PRICING EURODOLLAR FUTURES OPTIONS WITH THE HEATH–JARROW–MORTON MODEL

NUSRET CAKICI*
JINTAO ZHU

This article uses the algorithm developed by Ritchken and Sankarasubramanian (1995) to make comparisons among the Heath–Jarrow–Morton (HJM) models (Heath, Jarrow, & Morton, 1992) with different volatility structures in pricing the Eurodollar futures options. We show that the differences among the HJM models as well as the difference between the HJM models and Black's model can be insignificant when the volatility of the forward rate is relatively small. Moreover, our findings imply that the difference between the American-style and European-style options is insignificant for options with a life of less than 1 year. However, the difference can be significant for options with a 1-year maturity, the difference depending on the exercise price. Finally, our tests indicate that the difference between the forward price and the futures price is insignificant if the volatility parameter is low enough and when the volatility of the spot rate is proportional to the spot

Nusret Cakici gratefully acknowledges support from the PSC-CUNY research foundation. The authors thank an anonymous referee for helpful comments.
*Correspondence author, Department of Economics, City College and Graduate Center of the City University of New York, Convent Avenue at 138th Street, New York, New York 10031; e-mail: ncakici@yahoo.com

Received January 2000; Accepted October 2000

- *Nusret Cakici is a Professor in the Department of Economics at the City College and Graduate Center of the City University of New York in New York, New York.*
- *Jintao Zhu is at Citibank, N.A., in Long Island City, New York.*

The Journal of Futures Markets, Vol. 21, No. 7, 655–680 (2001)

rate. A higher volatility parameter can lead to a significant difference between the forward price and the futures price, although its impact on the price of the options will still be trivial.
Jrl Fut Mark 21: 655–680, 2001

INTRODUCTION

The evolution of the term structure has become the focal point in pricing interest-rate contingent claims. Equilibrium models such as the Rendleman–Bartter model (Rendleman & Bartter, 1980), the Vasicek model (Vasicek, 1977), the Cox–Ingersoll–Ross (CIR) model (Cox, Ingersoll, & Ross, 1985), and the Longstaff–Schwartz model (Longstaff & Schwartz, 1985) all begin with specific assumptions about the economy and derive the term structure of the interest rate endogenously. A shortcoming of this methodology is that the initial term structure cannot be matched. In contrast, the Heath–Jarrow–Morton (HJM) model (Heath, Jarrow, & Morton, 1992), which generalizes the methodology pioneered by Ho and Lee (1986), makes assumptions about the evolution of the term structure and imposes arbitrage-free conditions to price the contingent claims. The HJM model has, indeed, several advantages over the general equilibrium models. First, the initial term structure is an input of the model, not an endogenous result. Second, the evolution of the term structure and contingent claim prices are independent of the investor's preferences. Finally, estimates of the drifts are not needed. Therefore, the HJM model has found increasing favor in both theoretical research and practical trading.

Unfortunately, the computation of prices is complex because, without the term structure being severely restricted, the evolution of the term structure under the martingale measure is usually not Markovian with respect to a finite-dimensional state space. Heath, Jarrow, and Morton (1990) used the binomial method to price contingent claims. Because of path dependence, the lattice does not recombine, and it grows exponentially. To our knowledge, there was no powerful algorithm for implementing the HJM model in pricing interest-rate derivatives before Ritchken and Sankarasubramanian (1995) published their article, "Volatility Structures of Forward Rates and the Dynamics of the Term Structure."

The conditions on the volatility structure identified by Ritchken and Sankarasubramanian (1995) fail to remove the path dependence completely but instead manage to capture the term structure by two state variables instead of the entire term structure. Interestingly, their restrictions are imposed on the volatility structure of the forward rate, not the spot rate, which is why the Ritchken–Sankarasubramanian (RS) algorithm still permits a broad set of volatility structures.

The goal of this article is to use the HJM model, combined with the RS algorithm, to price the Eurodollar futures options and test the differences among the HJM models with different term structures in pricing Eurodollar futures options. Flesaker (1993b) studied the HJM model with constant volatility of the forward rate. Bühler, Uhrig-Homburg, Walter, and Weber (1999) presented empirical studies of the spot-rate models and the forward-rate models. In this article, we examine the pricing difference between the constant-volatility HJM model and the nonconstant HJM models. Our findings indicate that under certain conditions, the Ho and Lee model and the HJM model with nonconstant volatility could yield fairly similar results.

