MULTIPERIOD HEDGING WITH FUTURES CONTRACTS

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The hedging problem is examined where futures prices obey the cost-of-carry model. The resultant hedging model explicitly incorporates maturity effects in the futures basis. Formulas for the optimal static and dynamic hedges are derived. Although these formulas are developed for the case of direct hedging, the framework used is sufficiently flexible so that these formulas can be applied to many cross-hedging situations. The performance of the model is compared with that of several other models for two hedging scenarios: one involving a financial asset and the other involving a commodity. In both cases, significant maturity effects were found in the first and second moments of the futures basis. Our hedging formulas outperformed other hedging strategies on an ex-ante basis. © 2002 Wiley Periodicals, Inc. Jrl Fut Mark 22:1179–1203, 2002

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INTRODUCTION

Futures contracts are one of the main financial instruments used to hedge risk. The conventional method of futures hedging was developed by Johnson (1960) and Ederington (1979). Under this approach, the hedge ratio is set equal to the slope coefficient from a regression of changes in the spot price on changes in the futures price. This produces the minimum-variance futures hedge when spot and futures price changes are independently and identically distributed through time. In many situations, however, spot and futures price changes display predictible time patterns. In the case of direct hedging, for example, convergence of the basis will cause spot and futures prices to move more closely together as the time to maturity decreases.

Recently, several alternatives to the conventional hedging model have been proposed. Cecchetti, Cumby, and Figlewski (1988); Baillie and Myers (1991); and Myers (1991) examine the hedging problem when variances are conditionally autoregressive. Howard and D'Antonio (1991) determine the optimal multiperiod hedge when spot returns are autocorrelated but futures returns are not. McNew and Fackler (1994) estimate the dynamic hedge ratio under the assumption that it is linearly related to a specified set of informational variables. Because none of these models allows for convergence of the basis, they are appropriate only for situations involving cross-hedging.

Another group of articles incorporates basis convergence into its models. Kroner and Sultan (1993), Gagnon and Lypny (1995), and Park and Switzer (1995) present hedging models in which spot and futures prices are cointegrated in their logs and return disturbances are GARCH. Lien and Luo (1993) determine the optimal dynamic hedge when spot and futures prices are cointegrated in their levels and variances are homoskedastic. In a follow-up article, they extend this model to the case in which variances are conditionally autoregressive (Lien & Luo, 1994). Geppert (1995) solves for the optimal static hedge when spot and futures prices are cointegrated in their logs and variances are homoskedastic.[1] Chang, Chang, and Fang (1996) describe a hedging

[1]In deriving the optimal hedge, Geppert (1995) confuses log and absolute prices. The hedger's objective function, given by Equation 1 in his article, is written in terms of log prices. However, this equation requires that the change in the value of a portfolio equals the sum of the changes in value of its individual components. This assumption is valid when values are expressed in terms of absolute prices but not when they are expressed in terms of log prices. The impact of this error on the reported hedging results is unclear. Ceccetti, Cumby, and Figlewski (1988); Baillie and Myers (1991); Kroner and Sultan (1993); Gagnon and Lypny (1995); and Park and Switzer (1995) commit similar errors.

model in which the spot price follows a standard Weiner process and the futures basis follows a brownian bridge. Finally, Lien and Tse (1999) determine the optimal dynamic hedge when spot and futures prices are fractionally cointegrated in their levels and unexpected price changes are bivariate GARCH. Because these models require convergence of the futures basis, they apply only to situations of direct hedging.

Stultz (1984) and Adler and DeTemple (1988) examine hedging when spot and futures prices follow correlated diffusion processes. In both models, the hedger seeks to maximize expected utility rather than minimize price variance. However, Duffie (1989) shows that the optimal hedge ratio for a mean-variance utility maximizer can be broken into two portions: one reflecting speculative demand (which varies across individuals according to their risk adversion) and another reflecting a pure hedge (which is the same for all mean-variance utility hedgers). Thus, utility maximization and variance minimization are closely linked in these settings. Because the pure hedge term is common to all hedgers and the speculative demand term is both difficult to estimate and often close to zero, Duffie (1989) argues that that it is reasonable to focus attention on the pure hedge. Neither model allows for convergence of the basis except for the special case in which the futures and spot prices are perfectly correlated, which implies that the basis is deterministic. Consequently, they are appropriate only for situations involving cross-hedging.

These alternative hedging models have several limitations. First, all of these models assume a particular relationship between spot and futures prices rather than deriving one directly from futures pricing theory. This leads to conflicting price processes across the models. For example, does convergence of the basis imply that spot and futures prices should be cointegrated in their logs (Kroner & Sultan, 1993; Gagnon & Lypny, 1995; Park & Switzer, 1995; Geppert, 1995) or in their levels (Lien & Luo, 1993, 1994)? As another example, the model of Howard and D'Antonio (1991) assumes that spot returns are autocorrelated and futures returns are random, whereas the model of Chang, Chang, and Fang (1996) implies that futures returns are autocorrelated and spot returns are random. A second weakness is that each model is limited in scope either to direct hedging scenarios or to cross-hedging scenarios. Third, most of these hedging models are myopic. When the joint distribution of spot and futures price changes has a predictable component, myopic hedging models are suboptimal when the hedger's horizon extends over multiple periods. The models of Howard and D'Antonio (1991), Lien and Luo (1993, 1994, 1999), and Geppert

(1995) are nonmyopic and therefore not subject to this criticism. Finally, with the exception of the model by Chang, Chang, and Fang (1996), these models do not consider the impact of the time to maturity of the futures contract on the optimal hedge ratio.[2] It is generally believed that basis risk tends to decrease as the time to maturity shortens. For example, both Duffie (1989) and Hull (1997) recommend using the futures contract whose maturity least exceeds the hedger's horizon to minimize the basis risk associated with a simple hedge.

In this article, we examine the hedging problem when futures prices obey the cost-of-carry model. The resultant hedging model explicitly incorporates maturity effects in the futures basis. Formulas for the optimal static and dynamic hedges are derived. Although these formulas are developed for the case of direct hedging, the framework used is sufficiently flexible that these formulas can be applied to many cross-hedging situations. The performance of the model is measured for two of the following hedging scenarios: one involving a financial asset and the other involving a commodity. In both cases, significant maturity effects are found in the first and second moments of the futures basis. The static cost-of-carry hedge outperforms the dynamic cost-of-carry hedge in both scenarios. This is attributed to the lower sensitivity of the static hedge to estimation errors. Both cost-of-carry hedges outperform the cointegrated price hedge [Lien and Luo (1993)], the GARCH hedge [e.g., Kroner & Sultan (1993)], and the convention hedge [e.g., Ederington (1970)] on an ex-ante basis. Additionally, the effectiveness of the hedge increases with its duration.

THE MINIMUM-VARIANCE HEDGE

Consider an individual who has taken a fixed position in some asset. Without loss of generality, assume that this person is long one unit of the asset. A decision is made to hedge this position for the next T periods using a particular futures contract. Let h_t represent the short position taken in the futures market at time t under the adopted hedging strategy. Ignoring daily resettlement, the net payoff at time T to the hedger will be

$$P_T = S_T - \sum_{t=0}^{T-1} h_t(F_{t+1} - F_t)$$

[2]McNew and Fackler (1994) incorporate a time to a roll-over variable in their hedging model. Because they roll over each futures contract in the week that it expires, this variable is nearly identical to the time to maturity. Their model only allows the optimal hedge ratio to be a linear function of the time to roll over, however.

where S_t and F_t are the respective spot and futures prices at time t.[3] We assume that the objective of the hedger is to minimize the variance of the net payoff at time T. Duffie (1989) shows that the optimal hedge ratio for a person with mean-variance utility can be broken into two of the following portions: one reflecting speculative demand (which varies across individuals according to their risk adversion) and another reflecting a pure hedge (which is the same for all mean-variance utility hedgers). Because the pure hedge term is common to all hedgers and the speculative demand term is both difficult to estimate and often close to zero, Duffie argues that that it is reasonable to focus attention on the pure hedge.

