transformed trinomial process still has the characteristics of the explicit finite difference method and reduces the number of nodes for calculation to improve efficiency.[10] Geske and Shastri (1985) concluded:

In addition, when transformed logarithmically, the explicit method is more efficient than the implicit method because it does not require the solution of a set of simultaneous equations. The binomial technique is more intuitive and also may be more readily implemented than the finite difference methods. Thus, it is pedagogically superior.

The log-transformed trinomial process specified by equations (19) through (24) is then more efficient, intuitive, and easily implemented than the grid of the explicit finite difference method. The degree of freedom k allows us to impose one more restriction. When the requirement of the third moment is imposed using equation (9), this implies

$$k = \sqrt{\nu(3 - 2e^2\nu)} \tag{25}$$

[9]See Brennan and Schwartz (1978, pp. 464–465).
[10]See Geske and Shastri (1985, p. 51).

To keep equations (19) through (24) meaningful, some restrictions are imposed on e and ν. If $e \geq 0$, to assure the positive risk-neutral probability P_3, we have

$$0 < \frac{\nu}{2k^2} - \frac{e\nu}{2k} \tag{26}$$

Restrictions (25) and (26) imply

$$e^2\nu < \frac{1}{2} \tag{27}$$

Inequality (27) not only assures that the three movements in equations (22) to (24) are well defined but also implies

$$k > \sqrt{2\nu} \tag{28}$$

Furthermore, it is straightforward to check

$$0 < P_1 = \frac{\nu}{2k^2} + \frac{e\nu}{2k} < 1$$
$$P_3 = \frac{\nu}{2k^2} - \frac{e\nu}{2k} < 1$$

and

$$0 < P_2 = 1 - \frac{\nu}{k^2} < 1, \text{ respectively}$$

If $e < 0$, the same argument also implies inequality (27). This restriction is similar to the stability conditions of the explicit finite difference method.[11]

Interestingly, if $e^2\nu \to 0$, this implies

$$k \to \sqrt{3\nu} \quad \text{and} \quad (P_1, P_2, P_3) = \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$$

The limiting case is exactly Case 5(c).

4. NUMERICAL EVALUATIONS

The unified approach proposed in this paper allows us to run a horse race to compare the numerical performances among numerous lattice approaches and explicit finite difference methods. The numerical performances are measured in terms of the numerical accuracy and the

[11]See Brennan and Schwartz (1978, p. 464).

convergence property. We follow the procedure in Broadie and Detemple (1996) to investigate the numerical accuracy of each method. We first choose a large set of European options (625 options) with practical parameters: $S/X = 0.8, 0.9, \ldots, 1.2$, $r = 0.02, 0.04, \ldots, 0.1$, $q = 0, 0.015, \ldots, 0.06$, $\sigma = 0.2, 0.3, \ldots, 0.6$, and $T = 1.0$, where q is the dividend yield. Then for each method we price the test set of options and compute the error measures. Both European calls and puts are being evaluated. The root-mean-squared (RMS) relative error is used as the error measure and is defined by:

$$RMS = \sqrt{\frac{1}{m}\sum_{i=1}^{m} e_i^2}$$

where m is the number of options considered, $e_i = \frac{(P_i^* - P_i)}{P_i}$ is the relative error, P_i is the true option price (Black–Scholes), and P_i^* is the estimated option price.

Tables I and II report the RMS relative errors of the considered methods for pricing European call options and put options, respectively. Many findings can be drawn from Tables I and II. First of all,

TABLE I

Root-Mean-Squared (RMS) Relative Errors of Methods Considered for Pricing European Call Options