As is well known, the simplest way to price the Eurodollar futures option is to use Black's model. When the latter is used to price the interest-rate derivatives, there are two approximations. First, the futures price and forward price are unrealistically assumed to be equal. Second, interest rates are assumed to be constant for discounting purposes, even though they are assumed to be stochastic when the payoff from the option is calculated. Because the two effects tend to cancel each other out, Black's model is more powerful in pricing the interest-rate derivatives than it appears to be. In this article, we test the difference between the HJM model and Black's model through simulations. The results show that when the volatility of the spot rate is proportional to the spot rate, the difference in pricing Eurodollar futures options is insignificant. However, when the volatility structure of the spot rate is in the same class considered by the CIR model, the difference between Black's model and the HJM model becomes significant.

Recently, Flesaker (1993a) used the HJM model with constant volatility and provided closed-form equations for both the Eurodollar futures price and forward price. In this article, we also use the HJM models with various volatility structures to examine the difference between the forward price and the futures price and its effects on the Eurodollar futures options. We find that the difference between the forward price and the futures price can be significant, but its impact on the options price is trivial.

The article proceeds as follows. The second and third sections provide brief descriptions of the HJM model and the RS algorithm, respectively. The theories of pricing the Eurodollar futures are discussed in the fourth section. The sensitivity of option prices to the parameters is explained in the fifth section. The sixth section compares the HJM model to Black's model. The differences among the HJM models with different term structures are examined in the seventh section. The eighth section tests the differences between the European options and American options. The difference between the forward price and the futures price and its effects on the option prices is discussed in the ninth section. Finally, the tenth section concludes the article.

HJM MODEL

The following equations construct the one-factor HJM model:

$$df(t, T) = \mu_f(t, T)\, dt + \sigma_f(t, T)\, dw(t) \tag{1}$$

$$P(t, T) = e^{-\int_t^T f(t, s)\, ds} \tag{2}$$

$$\mu_f(t, T) = \sigma_f(t, T) \int_t^T \sigma_f(t, s)\, ds \tag{3}$$

$$g(0) = E_0[e^{-\int_0^s r(t)\, dt} g(s)] \tag{4}$$

In this model, $f(t, T)$ is the forward rate at date t for instantaneous and riskless borrowing or lending at date T. Equation 1 is the stochastic process of the forward rate of every maturity T. $\mu_f(t, T)$ and $\sigma_f(t, T)$ are the drift and volatility of this stochastic process, respectively, and can be functions of the term structure itself. $dw(t)$ is a Brownian motion. The spot rate $r(t)$ is given by $f(t, t)$. Equation 2 gives the definition for the price of a pure discount bond with maturity T at time t. The HJM model proves that under the no-arbitrage condition, the drift term $\mu_f(t, T)$ and volatility parameter $\sigma_f(t, T)$ are not independent, and their relationship is described by Equation 3. In Equation 4, $g(0)$ represents the value of a European claim at date 0 with a terminal payout at date s that is fully determined by the yield curve at that time. The expectation is taken in a risk-neutral world.

The volatility function $\sigma_f(t, T)$ is selected arbitrarily. As long as it is given, we can get the drift term $\mu_f(t, T)$ directly from Equation 3. Then, the whole stochastic process of the term structure as well as the value of the European claim is determined. Such an arbitrary property leads to a generality of the model and can be used to consistently price all contingent claims theoretically. However, the same property also makes the term structure not follow a Markovian process with respect to finite state variables, which leads to difficulties in implementing the model in practice.

Recently, Ritchken and Sankarasubramanian (1995) put a restriction on the volatility function. Although this restriction does sacrifice some part of the generality of the HJM model, it permits the term structure to be represented by a two-state Markovian model. It is described by

$$\sigma_f(t, T) = \sigma_f(t, t)k(t, T) \tag{5}$$

with

$$k(t, T) = e^{-\int_t^T \kappa(x)\,dx}$$

Here, $\sigma_f(t, t)$ is the volatility of the spot rate, and $\kappa(x)$ is a deterministic function given exogeneously. Very interestingly, the spot-rate volatility $\sigma_f(t, t)$ is still selected arbitrarily. This allows the class of term structure characterized by Equation 5 to be wide enough to include models such as Vasicek's model (Vasicek, 1977) or the CIR model (Cox et al., 1985) as some of its special cases. The implication is that its practical application could be quite broad.

