A NOTE ON THE VALUATION OF COMPOUND OPTIONS

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The value of a compound option, *an option on an option*, has been derived by Geske (1976) using Fourier integrals. This article presents two alternative proofs to derive the value of a compound option. One proof is based on the martingale approach, which provides a simple and powerful tool for valuing contingent claims. The second proof uses the expectation of a truncated bivariate normal variable. These proofs allow for an intuitive interpretation of the three elements constituting the value of a compound option. © 2002 Wiley Periodicals, Inc. Jrl Fut Mark 22:1103–1115, 2002

INTRODUCTION

The value of a compound option, *an option on an option*, has been derived by Geske (1976) using Fourier integrals. Rubinstein (1991) suggests using the risk-neutral principle to derive the same formula. However, he does not go through the formal proof. This article presents detailed proofs that allow an intuitive interpretation of the three probability factors in Geske's formula, and explains why they are different from each other. Nielsen (1993) explains the probability factors in the Black–Scholes formula (1973) by relating them to the risk adjusted probabilities. A similar procedure is undertaken in this article by linking the probability factors in Geske's formula to the risk adjusted probabilities.

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The article is organized as follows. The next section presents Geske's formula, then I present a proof based on the martingale approach. The interpretation section shows why the probability factor of the contingent receipt of the stock is higher than the risk-adjusted probability of exercising both options. Then I decompose the payoff of the compound option into three components that compute the discounted expected values using the risk-neutral principle and the expectation of a truncated bivariate normal variable.

GESKE'S FORMULA

Geske's formula values a call option that has an exercise price of K_1 and a maturity of T_1. At time T_1, the option gives the right to buy another option, the underlying option, that has an exercise price of K_2 and a maturity of T_2. At time T_2, the underlying option gives the right to buy the stock. The value of the compound option is given by

$$
\begin{aligned}
C_1(t) &= S(t)N_2(D_1 + s_1, D_2 + s_2; \rho) \\
&\quad - K_2 e^{-r(T_2-t)} N_2(D_1, D_2; \rho) - K_1 e^{-r(T_1-t)} N_1(D) \qquad (1)
\end{aligned}
$$

where $C_1(t)$ is the value of the compound option; $S(t)$ is the price of the stock; $K_1(t)$ is the exercise price of the compound option; $K_2(t)$ is the exercise price of the underlying option; $T_1(t)$ is the maturity of the compound option; $T_2(t)$ is the maturity of the underlying option; r is the risk free interest rate, $D_1 = [\ln(S(t)/S^*) + (r - \frac{1}{2}\sigma^2)(T_1 - t)]/(\sigma\sqrt{T_1 - t})$; $D_2 = [\ln(S(t)/K_2) + (r - \frac{1}{2}\sigma^2)(T_2 - t)]/(\sigma\sqrt{T_2 - t})$; $D = [\ln(S(T_1)/K_2) + (r - \frac{1}{2}\sigma^2)(T_2 - T_1)]/(\sigma\sqrt{T_2 - T_1})$; and S^* is the price of the stock such that the underlying option is at the money at time T_1.

MARTINGALE DERIVATION

First, we assume that the price process is a geometric Brownian Motion given by the following:

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW(t)$$

where $W(t)$ is a standard Brownian Motion. Note that $S(t)$ can be written as

$$S(t) = \exp\{(r - \tfrac{1}{2}\sigma^2)t + \sigma W^*(t)\}$$

where

$$W^*(t) = W(t) + \frac{\mu - r}{\sigma}$$

is the Girsanov transform of $W(t)$.