Ferguson and Leistikow (1999) argue that the spot price in this variance minimization problem should be adjusted for its cost of carry. This is not done here for two reasons. First, adjusting the spot price for its cost of carry will have no impact on the optimal hedge ratios when the cost-of-carry rate is constant. Second, the net cost of carry is difficult to measure with precision for many assets. In the case of a commodity, for example, the net cost of carry includes the convience yield of the commodity, which is unobservable. As a result, some proxy for the net cost of carry must be used in the hedging problem. This subjects the computed hedge ratios to an additional source of estimation error. Empirically, Ferguson and Leistikow (1999) discovered that, for hedging the price of gold, their approach does not outperform a myopic error correction model that does not adjust spot prices for their cost of carry.

Suppose the hedger decides to pursue a dynamic hedging strategy. The optimal dynamic hedge is determined using backward iteration. At time $T - 1$, the conditional variance of the preceding payoff is

$$\mathrm{Var}_{T-1}(P_T) = \mathrm{Var}_{T-1}(S_T) - 2h_{T-1}\,\mathrm{Cov}_{T-1}(S_T, F_T) + h_{T-1}^2\,\mathrm{Var}_{T-1}(F_T)$$

which is minimized by choosing a hedge ratio of

$$h_{T-1}^* = \frac{\mathrm{Cov}_{T-1}(S_T, F_T)}{\mathrm{Var}_{T-1}(F_T)} \tag{1}$$

This corresponds to the conventional hedge ratio when changes in both spot and futures prices are homoskedastic. Next, consider the hedger's

[3]When the interest rate is fixed, the hedging formulas presented in this section can easily be adjusted to account for daily resettlement. When the daily interest rate is r, the optimal hedge ratio under daily resettlement, H_t^*, is related to the optimal hedge ratio with no daily resettlement, h_t^*, by the following formula: $H_t^* = h_t^*/(1 + r)^{M_t - 1}$, where M_t represents the number of days to maturity for the futures contract at time t. This adjustment for daily resettlement is commonly referred to as "tailing the hedge."

problem at time $T - k$. Assume that the hedger has already solved for the optimal hedge ratios $h^*_{T-2}, h^*_{T-3}, \ldots, h^*_{T-k+1}$. These hedge ratios may depend on price information after $T - K$. In such situations, these hedge ratios must be treated as stochastic variables at time $T - K$. The conditional variance at $T - k$ of the hedger's payoff is

$$\begin{aligned} \mathrm{Var}_{T-k}(P_T) &= \mathrm{Var}_{T-k}\left[S_T - \sum_{t=T-k+1}^{T-1} h^*_t(F_{t+1} - F_t)\right] \\ &\quad + 2h_{T-k}\,\mathrm{Cov}_{T-k}\left[F_{T-k+1}, S_T - \sum_{t=T-k+1}^{T-1} h^*_t(F_{t+1} - F_t)\right] \\ &\quad + h^2_{T-k}\,\mathrm{Var}_{T-k}(F_{T-k+1}) \end{aligned}$$

Because the hedge ratios $h^*_{T-2}, h^*_{T-3}, \ldots, h^*_{T-k+1}$ occur within the conditional covariance operator, they can be either stochastic or deterministic at time $T - K$. The conditional variance is minimized by choosing a hedge ratio of

$$h^*_{T-k} = \frac{\mathrm{Cov}_{T-k}[F_{T-k+1}, S_T - \sum_{t=T-k+1}^{T-1} h^*_t(F_{t+1} - F_t)]}{\mathrm{Var}_{T-k}(F_{T-k+1})} \tag{2}$$

By using Equation 1 to compute the hedge ratio in the final period and then applying Equation 2 recursively to solve for the hedge ratio in each prior period, the optimal dynamic hedge can be completely determined. Inspection of these two equations reveals that the hedge ratios will be deterministic at time 0 when spot and futures price changes are homoskedastic. When price changes are conditionally heteroskedastic, each hedge ratio will depend on the current (and possibly past) spot and futures prices at that time. A simple example would be a situation in which spot and futures returns are homoskedastic. In this case, conditional price variances would be proportional to current price levels; therefore, the optimal hedge ratio would depend on the relative spot and futures price levels at the time.

In many situations, the hedger may prefer to use a static hedging strategy. Static hedging involves much lower transaction costs than dynamic hedging. Additionally, static hedging strategies are less sensitive to estimation and modeling errors than dynamic hedging strategies. As a result, static hedges may frequently outperform dynamic hedges in practice. Suppose the hedger decides to pursue a static hedging strategy. When the hedge ratio is held constant, the net payoff to the hedger simplifies to

$$P_T = S_T - h(F_T - F_0)$$

The conditional variance of this payoff is

$$\mathrm{Var}_0(P_T) = \mathrm{Var}_0(S_T) - 2h\,\mathrm{Cov}_0(S_T, F_T) + h^2\,\mathrm{Var}_0(F_T)$$

which is minimized by setting the hedge ratio to

$$h^{**} = \frac{\mathrm{Cov}_0(S_T, F_T)}{\mathrm{Var}_0(F_T)} \tag{3}$$

This static hedge is identical to the conventional hedge when spot and futures price changes are independently and identically distributed through time.

THE COST-OF-CARRY MODEL

To implement the hedging formulas in the previous section, a model of the joint process for spot and futures prices is required. Consider the following cost-of-carry model:

$$F_t = S_t \cdot \exp\left\{ \sum_{k=1}^{M_t} c_{t+k} \right\}$$

where c_i represents the expected net cost of carry for period i, and M_t represents the time to maturity of the futures contract. By taking logs, this formula can be rewritten as

$$\ln(F_t) - \ln(S_t) = \sum_{k=1}^{M_t} c_{t+k} \tag{4}$$

For ease of exposition, we will refer to $\ln(F_t) - \ln(S_t)$ as the log-basis. Suppose market expectations about the cost of carry for each period are drawn from independent and identical normal distributions, each with a mean of $\bar{c}$ and a variance of σ^2. Then, the log-basis will be normally distributed with a mean of $\bar{c}M_t$ and a variance of $\sigma^2 M_t$. This implies that both the first and second moments of the log-basis will be linear functions of the time to maturity.

Such maturity effects are readily observed in futures prices. Figure 1 portrays the log-basis for the Nikkei 225 Index futures contract traded on the Singapore International Monetary Exchange (SIMEX) as a function of time to maturity.[4] The average log-basis is positive, indicating that the futures price tends to be higher than the spot price. More importantly, the average log-basis increases linearly with the time to

[4]Details about the data used in this article are provided in a later section.

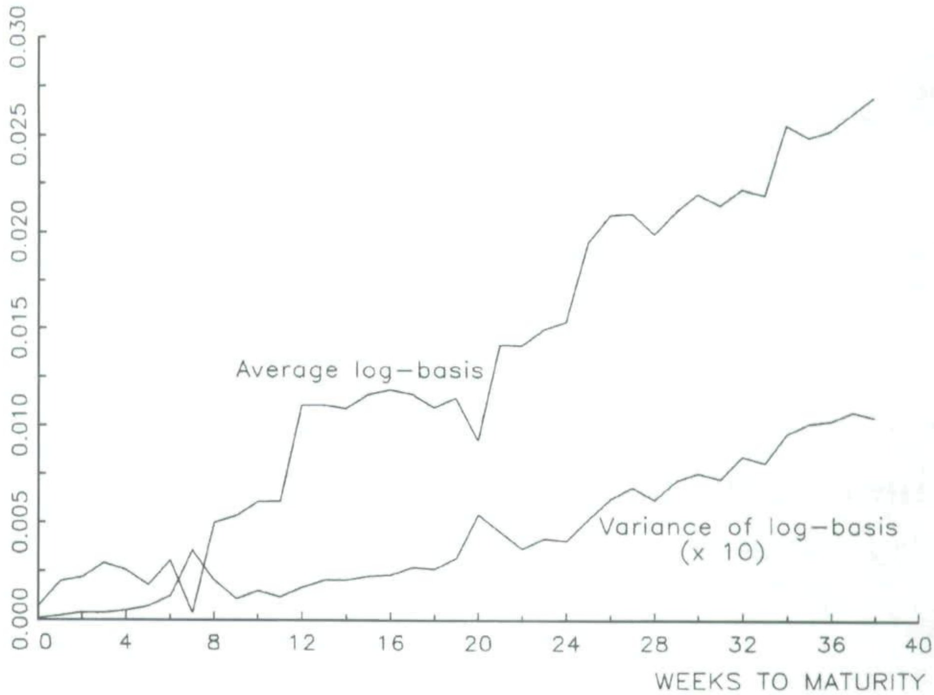

FIGURE 1
The log-basis for the Nikkei 225 Index.