Under Condition 5, the term structure could be captured by two state variables. One is the spot rate $r(t)$, and another is $\phi(t)$, which represents the accumulated variance for the forward rate to date t. The stochastic process of these two state variables are described by the following two equations:

$$dr(t) = \mu(r, \phi, t)\,dt + \sigma_f(t, T)\,dw(t) \tag{6}$$

$$d\phi(t) = (\sigma_f^2(t, t) - 2\kappa(t)\phi(t))\,dt \tag{7}$$

with

$$\mu(r, \phi, t) = \kappa(t)[f(0, t) - r(t)] + \phi(t) + \frac{d}{dt}f(0, t)$$

In the class of volatility structure of Equation 5, the price of a pure discount bond is given by

$$P(t, T) = \left(\frac{P(0, T)}{P(0, t)}\right) e^{-\beta(t, T)(r(t) - f(0, t)) - \frac{1}{2}\beta^2(t, T)\phi(t)} \tag{8}$$

where

$$\beta(t, T) = \int_t^T k(t, u)\,du$$

With regard to the spot-rate volatility $\sigma_f(t, t)$ in Equation 5, we follow the academic convention and assume constant elasticity:

$$\sigma_f(t, t) = \sigma[r(t)]^\gamma; \quad \gamma \geq 0 \tag{9}$$

where γ is the spot-rate volatility elasticity. Different γ values indicate a different volatility structure of the spot rate and, thus, different term structures. They also imply that the transformations derived from Equation 9 will differ. In this study, we consider three different cases to test the sensitivity of Eurodollar futures options to γ. When γ is fixed, there are only two parameters, σ and κ.

In Case 1, $\gamma = 0$, which implies a constant spot-rate volatility. In this case, the appropriate transformation would be

$$Y(t) = r(t)/\sigma \tag{10a}$$

and

$$m(Y, \phi, t) = \frac{1}{\sigma}\left[\kappa(t)(f(0, t) - r(t)) + \phi(t) + \frac{df(0, t)}{dt}\right] \tag{10b}$$

In Case 2, where $\gamma = 0.5$, the appropriate transform would now be

$$Y(t) = 2\frac{\sqrt{r(t)}}{\sigma} \tag{11a}$$

and

$$m(Y, \phi, t) = \frac{1}{Y(t)}\left[\frac{\kappa}{2}\left(\frac{4v(\phi, t)}{\sigma^2} - [Y(t)^2]\right) - \frac{1}{2}\right] \tag{11b}$$

where

$$v(\phi, t) = f(0, t) + \frac{1}{\kappa}\frac{d}{dt}f(0, t) + \frac{\phi(t)}{\kappa}$$

In Case 3, where $\gamma = 1$, the transform is

$$Y(t) = \frac{\log(r(t))}{\sigma} \tag{12a}$$

and

$$m(Y, \phi, t) = \frac{1}{\sigma}\left[v(Y, \phi, t) - \frac{1}{2}\sigma^2\right] \tag{12b}$$

where

$$v(Y, \phi, t) = \left\{ \kappa[f(0, t) - r(t)] + \phi(t) + \frac{d}{dt} f(0, t) \right\} e^{-r(t)}$$

ALGORITHM

The spot-rate volatility structure $\sigma_f(t, t)$ is selected arbitrarily and can depend on both state variables, $r(t)$ and $\phi(t)$. Because it is generally not constant, to use the lattice approach to generate simulations, Li, Ritchken, and Sankarasubramanian (1995) followed Nelson and Ramaswamy (1990) in making the following transformation:

$$Y(t) = \int \frac{1}{\sigma[r(t), \phi(t), t]} \, dr(t) \tag{13}$$

Let $r(t) = h[Y(t)]$ be the inverse function. Then we have

$$dY(t) = m(Y, \phi, t) \, dt + dw(t) \tag{14}$$

and

$$d\phi(t) = (\sigma^2[r(t), \phi(t), t)] - 2\kappa(t)\phi(t)) \, dt \tag{15}$$

where

$$m(Y, \phi, t) = \frac{\partial Y(t)}{\partial t} + \mu(r, \phi, t)\frac{\partial Y(t)}{\partial r(t)} + \frac{1}{2}\sigma^2[r(t), \phi(t), t]\frac{\partial^2 Y(t)}{\partial r(t)^2}$$