Under the martingale approach, the value at time t of the compound option is given by the following conditional expectation under the risk-neutral probability measure,

$$C_1(t) = E^*[\max\{C_2(T_1) - K_1, 0\}e^{-r(T_1-t)} \mid \mathcal{F}_t] \tag{2}$$

We know that $C_2(T_1)$ is given by a straightforward application of Black and Scholes (1973) formula. Thus,

$$C_2(T_1) = S(T_1)N_1(D + s) - K_2e^{-r(T_2-T_1)}N_1(D) \tag{3}$$

where

$$D = \frac{\ln(S(T_1)/K_2) + (r - \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}$$
$$s = \sigma\sqrt{T_2 - T_1}$$

and where N_1 is the cumulative standard normal distribution function.

Replacing formula (3) in Equation (2) we get

$$C_1(t) = E^*[\max\{S(T_1)N_1(D + s) - K_2e^{-r(T_2-T_1)}N_1(D) - K_1, 0\}e^{-r(T_1-t)} \mid \mathcal{F}_t] \tag{4}$$

Let S^* be the value of the stock such that the underlying option is at the money at time T_1, i.e.,

$$S(T_1) = S^*$$

where S^* solves

$$C_2(T_1) - K_1 = 0$$

Let

$$Y_1 = W^*(T_1) - W^*(t)$$

and

$$Z_1 = \frac{Y_1}{\sigma\sqrt{T_1 - t}}$$

Note that

$$S(T_1) = S(t)\exp\{(r - \tfrac{1}{2}\sigma^2)(T_1 - t) + \sigma Y_1\}$$

and therefore

$$Z_1 = \frac{\ln(S(T_1)/S^*) - (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}$$

Let

$$D_1 = \frac{\ln(S(t)/S^*) + (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}$$

Note also that

$$S(T_1) > S^*$$

is equivalent to

$$Z_1 > -D_1$$

Thus, Equation (4) can be written as

$$\begin{aligned} C_1(t) = &\int_{-D_1}^{+\infty} S(T_1)N_1(D + s)\frac{e^{-Z_1^2/2}}{\sqrt{2\pi}}e^{-r(T_1-t)}\,dZ_1 \\ &- \int_{-D_1}^{+\infty} K_2 e^{-r(T_2-T_1)}N_1(D)\frac{e^{-Z_1^2/2}}{\sqrt{2\pi}}e^{-r(T_1-t)}\,dZ_1 \\ &- \int_{-D_1}^{+\infty} K_1\frac{e^{-Z_1^2/2}}{\sqrt{2\pi}}e^{-r(T_1-t)}\,dZ_1 \end{aligned} \tag{5}$$

Equation (5) can be written as

$$C_1(t) = C_{11} - C_{12} - C_{13}$$

Each component will be computed separately starting with the third one.

$$C_{13} = \int_{-D_1}^{+\infty} K_1\frac{e^{-Z_1^2/2}}{\sqrt{2\pi}}e^{-r(T_1-t)}\,dZ_1 = K_1 e^{-r(T_1-t)}N_1(D_1) \tag{6}$$

The second component in Equation (5) is

$$C_{12} = \int_{-D_1}^{+\infty} K_2 e^{-r(T_2-T_1)} N_1(D) \frac{e^{-Z_1^2/2}}{\sqrt{2\pi}} e^{-r(T_1-t)}\, dZ_1 \tag{7}$$

Equation (7) is the density of a bivariate normal distribution. In the following, we will specify its parameters, and we will show that the relevant correlation coefficient is $\rho(Z_1, Z_2)$ where Z_2 is defined below.

Let

$$Y_2 = W^*(T_2) - W^*(t)$$

and

$$Z_2 = \frac{Y_2}{\sigma\sqrt{T_2 - t}}$$

Recall that

$$Y_1 = W^*(T_1) - W^*(t)$$

and

$$Z_1 = \frac{Y_1}{\sigma\sqrt{T_1 - t}}$$

Let

$$\rho = \rho(Z_1, Z_2)$$

Because Z_1 and Z_2 are standardized normal variables

$$\rho = \text{cov}(Z_1, Z_2) = \text{cov}\left(\frac{Y_1}{\sqrt{T_1 - t}}, \frac{Y_2}{\sqrt{T_2 - t}}\right) = \frac{1}{\sqrt{T_1 - t}\,\sqrt{T_2 - t}}\,\text{cov}(Y_1, Y_2)$$