	Method								
n	*1*	*2*	*3*	*4*	*5a*	*5c*	*5d*	*6a*	*6b*
20	1.07E−02	9.51E−03	7.99E−03	5.17E−03	3.81E−03	6.74E−03	4.01E−03	8.92E−03	7.97E−03
30	9.31E−03	6.18E−03	5.12E−03	3.21E−03	2.71E−03	4.53E−03	2.61E−03	5.26E−03	5.11E−03
40	5.17E−03	4.73E−03	4.43E−03	3.57E−03	1.88E−03	3.32E−03	1.93E−03	4.79E−03	4.43E−03
50	4.97E−03	3.82E−03	3.39E−03	1.75E−03	1.53E−03	2.66E−03	1.67E−03	3.56E−03	3.38E−03
60	3.21E−03	3.03E−03	3.43E−03	1.81E−03	1.33E−03	2.34E−03	1.29E−03	3.73E−03	3.42E−03
70	2.71E−03	2.68E−03	2.16E−03	1.38E−03	1.13E−03	1.99E−03	1.12E−03	2.34E−03	2.16E−03
80	3.57E−03	2.45E−03	1.68E−03	1.52E−03	9.40E−04	1.64E−03	1.00E−03	1.99E−03	1.68E−03
90	2.19E−03	2.06E−03	2.05E−03	1.34E−03	8.68E−04	1.47E−03	8.80E−04	2.36E−03	2.05E−03
100	1.75E−03	1.85E−03	1.70E−03	1.03E−03	7.72E−04	1.33E−03	7.69E−04	1.87E−03	1.70E−03
200	1.03E−03	9.66E−04	8.36E−04	5.53E−04	3.73E−04	6.67E−04	4.22E−04	9.23E−04	8.36E−04
300	6.48E−04	6.03E−04	4.86E−04	3.39E−04	2.73E−04	4.29E−04	2.48E−04	5.17E−04	4.86E−04
400	5.53E−04	4.41E−04	4.17E−04	2.60E−04	1.93E−04	3.33E−04	2.04E−04	4.64E−04	4.17E−04
500	3.74E−04	3.81E−04	2.85E−04	1.76E−04	1.57E−04	2.68E−04	1.53E−04	3.06E−04	2.85E−04

Note. The methods correspond to the cases described in the text. The root-mean-squared relative errors are defined as follows:

$$RMS = \sqrt{\tfrac{1}{m}\textstyle\sum_{i=1}^{m} e_i^2}$$

where $e_i = (P_i^* - P_i)/P_i$ is the relative error, P_i is the true option price (Black–Scholes), and P_i^* is the estimated option price. To conduct a comprehensive analysis, we consider 625 options whose parameters cover a wide range: $S/X = 0.8, 0.9, \ldots, 1.2$; $r = 0.02, 0.04, \ldots, 0.1$; $q = 0, 0.015, \ldots, 0.06$; $\sigma = 0.2, 0.3, \ldots, 0.6$; and $T = 1.0$.

TABLE II

Root-Mean-Squared (RMS) Relative Errors of Methods Considered for Pricing European Put Options

	Method								
n	*1*	*2*	*3*	*4*	*5a*	*5c*	*5d*	*6a*	*6b*
20	1.39E−02	1.10E−02	8.64E−03	6.15E−03	4.86E−03	7.71E−03	4.92E−03	8.71E−03	8.62E−03
30	9.00E−03	7.24E−03	6.67E−03	4.07E−03	3.27E−03	4.98E−03	3.31E−03	6.91E−03	6.66E−03
40	6.15E−03	5.38E−03	3.98E−03	2.99E−03	2.45E−03	3.69E−03	2.47E−03	4.01E−03	3.97E−03
50	4.23E−03	4.34E−03	3.68E−03	2.62E−03	1.99E−03	3.04E−03	2.01E−03	3.77E−03	3.68E−03
60	4.07E−03	3.66E−03	3.35E−03	2.46E−03	1.61E−03	2.54E−03	1.65E−03	3.51E−03	3.35E−03
70	3.52E−03	3.08E−03	2.57E−03	1.64E−03	1.41E−03	2.16E−03	1.38E−03	2.58E−03	2.57E−03
80	2.99E−03	2.74E−03	2.20E−03	1.40E−03	1.23E−03	1.91E−03	1.21E−03	2.21E−03	2.20E−03
90	2.41E−03	2.34E−03	2.66E−03	1.21E−03	1.09E−03	1.66E−03	1.10E−03	2.78E−03	2.65E−03
100	2.62E−03	2.22E−03	1.70E−03	1.25E−03	9.74E−04	1.54E−03	9.87E−04	1.75E−03	1.70E−03
200	1.25E−03	1.07E−03	1.05E−03	5.69E−04	4.91E−04	7.46E−04	4.94E−04	1.06E−03	1.05E−03
300	6.92E−04	7.18E−04	6.19E−04	3.77E−04	3.28E−04	5.14E−04	3.39E−04	6.48E−04	6.19E−04
400	5.69E−04	5.43E−04	5.04E−04	3.09E−04	2.42E−04	3.79E−04	2.49E−04	5.01E−04	5.04E−04
500	4.21E−04	4.31E−04	3.32E−04	2.63E−04	2.01E−04	3.13E−04	1.98E−04	3.48E−04	3.32E−04