As can be seen, the transformed process $dY(t)$ and $d\phi(t)$ have constant volatility, and a lattice approximation can now be established. Suppose the maturity of the option is T. Then we divide T into n intervals, $\Delta t = T/n$. Let the approximating variables be y^a and ϕ^a at the beginning of some time increment. In the next time increment, the variables move to either (y^{a+}, ϕ^{a+}) or (y^{a-}, ϕ^{a-}), where

$$y^{a+} = y^a + J\sqrt{\Delta t} \tag{16a}$$

$$y^{a-} = y^a - J\sqrt{\Delta t}. \tag{16b}$$

J is an integer that can be different at each node. It is chosen specifically to make the probability of up jump or down jump be positive. The up-jump probability is given by the following equation:

$$p = \frac{m(y^a, \Phi^a, t)\Delta t + (y^a - y^{a-})}{(y^{a+} - y^{a-})} \tag{17}$$

From the relationship set up by Equation 13, we can easily get the corresponding tree for the spot rate $r(t)$. The process of $\phi(t)$ has two different properties from that of the spot rates $r(t)$ or $y(t)$. First, $\phi(t)$ is locally deterministic, which implies

$$\phi^{a+} = \phi^{a-} = \phi^{a} + [\sigma^2(r(t), \phi^a t) - 2\kappa(t)\phi^a]\Delta t \tag{18}$$

Second, the number of distinct Φ values at each node will equal the number of unique paths leading to that node. Rather than keeping track of all distinct paths, we just identify two paths at each node, which give the maximum value $\overline{\phi}^a$ and the minimum value $\underline{\phi}^a$. We then partition the interval $[\underline{\phi}^a, \overline{\phi}^a]$ into m equidistant points with $\overline{\phi}^a(k)$, $k = 1 \ldots m$, representing kth points and

$$\phi^a(k) = \underline{\phi}^a + \frac{k-1}{m-1}(\overline{\phi}^a - \underline{\phi}^a) \tag{19}$$

When we finish the lattices of the spot rates $r(t)$ and $\phi(t)$, the lattice for the underlying assets can be established with Equation 8. The terminal value of the contingent claim can then be captured.

VALUATION OF OPTIONS ON EURODOLLAR FUTURES AND TESTS OF CONVERGENCE

To price the Eurodollar futures options, we use the forward price as a substitute for the futures price. We also assume that the Eurodollar rates are default-free. Let τ be the maturity of the option on Eurodollar futures, and $T = \tau + 0.25$. Then the forward price for the time period from τ to T at time t is given by the following equation:

$$G(t, \tau, T) = \frac{P(t, T)}{P(t, \tau)}, \tag{20}$$

where $G(t, \tau, T)$ is the forward price at time t starting from time τ, $P(t, T)$ is the price of a pure discount bond at time t with a payoff of \$1 at maturity date T, and $P(t, \tau)$ is the price of a pure discount bond with a payoff of \$1 at maturity date τ. The Eurodollar futures contract is quoted on the base of a 90-day quarterly compounding rate, and the quoted Eurodollar futures price equals 100 minus the 90-day quarterly compounding rate. Thus, we have

$$efp = 100\left[1 - 4\left(\frac{P(t, \tau) - P(t, T)}{P(t, T)}\right)\right], \tag{21}$$

where efp is the Eurodollar futures price.

Now, *efp* in Equation 21 is, in fact, the Eurodollar forward price. In the ninth section, we examine the difference between the forward price and futures price and its effect on the options.

The convergence behavior of the RS algorithm depends on the number of ϕ values and the number of time partitions n. Li et al. (1995) concluded that 25 partitions for ϕ and 50 partitions for time to maturity is enough to price the options on the long-term bonds accurately. The convergence test on pricing the Eurodollar futures options shows that 50 time partitions are necessary and sufficient to provide accurate results. However, with regard to the partitions on the state variable ϕ for pricing the options on the Eurodollar futures, 25 partitions are not necessary. X is the exercise price, *efp* is the current Eurodollar futures price, τ is the time to maturity, and n and m denote the time partition and the number of partitions for the state variable ϕ, respectively.

Table I shows the convergence results for options with different exercise prices and different times to maturity at various volatility parameter levels when $\gamma = 1$. When $\sigma = 0.1$, 3 partitions for ϕ give us the same option prices as 25 partitions for ϕ. When the volatility parameter is as large as 0.5 and the time to maturity equals 1 year, the deviation of 3 partitions for ϕ from 25 partitions is within 0.03 basis point. For the whole table, 5 partitions for ϕ provide results whose deviation from 25 partitions is within 0.01 basis point. Therefore, 3 or 5 partitions for ϕ provide sufficient accuracy.