We know that

$$Y_1 = W^*(T_1) - W^*(t)$$

and

$$Y_2 = W^*(T_2) - W^*(T_1) + W^*(T_1) - W^*(t)$$

Because a Standard Brownian Motion has independent increments

$$\text{cov}(Y_1, Y_2) = \text{var}(Y_1) = T_1 - t$$

Thus

$$\rho = \frac{T_1 - t}{\sqrt{T_1 - t}\sqrt{T_2 - t}} = \frac{\sqrt{T_1 - t}}{\sqrt{T_2 - t}}$$

Note that the cumulative conditional normal distribution is defined by

$$N_1(D_1 \mid Z_1) = N_1\left(\frac{D_1 - \rho Z_1}{\sqrt{1 - \rho^2}}\right)$$

and the bivariate cumulative normal distribution is defined by

$$N_2(D_1, D_2; \rho) = \int_{-\infty}^{D_2} N_1(D_1 \mid Z_1)\frac{e^{-Z_1^2/2}}{\sqrt{2\pi}}\, dZ_1 = \int_{-D_2}^{+\infty} N_1\left(\frac{D_1 + \rho Z_1}{\sqrt{1 - \rho^2}}\right)\frac{e^{-Z_1^2/2}}{\sqrt{2\pi}}\, dZ_1$$

Note also that

$$\sqrt{1 - \rho^2} = \sqrt{\frac{T_2 - T_1}{T_2 - t}}$$

Simple algebraic manipulations allow us to write D as follows

$$D = \frac{D_2 + \rho\dfrac{Y_1}{\sqrt{T_1 - t}}}{\sqrt{1 - \rho^2}} = \frac{D_2 + \rho Z_1}{\sqrt{1 - \rho^2}}$$

where

$$D_2 = \frac{\ln(S(t)/K_2) + (r - \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}}$$

Thus, Equation (7) can be written as

$$\begin{aligned} C_{12} &= K_2 e^{-r(T_2 - t)} \int_{-D_1}^{+\infty} N_1(D)\frac{e^{-Z_1^2/2}}{\sqrt{2\pi}}\, dZ_1 \\ &= -K_2 e^{-r(T_2 - t)} \int_{-D_1}^{+\infty} N_1\left(\frac{D_2 + \rho Z_1}{\sqrt{1 - \rho^2}}\right)\frac{e^{-Z_1^2/2}}{\sqrt{2\pi}}\, dZ_1 \\ &= K_2 e^{-r(T_2 - t)}\, N_2(D_1, D_2; \rho) \end{aligned} \tag{8}$$

where D_1 and D_2 are defined above.

Now we will compute the first component in Equation (5). Before we do so, remember that $S(T_1)$ is lognormally distributed. Its mean is

$$m_1 = \ln(S(t)) + (r - \tfrac{1}{2}\sigma^2)(T_1 - t)$$

and its standard deviation is

$$s_1 = \sigma\sqrt{T_1 - t}$$

Therefore, $S(T_1)$ can be written as:

$$S(T_1) = \exp(s_1 Z_1 + m_1)$$

and

$$\begin{aligned} C_{11} &= \int_{-D_1}^{+\infty} \exp(s_1 Z_1 + m_1) N_1(D + s) \frac{e^{-Z_1^2/2}}{\sqrt{2\pi}} e^{-r(T_1 - t)} dZ_1 \\ &= S(t) \int_{-D_1}^{+\infty} \frac{e^{-(Z_1 - s_1)^2/2}}{\sqrt{2\pi}} N_1(D + s)\, dZ_1 \end{aligned}$$