maturity. The variance of the log-basis also linearly increases in the time to maturity (the downward spike in the average log-basis and upward in the variance of the log-basis at seven weeks to maturity is attributable to the October 1989 market crash). Similar patterns are observed in the log-basis of the High Sulphur Fuel Oil futures contract on SIMEX, which is presented in Figure 2. The only significant difference is that average log-basis for fuel oil is negative, reflecting its high convenience yield.

To determine whether these observed maturity effects are significant, the average and variance of the log-basis were regressed against the time to maturity for both futures contracts. The results are listed in Table I. For the Nikkei 225 Index contract, both the average log-basis ($t = 30.85$) and the variance of the log-basis ($t = 9.82$) significantly increased with the time to maturity. However, the average log-basis significantly decreased ($t = -9.56$), and the variance of the log-basis ($t = 11.63$) significantly increased with the time to maturity. None of the intercept terms were significant, indicating that the hypothesized proportional relationship between the first two moments of the log-basis and the time to maturity fits the data well.

TABLE I
Regression Results of Log-Basis Against Time to Maturity

	Nikkei 225 Index		*High Sulphur Fuel Oil*	
	Average Log-Basis	*Variance of Log-Basis*	*Average Log-Basis*	*Variance of Log-Basis*
α	−0.000471	0.000013	−0.000657	−0.000148
	(−0.91)	(0.55)	(−0.47)	(−0.76)
β	0.000720	0.000011	−0.002485	0.000419
	(30.85)	(9.82)	(−9.56)	(11.63)

Note. t Statistics are given in parentheses.

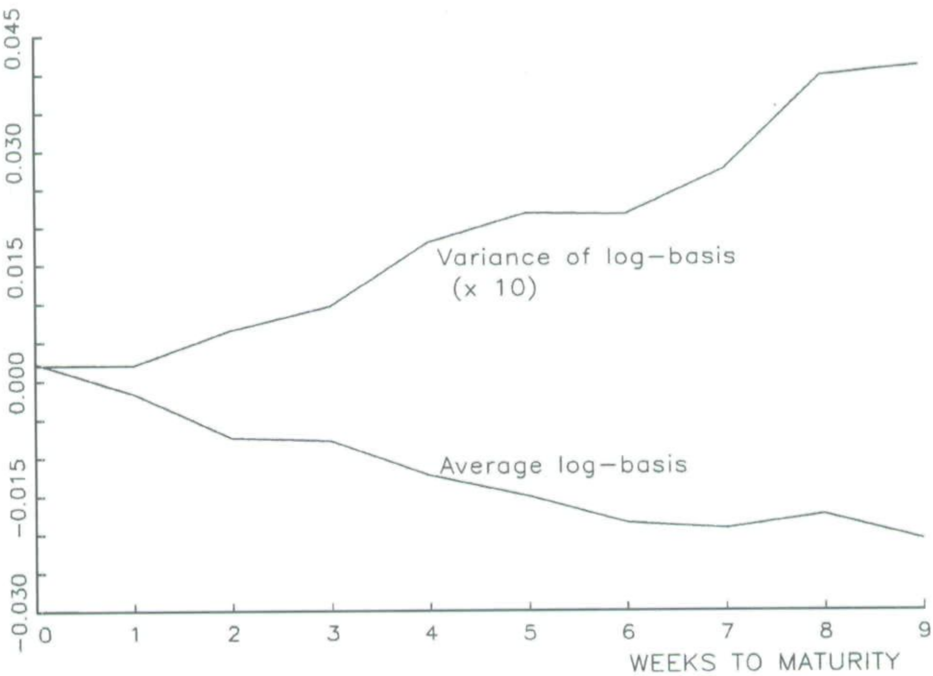

FIGURE 2
The log-basis for high sulphur fuel oil.

When the expected cost of carry for each period has a stationary distribution, Equation 4 implies that the futures and underlying spot prices will be cointegrated, with the cointegrating vector given by

$$Z_t = \ln(F_t) - \ln(S_t) - \bar{c}M_t$$

This differs from the usual cointegrating vector included in futures hedging models as a result of the presence of a time to maturity term [cf. Lien & Luo (1993, 1994), Kroner & Sultan (1993), Gagnon & Lypny (1995), and Park & Switzer (1995)]. The Granger Representation Theorem (Engle & Granger, 1987), which states that cointegrated price series can be represented by an error correction model, requires that each price series be purely nondeterministic. This implies that a time to maturity term must be included in any error-correction representation of spot and futures prices.[5] Several hedging models incorporate GARCH effects into futures and spot price changes [cf. Kroner & Sultan (1993), Lien & Luo (1994), Gagnon & Lypny (1995), and Park & Switzer (1995)]. When testing for conditional heteroskedasticity and estimating GARCH parameters, a futures return series based on the nearest futures contract in each period is normally used. As each successive maturity date is passed, the time to maturity of the nearest futures contract and thus the variance of the log-basis will jump upward. This induces a cyclical pattern in the conditional variance of the log-basis. Failure to account for this deterministic time pattern in the variance could result in the identification of spurious GARCH effects.

Granger's Representation Theorem implies that the futures and underlying spot prices can be written as

$$\Delta \ln(S_t) = \mu_s + \theta_s \cdot \mathbf{Z}_{t-1} + \sum_{n=1}^{N} [\gamma_{sn} \cdot \Delta \ln(S_{t-n}) + \delta_{sn} \cdot \Delta \ln(F_{t-n})] + \epsilon_{st} \tag{5a}$$

$$\Delta \ln(F_t) = \mu_f + \theta_f \cdot \mathbf{Z}_{t-1} + \sum_{n=1}^{N} [\gamma_{fn} \cdot \Delta \ln(S_{t-n}) + \delta_{fn} \cdot \Delta \ln(F_{t-n})] + \epsilon_{ft} \tag{5b}$$

where the cointegrating vector $\mathbf{Z}_t$ is defined as previously noted, and N is the number of lagged returns necessary to "whiten" the residuals. The variance–covariance matrix of the residual errors is assumed to be linear in the time to maturity

$$\Sigma_t = \begin{pmatrix} \boldsymbol{\alpha}_{ss} + \boldsymbol{\beta}_{ss} M_t & \boldsymbol{\alpha}_{sf} + \boldsymbol{\beta}_{sf} M_t \\ \boldsymbol{\alpha}_{sf} + \boldsymbol{\beta}_{sf} M_t & \boldsymbol{\alpha}_{ff} + \boldsymbol{\beta}_{ff} M_t \end{pmatrix}$$

[5]The precise definition of cointegration requires the cointegrating vector to be covariance stationary. Because Equation 4 implies that the variance of $\mathbf{Z}_t$ will be a function of the time to maturity, the futures and underlying spot price cannot be cointegrated in a strict sense. However, Hansen (1992) shows that much of the statistical theory developed under the strict definition of cointegration continues to hold when heteroskedasticity is permitted in the cointegrating vector. He refers to this more general definition of cointegration as heteroskedastic cointegration

Maximum likelihood estimates of the parameters in this error-correction representation are readily obtained from the log-likelihood function

$$LL = -\frac{1}{2}\sum_{t=1}^{N} \ln(2\pi) + \ln[\mathrm{Det}(\Sigma_t)] + (\epsilon_{st}\ \epsilon_{ft})\Sigma_t^{-1}(\epsilon_{st}\ \epsilon_{ft})'$$

The number of lagged returns included in the model cannot be determined a priori. Instead, it must be choosen using some empirically based decision rule such as the Akaike Information Criterion or the Schwartz Bayesian Criterion.