TABLE I

Results of Convergence Tests

		m ($\tau = 0.5$)				m ($\tau = 1$)			
σ	X	3	5	10	25	3	5	10	25
0.1	94	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	95	0.1378	0.1378	0.1378	0.1378	0.1905	0.1905	0.1905	0.1905
	96	0.0001	0.0001	0.0001	0.0001	0.0018	0.0018	0.0018	0.0018
0.3	94	1.1040	1.1040	1.1040	1.1040	1.2235	1.2234	1.2234	1.2234
	95	0.4115	0.4115	0.4115	0.4115	0.5665	0.5665	0.5665	0.5665
	96	0.0701	0.0700	0.0700	0.0700	0.1686	0.1686	0.1686	0.1686
0.5	94	1.3421	1.3420	1.3420	1.3420	1.5828	1.5825	1.5825	1.5825
	95	0.6818	0.6818	0.6818	0.6818	0.9332	0.9330	0.9329	0.9329
	96	0.2452	0.2451	0.2451	0.2451	0.4494	0.4493	0.4493	0.4492

Note: The initial term structure is flat at 5%. The current Eurodollar futures price is 95, and $\kappa = 0.01$. The options are American call options. $\gamma = 1$, and $n = 50$. n is the number of time partitions. σ is the volatility parameter, X is the exercise price, and τ is the life of the option. κ is the scaling factor of the forward-rate volatilities. γ is the spot-rate volatility elasticity with respect to the spot rate. m is the number of partitions of the state variable $\phi(t)$.

TABLE II
Results of Convergence

	$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
n	*94.00*	*95.00*	*96.00*	*94.00*	*95.00*	*96.00*	*94.00*	*95.00*	*96.00*
50	1.0771	0.3971	0.0857	1.1842	0.5560	0.2016	1.3540	0.7718	0.3885
100	1.0758	0.3980	0.0843	1.1833	0.5573	0.2007	1.3538	0.7736	0.3886
200	1.0759	0.3985	0.0844	1.1821	0.5579	0.1993	1.3530	0.7745	0.3879
300	1.0764	0.3986	0.0850	1.1820	0.5581	0.1995	1.3525	0.7748	0.3874
400	1.0762	0.3987	0.0848	1.1821	0.5582	0.1995	1.3521	0.7749	0.3867

Note: The initial term structure is flat at 5%. The current Eurodollar futures price is 95. The options are American call. $\gamma = 0$, $\kappa = 0$, and $\sigma = 0.02$. n is the number of time partitions. σ is the volatility parameter, X is the exercise price, and τ is the life of the option. κ is the scaling factor of the forward-rate volatilities. γ is the spot-rate volatility elasticity with respect to the spot rate.

Table II presents the convergence results for options with different exercise prices and different times to maturity when $\kappa = 0$ and $\gamma = 0$. In this case, the forward-rate term structure follows a constant-volatility process, which happens to be the simplest case of the HJM model, and the continuous time limit of the Ho and Lee model. The parameter is deterministic, and the convergence behavior depends only on the time partitions. Again, it can easily be seen that 50 time partitions give us very accurate results.

We also test the convergence behavior of the model when $\gamma = 0.5$. In this case, the convergence behavior is quite similar to the case in which $\gamma = 1$. Therefore, 50 time partitions and 3 or 5 partitions for the state variable ϕ are going to provide results accurate enough for pricing the Eurodollar futures options.

SENSITIVITY OF OPTION PRICES TO MODEL PARAMETERS

Let the volatility structure follow Equation 9 and $\gamma = 1$. The price sensitivity of American call options to parameters is shown in Table III.

First, when $\sigma < 10\%$, all the prices of options with exercise price 94 are completely determined by the intrinsic value. Second, it can easily be seen that, once again, when $\sigma < 10\%$, all the prices of options with exercise price 96 are zero, except for the option with 1 year to maturity and with σ at the 30% level. These option prices carry little information about the model parameters, to which they obviously are not sensitive. It is also easy to see that with exercise price 95, all the prices of options neither are zero nor are determined by the intrinsic value, regardless of the value of σ.