Note that

$$D + s = \frac{D_2 + s_2 + \rho Z_1'}{\sqrt{1 - \rho^2}}$$

where

$$s_2 = \sigma\sqrt{T_2 - t}$$

and

$$Z_1' = Z_1 - s_1$$

Thus,

$$N_1(D + s) = N_1\left(\frac{D_2 + s_2 + \rho Z_1'}{\sqrt{1 - \rho^2}}\right)$$

Replacing in Equation (5) implies

$$\begin{aligned} C_{11} &= S(t) \int_{-(D_1 + s_1)}^{+\infty} \frac{e^{-Z_1'^2/2}}{\sqrt{2\pi}} N_1\left(\frac{D_2 + s_2 + \rho Z_1'}{\sqrt{1 - \rho^2}}\right) dZ_1' \\ &= S(t) N_2(D_1 + s_1, D_2 + s_2; \rho) \end{aligned} \tag{9}$$

where D_1 and D_2 are as defined above.

Conclusion

$$
\begin{aligned}
C_1(t) &= C_{11} - C_{12} - C_{13} \\
&= S(t)N_2(D_1 + s_1, D_2 + s_2; \rho) \\
&\quad - K_2 e^{-r(T_2-t)} N_2(D_1, D_2; \rho) \\
&\quad - K_1 e^{-r(T_1-t)} N_1(D)
\end{aligned}
\tag{10}
$$

INTERPRETATION

The first component can be seen as the receipt of the stock contingent on both the compound and the underlying options finishing in the money. Similarly, the second component can be seen as the payment of the exercise price K_2 contingent once again on both options finishing in the money. Finally, the third component can be interpreted as the payment of the exercise price K_1 contingent on the compound option finishing in the money.

Factor $N_2(D_1, D_2; \rho)$ can be interpreted as the joint probability that both the compound option and the underlying option are exercised. Factor $N_2(D_1 + s_1, D_2 + s_2; \rho)$ is not as straightforward. The expected value, computed using risk adjusted probabilities, of receiving the stock at expiration of the option, contingent upon both the compound option and the underlying option finishing in the money, is $N_2(D_1 + s_1, D_2 + s_2; \rho)$ multiplied by the stock price. Thus, $N_2(D_1 + s_1, D_2 + s_2; \rho)$ is the factor by which the present value of the contingent receipt of the stock falls short of the current stock price.

Why is the present value of unconditionally receiving the stock at time T_2 not equal to $S(t)N_2(D_1, D_2; \rho)$ corresponding to an expected future value of $e^{r(T_2-T_1)}S(t)N_2(D_1, D_2; \rho)$? The argument is the same as in Nielsen (1993) for the Black–Scholes formula. The present value of unconditionally receiving the stock at time T_2 is equal to S, the current stock price. Therefore, the expected future value of unconditionally receiving the stock has to be $e^{r(T_2-T_1)}S$. If the stock is received conditionally on an event that has probability $N_2(D_1, D_2; \rho)$, then the expected value should be $e^{r(T_2-T_1)}S(t)N_2(D_1, D_2; \rho)$ and the present value should be $S(t)N_2(D_1, D_2; \rho)$.

The present value of contingent receipt of the stock is strictly larger than $S(t)N_2(D_1, D_2; \rho)$ because $S(t)N_2(D_1 + s_1, D_2 + s_2; \rho) > S(t)N_2(D_1, D_2; \rho)$. If the present value were $S(t)N_2(D_1, D_2; \rho)$, then the value of the option would be $(S(t) - e^{r(T_2-t)}K_2)N_2(D_1, D_2; \rho) - K_1 e^{r(T_1-t)}N_1(D)$.

For K_1 being very small, this quantity could be negative when the compound option is out of the money, which clearly cannot be the case.