HEDGING FORMULAS

To derive explicit hedging formulas, it is helpful to rewrite the preceding error-correction representation as follows. First, Equation 5 is expressed in terms of individual log-prices

$$\begin{aligned} \ln(S_t) = {} & \mu_s - \theta_s \bar{c} M_{t-1} + [1 + \gamma_{s1} - \theta_s]\ln(S_{t-1}) \\ & + [\delta_{s1} + \theta_s]\ln(F_{t-1}) + [\gamma_{s2} - \gamma_{s1}]\ln(S_{t-2}) \\ & + \cdots + [\gamma_{sN} - \gamma_{sN-1}]\ln(S_{t-N}) - \gamma_{sN}\ln(S_{t-N-1}) \\ & + [\delta_{s2} - \delta_{s1}]\ln(F_{t-2}) + \cdots + [\delta_{sN} - \delta_{sN-1}]\ln(F_{t-N}) \\ & - \delta_{sN}\ln(F_{t-N-1}) + \epsilon_{st} \end{aligned} \tag{6a}$$

and

$$\begin{aligned} \ln(F_t) = {} & \mu_f - \theta_f \bar{c} M_{t-1} + [\gamma_{f1} - \theta_f]\ln(S_{t-1}) \\ & + [1 + \delta_{f1} + \theta_f]\ln(F_{t-1}) + [\gamma_{f2} - \gamma_{f1}]\ln(S_{t-2}) \\ & + \cdots + [\gamma_{fN} - \gamma_{fN-1}]\ln(S_{t-N}) - \gamma_{fN}\ln(S_{t-N-1}) \\ & + [\delta_{f2} - \delta_{f1}]\ln(F_{t-2}) + \cdots + [\delta_{fN} - \delta_{fN-1}]\ln(F_{t-N}) \\ & - \delta_{fN}\ln(F_{t-N-1}) + \epsilon_{ft} \end{aligned} \tag{6b}$$

Equation 6 implies that

$$\mathbf{X}_t = A X_{t-1} + \epsilon_t \tag{7}$$

where the $2(N+2)\times 1$ vectors $\mathbf{X}_t$ and ϵ_t are given by

$$\mathbf{X}_t = [1 \quad M_t \quad \ln(S_t) \quad \ln(F_t) \quad \ln(S_{t-1}) \quad \ln(F_{t-1}) \quad \cdots \quad \ln(S_{t-N}) \quad \ln(F_{t-N})]'$$

and

$$\epsilon_t = (0 \quad 0 \quad \epsilon_{st} \quad \epsilon_{ft} \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0)'$$

The transition matrix $\mathbf{A}$ is specified by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu_s & -\theta_s\bar{c} & 1+\gamma_{s1}-\theta_s & \delta_{s1}+\theta_s & \cdots & \delta_{sN}-\delta_{sN-1} & -\gamma_{sN} & -\delta_{sN} \\ \mu_f & -\theta_f\bar{c} & \gamma_{f1}-\theta_f & 1+\delta_{f1}+\theta_f & \cdots & \delta_{fN}-\delta_{N-1f} & -\gamma_{fN} & -\delta_{fN} \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}$$

and the variance–covariance matrix for ϵ_t is given by the partitioned matrix

$$\mathbf{V}_t = \begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times N} \\ \mathbf{0}_{2\times 2} & \Sigma_t & \mathbf{0}_{2\times N} \\ \mathbf{0}_{N\times 2} & \mathbf{0}_{N\times 2} & \mathbf{0}_{N\times N} \end{pmatrix}$$

By applying Equation 7 iteratively, the conditional distribution of spot and futures prices can be determined for any future data. Specifically

$$S_{t+k} = \exp\{\mathbf{1}_s\mathbf{A}^k\mathbf{X}_t\}\cdot\exp\left\{\mathbf{1}_s\sum_{i=1}^{k}\mathbf{A}^{k-i}\epsilon_{t+i}\right\} \tag{8a}$$

and

$$F_{t+k} = \exp\{\mathbf{1}_f\mathbf{A}^k\mathbf{X}_t\}\cdot\exp\left\{\mathbf{1}_f\sum_{i=1}^{k}\mathbf{A}^{k-i}\epsilon_{t+i}\right\} \tag{8b}$$

where the vectors $\mathbf{1}_s$ and $\mathbf{1}_f$ are given by

$$\mathbf{1}_s = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0)$$

and

$$\mathbf{1}_f = (0 \quad 0 \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0 \quad 0)$$

Equation 8 expresses the spot and futures prices at time $t+k$ as the product of two of the following components: a deterministic component that incorporates the information available at time t and a stochastic component that captures the random price shocks between times t and $t + k$. Because these random shocks are assumed to be normally distributed

$$E[\exp\{B\epsilon_t\}] = \exp\{B\, VB'/2\} \tag{9}$$

for any constant B. By applying this formula to Equation 8, the first and second moments required to determine the optimal dynamic and static hedges can be computed. These optimal hedges are summarized in the following two propositions.

Proposition 1. *The optimal static hedge ratio is given by*

$$h^{**} = a \cdot \exp\{(\mathbf{1}_s - \mathbf{1}_f)\mathbf{A}^T\mathbf{X}_0 + \boldsymbol{b}\} \tag{10}$$

where a and b are constants given by

$$a = \frac{\exp\{\mathbf{1}_s \boldsymbol{W} \mathbf{1}_f'\} - 1}{\exp\{\mathbf{1}_f \boldsymbol{W} \mathbf{1}_f'\} - 1}$$

and

$$b = [1_s W 1_s' - 1_f W 1_f']/2$$

and W *is defined by*

$$W = \sum_{t=0}^{T-1} \mathrm{A}^t V_{T-t} \mathrm{A}^t$$

Proposition 2. *The optimal dynamic hedge ratios are given by*

$$h_{T-k}^{*} = \sum_{i=1}^{2^{k-1}} \boldsymbol{a}(\boldsymbol{i}, \boldsymbol{k}) \cdot \exp\{[\boldsymbol{J}(\boldsymbol{i}, \boldsymbol{k}) - \mathbf{1}_f]\mathbf{A}\mathbf{X}_{T-k} + \boldsymbol{b}(\boldsymbol{i}, \boldsymbol{k})\} \tag{11}$$

where the vector $\boldsymbol{J}(\boldsymbol{i}, \boldsymbol{k})$ *and the scalars* $\boldsymbol{a}(\boldsymbol{i}, \boldsymbol{k})$ *and* $\boldsymbol{b}(\boldsymbol{i}, \boldsymbol{k})$ *are defined recursively as*

$$\boldsymbol{J}(1, 1) = \mathbf{1}_s$$

$$\boldsymbol{J}(\boldsymbol{i}, \boldsymbol{k} + 1) = \begin{cases} \boldsymbol{J}(\boldsymbol{i}, \boldsymbol{k})\mathrm{A} & \text{if } i \le 2^{k-1} \\ \boldsymbol{J}(\boldsymbol{i} - 2^{k-1}, \boldsymbol{k})\mathbf{A} + \mathbf{1}_f(\boldsymbol{I} - \mathrm{A}) & \text{if } 2^{k-1} < i \le 2^k \end{cases}$$