TABLE III
Sensitivity of Option Prices to Model Parameters

		$\tau = 0.25$		$\tau = 0.5$		$\tau = 1$	
σ	X	$\kappa = 0.01$	$\kappa = 0.1$	$\kappa = 0.01$	$\kappa = 0.1$	$\kappa = 0.01$	$\kappa = 0.1$
0.05	94	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	95	0.0493	0.0482	0.0689	0.0667	0.0954	0.0902
	96	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	94	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	95	0.0985	0.0963	0.1378	0.1332	0.1905	0.1801
	96	0.0000	0.0000	0.0000	0.0000	0.0018	0.0011
0.3	94	1.0365	1.0335	1.1040	1.0949	1.2234	1.1978
	95	0.2949	0.2884	0.4115	0.3979	0.5665	0.5358
	96	0.0199	0.0180	0.0700	0.0624	0.1686	0.1467

Note: The initial term structure is flat at 5%. The current Eurodollar futures price is 95. The options are American call. $\gamma = 1$, and $n = 50$. σ is the volatility parameter, X is the exercise price, and τ is the life of the option. κ is the scaling factor of the forward-rate volatilities.

As for all the other options that neither are completely determined by the intrinsic value nor are worthless, prices seem to be relatively sensitive to the volatility parameter σ but not quite so sensitive to the parameter κ. For example, when $X = 95$, $\tau = 0.5$, and $\kappa = 0.01$, the option price increases from 0.1378 to 0.4115 when σ goes up from 0.1 to 0.3. For the same exercise price and time to maturity, if we fix the parameter σ at 10% and let the parameter κ increase from 0.01 to 0.1, the option price changes only about 3.3%, from 0.1378 to 0.1332.

We also tested the sensitivity of the option price to parameters when $\gamma = 0.5$. The conclusions were very similar to those of the HJM model with $\gamma = 1$. The option prices are not so sensitive to the parameter κ but are relatively sensitive to the volatility parameter σ.

COMPARISON OF THE HJM MODEL WITH BLACK'S MODEL

In 1976, Black extended the well-known Black–Scholes model to value options on commodity futures. The application of the model is far beyond Black's original intention and is frequently used to value interest-rate options. When Black's model is applied to the interest-rate derivatives, such as the options on Eurodollar futures, we really assume the interest rates are constant for discounting purposes, even though they are assumed to be stochastic when the payoff from the options is calculated. Black has a closed-form equation for the European options. Let τ denote the maturity date of the option, *efp* the futures price, X the strike price, r the zero-coupon yield for maturity τ, and σ the volatility of the futures price. Then we have

$$c = e^{-r\tau}[efpN(d_1) - XN(d_2)] \tag{22a}$$

and

$$p = e^{-r\tau}[XN(-d_2) - efpN(-d_1)] \tag{22b}$$

where

$$d_1 = \frac{\log(efp/X) + \sigma^2\tau/2}{\sigma\sqrt{\tau}}$$

and

$$d_2 = \frac{\log(efp/X) - \sigma^2\tau/2}{\sigma\sqrt{\tau}}$$

In these expressions, c and p are the call-option price and put-option price, respectively.

When Black's model is used to price the Eurodollar futures options, we really use the put-option equation to calculate the call-option price and vice versa. Remember that the Eurodollar futures price is *efp* and the exercise price is X. Then, substituting (100 − *efp*) and (100 − X) for the futures price and exercise price, respectively, into the put-option equation, we can get the call-option price directly in basis points by multiplying the result by 100. The American option could then be valued with the lattice method.

Unlike the equilibrium models and arbitrage models, both of which center on the evolution of the term structure, the focal point of Black's model is the distribution of the futures prices directly. This property makes Black's model simple and clear. If Black's model and the HJM model do not provide significantly different results in pricing the Eurodollar futures options, then Black's model would make the pricing procedure simpler and more convenient. To determine if this is the case, we compare Black's model with the HJM model under different time-to-maturity (0.25, 0.5, and 1.0) and volatility (0.1 and 0.3) parameters. Our findings are presented in Table IV.