Note. The methods correspond to the cases described in the text. The root-mean-squared relative errors are defined as follows:

$$RMS = \sqrt{\tfrac{1}{m}\textstyle\sum_{i=1}^{m} e_i^2}$$

where $e_i = (P_i^* - P_i)/P_i$ is the relative error, P_i is the true option price (Black–Scholes), and P_i^* is the estimated option price. To conduct a comprehensive analysis, we consider 625 options whose parameters cover a wide range: $S/X = 0.8, 0.9, \ldots, 1.2$; $r = 0.02, 0.04, \ldots, 0.1$; $q = 0, 0.015, \ldots, 0.06$; $\sigma = 0.2, 0.3, \ldots, 0.6$; and $T = 1.0$.

the equal-probability trinomial specification of He (1990) and Tian (1993) (Method 5(a)) and the sharpened trinomial specification of Omberg (1988) (Methods 5(d)) are more accurate than other methods with the same number of time steps. For example, with 500 time steps to price European calls, the RMS relative errors of both Methods 5(a) and 5(d) (0.0157% and 0.0153%, respectively) are about two-fifths of the errors of binomial methods (1) and (2) (0.0374% and 0.0381%, respectively).

One possible reason for the above results is as follows.[12] Comparing with the other methods in this paper, He and Tian use the specification of equal probabilities, whereas the Omberg sharpened-trinomial process has close-to-equal probabilities. Furthermore, since the Omberg sharpened-trinomial process is optimally designed to account for the non-smooth nature of the option payoff function, this suggests that this is the fundamental reason for superior performance, with the specification of equal probabilities being ad hoc approximations to Omberg.

Second, the convergence speed of each method is roughly at the rate of $O(1/n)$, that is, doubling the number of time steps will half the errors. For example, using the sharpened trinomial process (Method 5(d)) with

[12] We thank the referee for offering this explanation.

20, 40, and 80 time steps to price European calls, the RMS relative errors are 0.4%, 0.2%, and 0.1%, respectively. In contrast, Heston and Zhou (2000) showed that the rate of convergence of the trinomial model is of $O(1/n^2)$ *if the payoff functions are smooth enough*.

Finally, on the contrary, corrections to higher moments (mainly to the third moment) do not necessarily improve the RMS relative errors in the lattice approaches. For example, the RMS relative errors of method 4 (a trinomial process matching the first two moments) are actually smaller than those of method 3 (a trinomial process matching the first three moments).

We also use the *minimum convergence step* (MCS) proposed by Alan White in Tian (1993) to examine the convergence property of each method. The minimum convergence step for an acceptable error, ε, is defined as follows:

$$N(\varepsilon) = \min\left\{ N : \left| \frac{P(n) - P^*}{P^*} \right| < \varepsilon, \forall n \geq N \right\}$$

where P^* is the accurate price of an option, and $P(n)$ is the price of the option obtained from an n-step lattice. This definition highlights the importance of the convergence stability. It identifies the minimum number of steps that are needed to ensure that the relative error be less than the desired accuracy for all subsequent number of steps.[13]

We first compute the value of an option for each method using time steps ranging from 4, 5, . . . up to 800. By comparing these option values with the accurate value, the MCS for a given precision level is then obtained. Tables III and IV report the MCS of each method for pricing European call options and put options, respectively. The results are similar to those of Tables I and II, for example, Methods 5(a) and 5(d) are better choices of parameters. For European calls, if the acceptable error is 1% of the true price, the average MCSs of Methods 5(a) and 5(d) are about 9 steps while the average MCSs of the other trinomial processes [Methods (3), (4), and 5(c)] are more than 12 steps. In general we can conclude that trinomial methods are superior to the explicit finite difference methods and the binomial methods, and that corrections to higher moments do not necessarily improve the pricing errors for each method.

[13]As suggested by the referee, the evaluation by MCS may be more important than that by RMS to option traders because they would like to be sure of accuracy for all options in a sample or within a range of parameters.

TABLE III
Minimum Convergence Step of European Call Options

$N(\varepsilon)$	$\varepsilon = 1\%$	$\varepsilon = 0.5\%$	$\varepsilon = 0.25\%$
Method 1			
average	26.27	53.93	108.15
min	4	4	8
max	102	210	440
Method 2			
average	23.80	50.52	105.65
min	4	5	15
max	131	282	559
Method 3			
average	18.94	40.12	90.57
min	4	4	10
max	91	178	362
Method 4			
average	12.38	24.62	50.13
min	4	4	4
max	51	105	220
Method 5a			
average	9.00	18.10	38.52
min	4	4	4
max	56	106	252
Method 5c			
average	15.55	33.06	72.87
min	4	4	6
max	86	175	392
Method 5d			
average	9.04	18.34	39.19
min	4	4	4
max	50	111	234
Method 6a			
average	19.42	41.03	94.40
min	4	4	6
max	130	232	520
Method 6b			
average	18.90	40.07	90.41
min	4	4	10
max	91	178	362