The event of exercise is not independent of $S(T_2)$. If exercise were completely random then the present value of contingent receipt of the stock would be $S(t)N_2(D_1, D_2; \rho)$. Because exercise depends on the future stock price and happens only when $S(T_2)$ is high, then $S(t)e^{r(T_2-t)}N_2(D_1, D_2; \rho)$ will be lower than the expected value.[1]

ALTERNATIVE DERIVATION

An alternative equivalent derivation can be done using the expectation of a truncated bivariate normal variable. This procedure builds on the Black–Scholes derivation in Nielsen (1993).

First, the payoffs from the compound options are presented. Then the expected values under a risk neutral probability measure of these payoffs are discounted and added to calculate the price of the compound option at any time t.

Payoffs

At time T_1, the value of the compound option is given by the following:

$$C_1(T_1) = \begin{cases} C_2(T_1) - K_1 & \text{if } C_2(T_1) > K_1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

At time T_2, the payoff of the option comprises three elements. The option can be valued by valuing each component and then adding up the different values. The value of each component will be the discounted expected value under the risk-adjusted probability distribution.

The three components are C_{11}, C_{12}, and C_{13}.

$$C_{11} = \begin{cases} S(T_2) & \text{if } S(T_2) > K_2 \quad \text{given that } C_2(T_1) > K_1 \text{ (equiv. to } S(T_1) > S^*) \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

$$C_{12} = \begin{cases} -K_2 & \text{if } S(T_2) > K_2 \quad \text{given that } C_2(T_1) > K_1 \text{ (equiv. to } S(T_1) > S^*) \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

$$C_{13} = \begin{cases} -K_1 & \text{if } C_2(T_1) > K_1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

[1]This paragraph is an adaptation of the argument in Nielsen (1993) extended to the case of a compound option.

Under the risk-adjusted probability distribution, the mean and variance of the lognormally distributed variable $S(t)$ is $\ln(S) + (r - \frac{1}{2}\sigma^2)$ and $\sigma\sqrt{t}$. The risk-neutral principle states that the current value of any stock price contingent claim equals the discounted value of the expected payoff. When the expectation is computed using the risk-adjusted probabilities, this leads to an intuitive derivation of the Geske formula.

The expected value of the first component is the expected value of the stock price conditional on the exercise of the two options multiplied by the probability of exercise of the two option.

The expected value of the second component is the expected value of exercise payment of the underlying option conditional on the exercise of the two options multiplied by the probability of exercise of the two option.

Finally, the expected value of the third component is the expected value of exercise payment of the compound option multiplied by the probability of its exercise.

$$
\begin{aligned}
E[C_{11}] &= E[E[S(T_2) \mid S(T_2) > K_2] \mid S(T_1) > S^*]P[S(T_2) > \\
&\qquad K_2 \mid S(T_1) > S^*]e^{-r(T_2-t)} \\
E[C_{12}] &= -K_2 P[S(T_2) > K_2 \mid S(T_1) > S^*]e^{-r(T_2-t)} \\
E[C_{13}] &= -K_1 P[S(T_1) > S^*]e^{-r(T_1-t)}
\end{aligned}
$$

where P is the risk-adjusted probability measure.

The value of the option at time t is simply the sum of the expected payoffs under a risk-adjusted probability measure.

Valuation Formula

An alternative equivalent derivation based on the expectation of a truncated bivariate normal distribution is given below.

Let

$$
\begin{aligned}
Z_1 &= \frac{\log(S(T_1)/S(t)) - (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\
Z_2 &= \frac{\log(S(T_2)/S(t)) - (r - \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}}, \\
D &= \frac{\log(S(T_1)/S(t)) - (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\
D_1 &= \frac{\log(S^*/S(t)) - (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}},
\end{aligned}
$$

and

$$D_2 = \frac{\log(K_2/S(t)) - (r - \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}}$$

Both Z_1 and Z_2 are standard normal distribution such that:

$$\rho(Z_1, Z_2) = \sqrt{\frac{T_1 - t}{T_2 - t}}$$

The compound option will be exercised if $S(T_1) > S^*$, which is equivalent to $Z_1 > -D_1$. Similarly, the underlying option will be exercised if $S(T_2) > K_2$, which is equivalent to $Z_2 > -D_2$.