$$\boldsymbol{a}(1, 1) = \boldsymbol{f}(1, 1)$$

$$\boldsymbol{a}(\boldsymbol{i}, \boldsymbol{k} + 1) = \begin{cases} \boldsymbol{f}(\boldsymbol{i}, \boldsymbol{k} + 1)[1/\boldsymbol{f}(\boldsymbol{i}, \boldsymbol{k}) - 1] \cdot \boldsymbol{a}(\boldsymbol{i}, \boldsymbol{k}) & \text{if } i \le 2^{k-1} \\ \boldsymbol{f}(\boldsymbol{i}, \boldsymbol{k} + 1) \cdot \boldsymbol{a}(\boldsymbol{i} - 2^{k-1}, \boldsymbol{k}) & \text{if } 2^{k-1} < i \le 2^k \end{cases}$$

and

$$b(1, 1) = [1_s V_T 1_s' - 1_f V_T 1_f']/2$$

$$b(i, k+1) = \begin{cases} b(i, k) + [g(i, k+1) + 1_f V_{T-k+1} 1_f']/2 & \text{if } i \le 2^{k-1} \\ b(i - 2^{k-1}, k) + g(i, k+1)/2 & \text{if } 2^{k-1} < i \le 2^k \end{cases}$$

and the functions $f(i, k)$ and $g(i, k)$ *are defined as*

$$f(i, k) = \frac{\exp\{J(i, k) V_{T-k+1} 1_s'\} - 1}{\exp\{1_f V_{T-k+1} 1_f'\} - 1}$$

and

$$g(i, k) = J(i, k) V_{T-k+1} J'(i, k) - 1_f V_{T-k+1} 1_f'$$

The proofs of both propositions are given in the appendix. Computationally, the optimal static hedge is much simpler than the optimal dynamic hedge. The formula for the optimal static hedge involves T terms that equal the hedger's time horizon. In contrast, the formula for the optimal dynamic hedge involves as many as 2^{T-1} terms.

EXTENSIONS OF THE MODEL

Propositions 1 and 2 do not depend on the exact specification of the transition matrix A. Consequently, they apply to any situation in which the process for spot and futures log prices can be expressed as a linear function of past spot and futures log prices, the time to maturity of the futures contract, and an i.i.d. error term. This will include many cross-hedging situations. For example, the existence of substitution effects between the asset being hedged and the asset underlying the futures contract would cause the spot and futures price changes to be cross-auto-correlated. As a result, spot and futures prices can be expressed in vector autoregressive form as

$$\Delta \ln(S_t) = \mu_s + \sum_{n=1}^{N} [\gamma_{sn} \cdot \Delta \ln(S_{t-n}) + \delta_{sn} \cdot \Delta \ln(F_{t-n})] + \epsilon_{st} \quad \text{(12a)}$$

$$\Delta \ln(F_t) = \mu_f + \sum_{n=1}^{N} [\gamma_{fn} \cdot \Delta \ln(S_{t-n}) + \delta_{fn} \cdot \Delta \ln(F_{t-n})] + \epsilon_{ft} \quad \text{(12b)}$$

This representation is identical to Equation 5 except for the omission of a cointegrating term. Following the same steps as before, this can be rewritten as

$$X_t = AX_{t-1} + \epsilon_t$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu_s & 0 & 1+\gamma_{s1} & \delta_{s1} & \cdots & \delta_{sN}-\delta_{sN-1} & -\gamma_{sN} & -\delta_{sN} \\ \mu_f & 0 & \gamma_{f1} & 1+\delta_{f1} & \cdots & \delta_{fN}-\delta_{N-1f} & -\gamma_{fN} & -\delta_{fN} \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}$$

and the variance–covariance matrix for $\boldsymbol{\epsilon}_t$ is unchanged from the direct hedging case. As before, we can use maximum likelihood techniques to estimate the transition matrix A and the variance–covariance matrix and then apply Propositions 1 and 2 to determine the optimal dynamic and static hedges.

The model can also be extended by allowing the transition matrix A to be time dependent. This could include the addition of nonlinear maturity effects, seasonalities (especially in the case of commodities), calendar anomalies (e.g., the January effect), or contract-specific effects. These effects could also be incorporated into the variance–covariance matrix V_t^s. The formulas for the optimal dynamic and static hedges would change slightly. However, the techniques outlined in this article could be used to determine these alternate formulas.

EMPIRICAL RESULTS

The effectiveness of the cost-of-carry hedge is examined for two direct hedging scenarios. In the first scenario, the Nikkei 225 Index futures contract on SIMEX is used to hedge a portfolio of the component stocks of this index. The second scenario involves hedging a spot position in fuel oil using the High Sulphur Fuel Oil contract on SIMEX. Wednesday settlement prices and closing spot prices are used in both studies, with Thursday prices being substituted when Wednesday prices are unavailable. Midweek prices were chosen to avoid potential weekend price effects [cf. French (1980) and Gibbons & Hess (1981)]. The sample period extends from September 1986 to April 1996 for the Nikkei 225 Index data and from February 1989 to June 1995 for the High Sulphur Fuel Oil data.[6]

[6]The data were provided by SIMEX. The start of each sample period coincides with the initial date of futures trading. The sample period for High Sulphur Fuel Oil ends in June 1995 because of significant nontrading of the futures contract in subsequent months.

Both sample periods are divided into overlapping three-year windows. The first two years in each window are used to estimate the parameters of the hedging models. These estimates are then used to form futures hedges over the third year of the window. For the Nikkei 225 Index, hedging horizons of 1, 2, 5, 10, and 15 weeks are considered. Hedges of between 1 and 5 weeks are analyzed for fuel oil. Shorter durations are used for the fuel oil hedges because of the fact that the High Sulphur Fuel Oil contract on SIMEX follows a monthly cycle, whereas the Nikkei 225 Index contract follows a quarterly cycle. Volume for both contracts is heavily concentrated in the nearest two contracts. For each hedging horizon considered, 52 hedges are computed within each window—one beginning each week of the third year of the window. No rolling over of hedges is permitted. Instead, the nearest contract whose maturity extends beyond the hedging horizon is used to hedge over the entire horizon. This avoids potential confounding of the results due to roll-over risk that is not formally modeled in any of the hedges that are considered.

The static cost-of-carry hedge is compared against the following alternatives: the conventional hedge [e.g., Ederington (1979)], the cointegrated price hedge (Lien & Luo, 1993), and the GARCH hedge [e.g., Kroner & Sultan (1993)]. No attempt is made to incorporate nonlinear maturity effects, seasonalities and other calendar pricing anomalies, or contract-specific effects into the static maturity effects model as this would unfairly bias the results toward this model. The number of lagged returns included in the cointegrated price, GARCH, and cost-of-carry hedging models is selected using the Schwartz Bayesian Criterion.[7] Hedging effectiveness is measured by the percentage reduction in price variance that is achieved under each of the four hedges.

Because each hedge is instituted using only data that were available at the time, this procedure provides a realistic framework for evaluating the effectiveness of the four hedging models. More frequent updating of the hedging parameters could be done; however, by only updating yearly, our procedure provides a test of the stability of the parameters of each model. Instability of the hedging parameters suggests possible model misspecification. The choice of a two-year estimation period for the

[7]Lien and Luo (1993) assume a single pair of lagged spot and futures returns in their model. However, their model is easily extended to allow for an arbitrary number of lagged terms. Similar comments apply to the GARCH hedging models cited herein. For the two hedging scenarios examined in this article, hedging performance is relatively insensitive to the exact number of lagged terms included in the models.

TABLE II
Maximum-Likelihood Estimates for the Cost-of-Carry Model

	Nikkei 225 Index			*High Sulphur Fuel Oil*		
Parameter	*Spot*		*Futures*	*Spot*		*Futures*
μ	−0.0014		0.0006	0.0018		0.0022
	(0.0004)		(0.0004)	(0.0004)		(0.0004)
θ	0.0400		−0.9991	0.3833		−0.2792
	(0.1225)		(0.1261)	(0.0259)		(0.0261)
c		0.0002			−0.0003	
		(0.0000)			(0.0001)	
α (×100)	0.0733		0.0754	0.1108		0.1087
	(0.0021)		(0.0022)	(0.0040)		(0.0042)
		0.0738			0.1069	
		(0.0021)			(0.0040)	
β (×100)	0.0021		0.0024	0.0475		0.0499
	(0.0003)		(0.0003)	(0.0023)		(0.0024)
		0.0022			0.0471	
		(0.0003)			(0.0023)	

Note. Standard errors are given in parentheses.

parameters was arbitrary. Attempts to find the optimal estimation period for each model would induce data-snooping biases into our procedure.