Note. The methods correspond to the cases described in the text. The *minimum convergence step* for an acceptable error, ε, is defined as follows: $N(\varepsilon) = Min\{N: \frac{P(n)-P^*}{P^*} < \varepsilon, \forall n \geq N\}$. To make sure the convergence in the definition for all $n > N$, we check $n = 4, 5, \ldots$ up to 800. To conduct a comprehensive analysis, we consider 625 options whose parameters cover a wide range: $S/X = 0.8, 0.9, \ldots, 1.2$; $r = 0.02, 0.04, \ldots, 0.1$; $q = 0, 0.015, \ldots, 0.06$; $\sigma = 0.2, 0.3, \ldots, 0.6$; and $T = 1.0$.

TABLE IV

Minimum Convergence Step of European Put Options

$N(\varepsilon)$	$\varepsilon = 1\%$	$\varepsilon = 0.5\%$	$\varepsilon = 0.25\%$
Method 1			
average	24.88	52.03	106.10
min	4	4	7
max	97	206	425
Method 2			
average	22.96	49.42	103.81
min	4	4	8
max	162	311	646
Method 3			
average	16.26	35.41	79.12
min	4	4	4
max	87	155	347
Method 4			
average	11.16	23.59	46.16
min	4	4	4
max	41	103	195
Method 5a			
average	8.27	16.84	36.04
min	4	4	4
max	75	147	299
Method 5c			
average	14.60	31.49	68.14
min	4	4	6
max	85	181	337
Method 5d			
average	8.57	17.42	37.26
min	4	4	4
max	64	133	283
Method 6a			
average	17.05	39.15	86.21
min	4	4	6
max	88	196	408
Method 6b			
average	16.25	35.56	79.09
min	4	4	4
max	87	155	347

Note. The methods correspond to the cases described in the text. The *minimum convergence step* for an acceptable error, ε, is defined as follows: $N(\varepsilon) = Min\{N : \frac{P(n) - P^*}{P^*} < \varepsilon, \forall n \geq N\}$. To make sure the convergence in the definition for all $n > N$, we check $n = 4, 5, \ldots$ up to 800. To conduct a comprehensive analysis, we consider 625 options whose parameters cover a wide range: $S/X = 0.8, 0.9, \ldots, 1.2$; $r = 0.02, 0.04, \ldots, 0.1$; $q = 0, 0.015, \ldots, 0.06$; $\sigma = 0.2, 0.3, \ldots, 0.6$; and $T = 1.0$.

5. CONCLUSION

In this paper, we have designed a log-transformed trinomial approach to unify various numerical procedures appearing in the option pricing literature. In fact, these procedures correspond to the proper choice of six parameters in the trinomial framework. For the explicit finite difference method, we also correct the downward biased estimate of the variance of the transformed diffusion process and discuss its limiting case. Interestingly, Omberg (1988) derived the limiting case as the trinomial process corresponding to Gauss–Hermite integration. Figlewski and Gao (1999) derived the limiting case directly by imposing the restriction of matching the fourth moment.

The log-transformed trinomial approach has an intuitive and pedagogical appeal for researchers and is easily implemented by practitioners. In addition to Geske and Shastri (1985), we make a comparison of numerous lattice approaches and explicit finite difference approaches under our unified framework. The performance is measured in terms of the numerical accuracy and the convergence property. We use the root-mean-squared (RMS) relative error as the error measure. The results show that corrections to higher moments (mainly to the third moment) do not necessarily improve the RMS relative errors in the lattice approaches. In particular, the equal-probability trinomial specification of He (1990) and Tian (1993) and the sharpened trinomial specification of Omberg (1988) are more accurate than other methods with the same number of time steps. We also use the minimum convergence step (MCS) to examine the convergence property of each method. It identifies the minimum number of steps that are needed to ensure that the relative error be less than the desired accuracy for all subsequent number of steps. The results show that the equal-probability trinomial specification and the sharpened trinomial specification have superior convergence performance. Consequently, consistent results are observed across the two methods of evaluation.

In conclusion, within the log-transformed trinomial framework, we find that the equal-probability trinomial specification of He (1990) and Tian (1993) and the sharpened trinomial specification of Omberg (1988) outperform other lattice approaches and explicit finite difference methods.

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