Variables $S(T_1)$ and $S(T_2)$ are both lognormally distributed. Therefore, they can be written as

$$\ln(S(T_1)) \sim N(m_1, s_1^2)$$
$$\ln(S(T_2)) \sim N(m_2, s_2^2)$$

Let us look at the following normal random variable:

$$\widetilde{S} = \begin{cases} S(T_2) & \text{if } S(T_2) > K_2 \quad \text{given that } S(T_1) > S^* \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Variable $\widetilde{S}$ is a truncated bivariate normal variable. At time T_1, the expected value of $\widetilde{S}$ is the expected value of S_2 truncated at K_2, i.e.,

$$E[\widetilde{S}] = E[S(T_2) \mid S(T_2) > K_2]$$

According to Nielsen (1993), this value should be equal to

$$\exp\left(m + \frac{s^2}{2}\right)N_1(D + s)$$

where

$$D = \frac{\ln(S(T_1)/K_2) + (r - \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}$$
$$s = \sigma\sqrt{T_2 - T_1}$$
$$m = \ln(S(T_1)) + (r - \tfrac{1}{2}\sigma^2)(T_2 - T_1)$$

Therefore, it is equal to

$$e^{r(T_2-T_1)}S(T_1)N_1(D + s)$$

Now, because $S(T_1)$ is truncated at S^*, then the expected value of $\widetilde{S}$ at time t should be

$$e^{r(T_2-T_1)}\int_{-D_1}^{\infty} S(T_1)N_1(D + s)e^{-Z_1^2/2}\,dZ_1$$

This is equal to

$$\begin{aligned} &e^{r(T_2-T_1)}e^{r(T_1-t)}S(t)N_2(D_1 + s_1, D_2 + s_2; \rho) \\ &= e^{r(T_2-t)}S(t)N_2(D_1 + s_1, D_2 + s_2; \rho) \end{aligned}$$

Therefore, the present value at time t should be

$$S(t)N_2(D_1 + s_1, D_2 + s_2; \rho)$$

Now, the probability for both options to be in the money is simply $N_2(D_1, N_2; \rho)$, and, the current present value of K_2 is simply

$$-K_2e^{-r(T_2-t)}N_2(D_1, D_2; \rho)$$

The last component is straightforward as, at time T_1, the probability of exercising the underlying option at time T_2 is $N_1(D)$, and the current expected value of K_1 is

$$-K_1e^{-r(T_1-t)}N_1(D)$$

CONCLUSION

Two alternative equivalent proofs based on risk-neutral probabilities are presented for the valuation of compound options. These proofs provide an intuitive interpretation of the probability factors in Geske's formula.

BIBLIOGRAPHY

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. Journal of Political Economy, 81, 637–654.

Cox, J. C., Ross, S. A., & Rubinstein, M. (1979). Option pricing: A simplified approach. Journal of Financial Economics, 7, 229–263.

Cox, J. C., & Rubinstein, M. (1985). Option markets. Englewood Cliffs, NJ: Prentice Hall.

Hull, J. (1991). Introduction to futures and option markets. Englewood Cliffs, NJ: Prentice Hall.

Geske, R. (1979). The valuation of compound options. Journal of Financial Economics, 7, 63–81.

Nielsen, L. T. (1993). Understanding $N(d_1)$ and $N(d_2)$: Risk adjusted probabilities in the Black-Scholes model. Finance, 14, 95–106.

Rubinstein, M. (1991). Exotic options (Finance Working Paper No. 220). Institute of Business and Economic Research, University of California at Berkeley.

Rubinstein, M. (1976). The valuation of uncertain income streams and the pricing of options. Bell Journal of Economics, 7, 407–425.