A set of representative parameter estimates for the cost-of-carry hedging model is presented in Table II. In both cases, no lagged returns were required to "whiten" the residuals in the error-correction representation. These estimates correspond to the last window examined under each of the two hedging scenarios. For Nikkei 225 Index futures, $\theta_s - \theta_f = 1.04$. This implies that any deviation in the log-basis from its conditional mean tends to be completely corrected within one week. An equivalent calculation for High Sulphur Fuel Oil futures indicates that 66% of any deviation from the conditional mean of its log-basis is corrected within one week. Maturity effects in the mean log-basis are significant for both futures contracts. The parameter c is positive for the Nikkei 225 Index that implies a positive cost of carry for its futures. In contrast, the high convenience yield of fuel oil causes the High Sulphur Fuel Oil contract to display a negative cost of carry. Significant maturity effects are also present in the residual variances of spot and futures returns. For both the Nikkei 225 Index and High Sulphur Fuel Oil, the magnitude of unexpected spot and futures returns increases with the time to maturity of the futures contract. The correlation between the spot and futures returns decreases slightly for both assets as the time to maturity of the futures contract is lengthened.

TABLE III
Comparison of Hedging Effectiveness

	Duration of Hedge (in Weeks)				
	1	2	5	*10*	*15*
A. Nikkei 225 Index					
Conventional hedge	94.72	96.25	96.56	96.85	96.88
Cointegrated price hedge	94.56	96.15	96.42	96.71	96.81
GARCH hedge	45.26	43.70	33.27	31.49	32.77
Static cost-of-carry hedge	94.99	97.27	98.86	99.18	98.19
Dynamic cost-of-carry hedge	94.99	95.61	97.52	98.73	97.67
B. High Sulphur Fuel Oil					
Conventional hedge	82.40	87.61	90.03	90.58	90.16
Cointegrated price hedge	82.64	88.71	90.80	91.84	91.56
GARCH hedge	n/a[a]	n/a	n/a	n/a	n/a
Static cost-of-carry hedge	82.22	88.12	91.82	93.22	93.03
Dynamic cost-of-carry hedge	82.20	86.76	90.70	91.97	91.85

[a]Efficiency is negative.

The effectiveness of the different hedging strategies is presented in Table III. With the exception of the GARCH hedge, the effectiveness of the hedging strategies initially increases with the hedging horizon and then level offs. For the Nikkei 225 Index, the static cost-of-carry hedge is the most effective hedging strategy irrespective of the hedging horizon. The dynamic cost-of-carry hedge is the second-best performer, followed by the conventional hedge and the cointegrated price hedge. The GARCH hedge performs poorly. A similar pattern is found for High Sulfur Fuel Oil. For hedging horizons beyond two weeks, the static cost-of-carry hedge performs best, followed by the dynamic cost-of-carry hedge and the cointegrated price hedge. However, for hedging horizons of two weeks or less, the cointegrated price hedge is the best performer. The GARCH hedge has a negative efficiency that implies that it performed worse than not hedging at all. This is probably due to oil price shocks caused by the invasion of Kuwait by Iraq and the subsequent Gulf War.

The fact that the static cost-of-carry hedge consistently outperforms the dynamic cost-of-carry hedge appears counterintuitive at first. Theoretically, dynamic hedges should outperform their static counterparts. However, as previously discussed, static hedging strategies are less sensitive to estimation and modeling errors than dynamic hedging strategies. This is the most plausible reason for the results found here.

Although it is unknown how well these results generalize to other scenarios, they suggest that hedgers should consider the use of static strategies rather than comparable dynamic strategies.

CONCLUSIONS

In this article, the hedging problem has been examined when futures prices follow a variant of the cost-of-carry model. The resultant hedging model explicitly incorporates maturity effects in the futures basis. Formulas for the optimal static and dynamic hedges have been derived. Although these formulas have been developed for the case of direct hedging, the framework used is sufficiently flexible that these formulas can be applied to many cross-hedging situations. The performance of the model has been compared with that of several other models for two of the following hedging scenarios: one involving a financial asset and the other involving a commodity. In both cases, significant maturity effects were found in the first and second moments of the futures basis. Our hedging formulas outperformed other hedging strategies on an ex-ante basis. Additionally, the effectiveness of the hedge increased with its duration.

APPENDIX

Proof of Proposition 1

Equation 8 implies that

$$\begin{aligned}
&\mathrm{Cov}_0(S_T, F_T)\\
&\quad = E_0[S_T F_T] - E_0[S_T]\cdot E_0[F_T]\\
&\quad = E_0\left[\exp\left\{(\mathbf{1}_s + \mathbf{1}_f)\left[\mathbf{A}^T\mathbf{X}_0 + \sum_{t=0}^{T-1}\mathbf{A}^t\boldsymbol{\epsilon}_{T-t}\right]\right\}\right]\\
&\qquad - E_0\left[\exp\left\{\mathbf{1}_s\left[\mathbf{A}^T\mathbf{X}_0 + \sum_{t=0}^{T-1}\mathbf{A}^t\boldsymbol{\epsilon}_{T-t}\right]\right\}\right]\cdot E_0\left[\exp\left\{\mathbf{1}_f\left[\mathbf{A}^T\mathbf{X}_0 + \sum_{t=0}^{T-1}\mathbf{A}^t\boldsymbol{\epsilon}_{T-t}\right]\right\}\right]
\end{aligned}$$

Applying Equation 9 to these conditional expectations

$$\begin{aligned}
\mathrm{Cov}_0(S_T, F_T) &= \exp\{(\mathbf{1}_s + \mathbf{1}_f)\mathbf{A}^T\mathbf{X}_0 + (\mathbf{1}_s + \mathbf{1}_f)W(\mathbf{1}_s' + \mathbf{1}_f')/2\}\\
&\quad - \exp\{\mathbf{1}_s\mathbf{A}^T\mathbf{X}_0 + \mathbf{1}_s W\mathbf{1}_s'/2\}\cdot\exp\{\mathbf{1}_f\mathbf{A}^T\mathbf{X}_0 + \mathbf{1}_f W\mathbf{1}_f'/2\}\\
&= \exp\{(\mathbf{1}_s + \mathbf{1}_f)\mathbf{A}^T\mathbf{X}_0 + [\mathbf{1}_s W\mathbf{1}_s' + \mathbf{1}_f W\mathbf{1}_f']/2\}[\exp\{\mathbf{1}_s W\mathbf{1}_f'\} - 1]
\end{aligned}$$

where

$$W = \sum_{t=0}^{T-1} A^t V_{T-t}(A')^t$$

Similarly, one can show that

$$\mathrm{Var}_0(F_T) = \exp\{2 \cdot \mathbf{1}_f \mathbf{A}^T \mathbf{X}_0 + \mathbf{1}_f \boldsymbol{W} \mathbf{1}_f'\}[\exp\{\mathbf{1}_f \boldsymbol{W} \mathbf{1}_f'\} - 1]$$

Substituting these two results into Equation 3 and simplifying, the optimal static hedge ratio is given by

$$h^{**} = \exp\{(\mathbf{1}_s - \mathbf{1}_f)\mathbf{A}^T\mathbf{X}_0 + [\mathbf{1}_s \boldsymbol{W} \mathbf{1}_s' - \mathbf{1}_f \boldsymbol{W} \mathbf{1}_f']/2\}\left[\frac{\exp\{\mathbf{1}_s \boldsymbol{W} \mathbf{1}_f'\} - 1}{\exp\{\mathbf{1}_f \boldsymbol{W} \mathbf{1}_f'\} - 1}\right]$$

$$= a \cdot \exp\{(\mathbf{1}_s - \mathbf{1}_f)\mathbf{A}^T\mathbf{X}_0 + \boldsymbol{b}\}$$

where

$$a = \frac{\exp\{\mathbf{1}_s \boldsymbol{W} \mathbf{1}_f'\} - 1}{\exp\{\mathbf{1}_f \boldsymbol{W} \mathbf{1}_f'\} - 1}$$

and

$$b = [1_s W 1_s' - 1_f W 1_f']/2$$

This completes the proof. ❑

The proof of Proposition 2 requires the following two lemmas, whose proofs are omitted but available upon request from the authors:

Lemma 1. For all $j \in \{1, \ldots, T-1\}$

$$a(1, j+1) = f(1, j+1)\left[1 - \sum_{k=1}^{j} a(1, k)\right]$$

Lemma 2. Let $k \leq T-2$ and $2^{k-1} < i \leq 2^k$. Then, for all $j \in \{k + 1, \ldots, T-1\}$

$$a(i, j+1) = f(1, j+1)\left[a(i - 2^{k-1}, k) - \sum_{n=k+1}^{j} a(1, n)\right]$$

Proof of Proposition 2

The proof is by induction. Equation 8 implies that

$$\begin{aligned}
&\mathrm{Cov}_{T-k}(F_{T-k+1}, S_T) \\
&= E_{T-k}\left[\exp\left\{[\mathbf{1}_s A^{k-1} + \mathbf{1}_f][\mathbf{A}X_{T-k} + \epsilon_{T-k+1}] + \mathbf{1}_s \sum_{t=1}^{k} \mathbf{A}^{t-1}\epsilon_{T-t+1}\right\}\right] \\
&\quad - E_{T-k}\left[\exp\left\{\mathbf{1}_s\left[A^k X_{T-k} + \sum_{t=1}^{k} \mathbf{A}^{t-1}\epsilon_{T-t+1}\right]\right\}\right] \\
&\quad \times E_{T-k}[\exp\{\mathbf{1}_f[\mathbf{A}X_{T-k}\epsilon_{T-k+1}]\}] \\
&= E_{T-k}\left[\exp\left\{[\boldsymbol{J}(1, \boldsymbol{k}) + \mathbf{1}_f][\mathbf{A}X_{T-k} + \epsilon_{T-k+1}] + \sum_{t=1}^{k} \boldsymbol{J}(1, \boldsymbol{t})\epsilon_{T-t+1}\right\}\right] \\
&\quad - E_{T-k}\left[\exp\left\{\boldsymbol{J}(1, \boldsymbol{k})\mathbf{A}X_{T-k} + \sum_{t=1}^{k} \boldsymbol{J}(1, \boldsymbol{t})\epsilon_{T-t+1}\right]\right\}\right] \\
&\quad \times E_{T-k}[\exp\{\mathbf{1}_f[\mathbf{A}X_{T-k} + \epsilon_{T-k+1}]\}]
\end{aligned}$$

Applying Equation 10 to these conditional expectations

$$\begin{aligned}
&\mathrm{Cov}_{T-k}(F_{T-k+1}, S_T) \\
&= \exp\left\{[\boldsymbol{J}(1, \boldsymbol{k}) + \mathbf{1}_f]\mathbf{A}X_{T-k} + [\boldsymbol{J}(1, \boldsymbol{k}) + \mathbf{1}_f]\boldsymbol{V}_{T-k+1}[\boldsymbol{J}'(1, \boldsymbol{k}) + \mathbf{1}_f']/2\right. \\
&\quad \left. + \frac{1}{2}\sum_{t=1}^{k} \boldsymbol{J}(1, \boldsymbol{t})\boldsymbol{V}_{T-t+1}\boldsymbol{J}'(1, \boldsymbol{t})\right\} \\
&\quad - \exp\left\{\boldsymbol{J}(1, \boldsymbol{k})\mathbf{A}X_{T-k} + \frac{1}{2}\sum_{t=1}^{k} \boldsymbol{J}(1, \boldsymbol{t})\boldsymbol{V}_{T-t+1}\boldsymbol{J}'(1, \boldsymbol{t})\right\} \\
&\quad \times \exp\{\mathbf{1}_f\mathbf{A}X_{T-k} + \mathbf{1}_f\boldsymbol{V}_{T-k+1}\mathbf{1}_f'/2\} \\
&= \exp\left\{[\boldsymbol{J}(1, \boldsymbol{k}) + \mathbf{1}_f]\mathbf{A}X_{T-k} + \frac{1}{2}\sum_{t=1}^{k} \boldsymbol{J}(1, \boldsymbol{t})\boldsymbol{V}_{T-t+1}\boldsymbol{J}'(1, \boldsymbol{t})\right. \\
&\quad \left. + [\mathbf{1}_f\boldsymbol{V}_{T-k+1}\mathbf{1}_f']/2\right\}[\exp\{(\boldsymbol{J}(1, \boldsymbol{k})\boldsymbol{V}_{T-k+1}\mathbf{1}_f'\} - 1] \\
&= \exp\{[\boldsymbol{J}(1, \boldsymbol{k}) + \mathbf{1}_f]\mathbf{A}X_{T-k} + \boldsymbol{b}(1, \boldsymbol{k}) \\
&\quad + \mathbf{1}_f\boldsymbol{V}_{T-k+1}\mathbf{1}_f'\}[\exp\{\boldsymbol{J}(1, \boldsymbol{k})\boldsymbol{V}_{T-k+1}\mathbf{1}_f'\} - 1] \qquad \text{(A1)}
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}&\mathrm{Var}_{T-k}(\mathrm{F}_{T-k+1})\\&\quad= \exp\{2 \cdot \mathbf{1}_f A X_{T-k} + \mathbf{1}_f V_{T-k+1}\mathbf{1}_f'\}[\exp\{\mathbf{1}_f V_{T-k+1}\mathbf{1}_f'\} - 1]\end{aligned} \tag{A2}$$

Let $k = 1$. Substituting Equations A1 and A2 into Equation 1, we find that

$$\begin{aligned}h^*_{T-1} &= \left[\frac{\exp\{J(1, 1)V_T\mathbf{1}_f'\} - 1}{\exp\{\mathbf{1}_f V_T\mathbf{1}_f'\} - 1}\right]\exp\{[J(1, 1) - \mathbf{1}_f]AX_{T-1} + b(1, 1)\}\\&= a(1, 1) \cdot \exp\{[J(1, 1) - \mathbf{1}_f]AX_{T-1} + b(1, 1)\}\end{aligned}$$

and so Equation 12 holds for $k = 1$.

Now, assume that Equation 12 holds for $k = \{1, \ldots, j\}$. For $k \leq j$

$$\begin{aligned}&\mathrm{Cov}_{T-j-1}(F_{T-j}, h^*_{T-k}F_{T-k+1})\\&\quad= \sum_{i=1}^{2^{k-1}} a(i, k) \cdot \mathrm{Cov}_{T-j-1}\Bigg(\exp\{\mathbf{1}_f[AX_{T-j-1} + \epsilon_{T-j}]\}, \exp\Bigg\{J(i, k)A^{j-k+2}X_{T-j-i}\\&\qquad+ \sum_{t=k}^{j} J(i, k)A^{1+t-k}\epsilon_{T-t} + \mathbf{1}_f\epsilon_{T-k+1} + b(i, k)\Bigg\}\Bigg)\\&\quad= \sum_{i=1}^{2^{k-1}} a(i, k) \cdot \exp\Bigg\{[J(i, k)A^{j-k+1} + \mathbf{1}_f]AX_{T-j-1} + b(i, k)\\&\qquad+ \frac{1}{2}\sum_{t=k}^{j} J(i, k)A^{1+t-k}V_{T-t}(A')^{1+t-k}J'(i, k) + [\mathbf{1}_f V_{T-j}\mathbf{1}_f' + \mathbf{1}_f V_{T-k+1}\mathbf{1}_f']/2\Bigg\}\\&\qquad\times [\exp\{J(i, k)A^{j-k+1}V_{T-j}\mathbf{1}_f'\} - 1]\\&\quad= \sum_{i=1}^{2^{k-1}} a(i, k) \cdot \exp\{[J(i, j+1) + \mathbf{1}_f]AX_{T-j} + b(i, j+1) + \mathbf{1}_f V_{T-j}\mathbf{1}_f'\}\\&\qquad\times [\exp\{J(i, j+1)V_{T-j}\mathbf{1}_f'\} - 1]\end{aligned} \tag{A3}$$

Similarly, we find that

$$
\begin{aligned}
&\mathrm{Cov}_{T-j-1}(F_{T-j}, h^*_{T-k}F_{T-k}) \\
&= \sum_{i=1}^{2^{k-1}} a(i,k)\cdot \exp\Big\{([\boldsymbol{J}(\boldsymbol{i},\boldsymbol{k})\mathbf{A} + \mathbf{1}_f(\boldsymbol{I}-\mathbf{A})]\mathbf{A}^{j-k+1}] + \mathbf{1}_f)\,\mathbf{A}\mathbf{X}_{T-j-1} \\
&\quad + [\mathbf{1}_f \boldsymbol{V}_{T-j}\mathbf{1}_f']/2 + \boldsymbol{b}(\boldsymbol{i},\boldsymbol{k}) + \frac{1}{2}\sum_{t=k}^{j}[\boldsymbol{J}(\boldsymbol{i},\boldsymbol{k})\boldsymbol{A} \\
&\quad + \mathbf{1}_f(\boldsymbol{I}-\mathbf{A})]\mathbf{A}^{t-k}\boldsymbol{V}_{T-t}(\mathbf{A}')^{t-k}[\boldsymbol{J}(\boldsymbol{i},\boldsymbol{k})\mathbf{A} + \mathbf{1}_f(\boldsymbol{I}-\mathbf{A})]'\Big\} \\
&\quad \times [\exp\{[\boldsymbol{J}(\boldsymbol{i},\boldsymbol{k})\mathbf{A} + \mathbf{1}_f(\boldsymbol{I}-\mathbf{A})]\mathbf{A}^{j-k}\boldsymbol{V}_{T-j}\mathbf{1}_f'\} - 1] \\
&= \sum_{i=2^{k-1}+1}^{2^k} a(i-2^{k-1},k)\cdot \exp\{[\boldsymbol{J}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1}) + \mathbf{1}_f]\mathbf{A}\mathbf{X}_{T-j} + \boldsymbol{b}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1}) \\
&\quad + \mathbf{1}_f \boldsymbol{V}_{T-j}\mathbf{1}_f'\}\,[\exp\{\boldsymbol{J}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1})\boldsymbol{V}_{T-j}\mathbf{1}_f'\} - 1] \qquad \text{(A4)}
\end{aligned}
$$

Substituting Equations A1–A4 into Equation 2, we find that

$$
\begin{aligned}
&h^*_{T-j-1} \\
&= \left[\frac{\exp\{\boldsymbol{J}(\mathbf{1},\boldsymbol{j}+\mathbf{1})\boldsymbol{V}_{T-j}\mathbf{1}_f'\} - 1}{\exp\{\mathbf{1}_f\boldsymbol{V}_{T-j}\mathbf{1}_f'\} - 1}\right]\exp\{[\boldsymbol{J}(\mathbf{1},\boldsymbol{j}+\mathbf{1}) + \mathbf{1}_f]\mathbf{A}\mathbf{X}_{T-j} \\
&\quad + \boldsymbol{b}(\mathbf{1},\boldsymbol{j}+\mathbf{1})\} - \sum_{k=1}^{j}\sum_{i=1}^{2^{k-1}} a(i,k)\left[\frac{\exp\{\boldsymbol{J}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1})\boldsymbol{V}_{T-j}\mathbf{1}_f'\} - 1}{\exp\{\mathbf{1}_f\boldsymbol{V}_{T-j}\mathbf{1}_f'\} - 1}\right] \\
&\quad \times \exp\{[\boldsymbol{J}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1}) + \mathbf{1}_f]\mathbf{A}\mathbf{X}_{T-j} + \boldsymbol{b}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1})\} \\
&\quad + \sum_{k=1}^{j}\sum_{i=2^{k-1}+1}^{2^k} a(i-2^{k-1},k)\left[\frac{\exp\{\boldsymbol{J}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1})\boldsymbol{V}_{T-j}\mathbf{1}_f'\} - 1}{\exp\{\mathbf{1}_f\boldsymbol{V}_{T-j}\mathbf{1}_f'\} - 1}\right] \\
&\quad \times \exp\{[\boldsymbol{J}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1}) + \mathbf{1}_f]\mathbf{A}\mathbf{X}_{T-j} + \boldsymbol{b}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1})\} \\
&= f(1,j+1)\left[1 - \sum_{k=1}^{j} a(1,k)\right]\exp\{[\boldsymbol{J}(\mathbf{1},\boldsymbol{j}+\mathbf{1}) + \mathbf{1}_f]\mathbf{A}\mathbf{X}_{T-j} + \boldsymbol{b}(\mathbf{1},\boldsymbol{j}+\mathbf{1})\} \\
&\quad + \sum_{k=1}^{j-1}\sum_{i=2^{k-1}+1}^{2^k} f(i,j+1)\left[a(i-2^{k-1},k) - \sum_{n=k+1}^{j} a(i,n)\right] \\
&\quad \times \exp\{[\boldsymbol{J}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1}) + \mathbf{1}_f]\mathbf{A}\mathbf{X}_{T-j} + \boldsymbol{b}(\boldsymbol{i},\boldsymbol{j}+\mathbf{1})\}
\end{aligned}
$$

$$+ \sum_{i=2^{j-1}+1}^{2^j} f(i, j+1) \cdot a(i - 2^{j-1}, j)$$

$$\times \exp\{[\boldsymbol{J}(\boldsymbol{i}, \boldsymbol{j}+1) + \mathbf{1}_f]\boldsymbol{A}\boldsymbol{X}_{T-j} + \boldsymbol{b}(\boldsymbol{i}, \boldsymbol{j}+1)\}$$

Applying Lemmas 1 and 2, this simplifies to

$$h^*_{T-j-1} = a(1, j+1) \cdot \exp\{[\boldsymbol{J}(1, \boldsymbol{j}+1) + \mathbf{1}_f]\boldsymbol{A}\boldsymbol{X}_{T-j} + \boldsymbol{b}(1, \boldsymbol{j}+1)\}$$

$$+ \sum_{k=1}^{j-1} \sum_{i=2^{k-1}+1}^{2^k} a(i, k+1) \cdot \exp\{[\boldsymbol{J}(\boldsymbol{i}, \boldsymbol{j}+1) + \mathbf{1}_f]\boldsymbol{A}\boldsymbol{X}_{T-j} + \boldsymbol{b}(\boldsymbol{i}, \boldsymbol{j}+1)\}$$

$$+ \sum_{i=2^{k-1}+1}^{2^k} a(i, k+1) \cdot \exp\{[\boldsymbol{J}(\boldsymbol{i}, \boldsymbol{j}+1) + \mathbf{1}_f]\boldsymbol{A}\boldsymbol{X}_{T-j} + \boldsymbol{b}(\boldsymbol{i}, \boldsymbol{j}+1)\}$$

$$= \sum_{i=1}^{2^j} a(i, j+1) \cdot \exp\{[\boldsymbol{J}(\boldsymbol{i}, \boldsymbol{j}+1) + \mathbf{1}_f]\boldsymbol{A}\boldsymbol{X}_{T-j} + \boldsymbol{b}(\boldsymbol{i}, \boldsymbol{j}+1)\}$$

Therfore, Equation 11 holds for $k = j + 1$. This completes the inductive proof.

❑

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