Assume that the current Eurodollar futures price equals 95, and note that κ is fixed at 0.01 for the HJM model. The call option with an exercise price of 95 is at-the-money. We first calculate the option prices with different exercise prices for the HJM model and then use the price

TABLE IV

Comparison of the HJM Model with Black's Model

		$\sigma = 0.1$ *for the HJM Model*			$\sigma = 0.3$ *for the HJM Model*		
τ	X	$\gamma = 1$	*Black*	Δ	$\gamma = 1$	*Black*	Δ
0.25	94.00	1.0000	1.0000	0.0000	1.0365	1.0366	0.0001
	94.50	0.5008	0.5008	0.0000	0.6197	0.6188	−0.0009
	95.00	0.0985	0.0985	0.0000	0.2949	0.2949	0.0000
	95.50	0.0015	0.0015	0.0000	0.1013	0.0993	−0.0020
	96.00	0.0000	0.0000	0.0000	0.0199	0.0197	−0.0002
0.5	94.00	1.0000	1.0000	0.0000	1.1040	1.1061	0.0021
	94.50	0.5090	0.5090	0.0000	0.7242	0.7221	−0.0021
	95.00	0.1378	0.1378	0.0000	0.4115	0.4115	0.0000
	95.50	0.0098	0.0097	−0.0001	0.1958	0.1931	−0.0027
	96.00	0.0001	0.0001	0.0000	0.0700	0.0686	−0.0014
1.0	94.00	1.0000	1.0000	0.0000	1.2234	1.2256	0.0022
	94.50	0.5323	0.5323	0.0000	0.8685	0.8686	0.0001
	95.00	0.1905	0.1905	0.0000	0.5665	0.5665	0.0000
	95.50	0.0341	0.0334	−0.0007	0.3342	0.3307	−0.0035
	96.00	0.0018	0.0018	0.0000	0.1686	0.1648	−0.0038

Note: The options are American call. σ is the volatility parameter, X is the exercise price, and τ is the life of the option. γ is the spot-rate volatility elasticity with respect to the spot rate. Δ is the difference between the option price from Black's model and that from the HJM model. For the HJM model, $\kappa = 0.01$, $n = 50$, and $m = 5$. For Black's model, $n = 400$, and $r = 0.0497$. The current Eurodollar futures price is 95. The initial term structure is flat at 5%.

of the at-the-money option from the HJM model as input for Black's model to obtain the implied volatility. This implied volatility is then used to calculate the option prices with different exercise prices. Table IV shows that when the volatility parameter for the HJM model is 0.1, the HJM model and Black's model yield virtually identical results under all three times to maturity. When the volatility parameter for the HJM model is as high as 0.3, the difference between the HJM model and Black's model is well within 0.4 basis point and often much smaller, which is not significant.

These findings do not necessarily lead us to conclude that the difference between Black's model and the HJM model is negligible in pricing the options on Eurodollar futures when $\sigma_f(t, t) = \sigma r$, because we assume that the initial term structure is flat. Hence, we also test the difference between these two models when the initial term structure is not flat. Assume that the 3-month Eurodollar rate equals 3%, the 6-month rate equals 4%, and the current 6-month Eurodollar futures price equals 95. The initial term structure is derived with the method of linear interpolation. For the HJM model, we have $\kappa = 0.01$ and $\sigma = 0.1$. Suppose the option prices for Black's model are discounted by the 6-month Eurodollar rate. The volatility for Black's model is still calculated by the at-the-money option price from the HJM model. For options with maturity within 1 year, the difference between the HJM model and Black's model is within half a basis point. Therefore, we conclude that even when the initial term structure is not flat, the difference between the two models is insignificant.

However, the conclusions about the difference between Black's model and the HJM model are different when $\sigma_f(t, t) = \sigma\sqrt{r}$. Assume that the initial term structure is flat at the interest rate implied by the current Eurodollar futures price of 95. We first use Black's model to calculate the option prices with different exercise prices and then calculate the implied volatility for the HJM model with the option price at the money. The results are given in Table V, where we find that when the volatility for Black's model is 0.1, the difference between Black's model and the HJM model is within 0.4 basis points even when the maturity is as long as 1 year. When the volatility for Black's model goes up to 0.3, the difference between the HJM model and Black's model is still negligible for the options whose maturity is 0.25 year. For the options with maturities that are longer or equal to half a year, the difference between Black's model and the HJM model could be larger than 1 basis point.

TABLE V

Comparison of the HJM Model with Black's Model

		$\sigma = 0.1$ *for Black's Model*			$\sigma = 0.3$ *for Black's Model*		
τ	X	$\gamma = 0.5$	*Black*	Δ	$\gamma = 0.5$	*Black*	Δ
0.25	94.00	1.0000	1.0000	0.0000	1.0306	1.0370	0.0064
	94.50	0.5005	0.5008	0.0003	0.6147	0.6195	0.0048
	95.00	0.0986	0.0986	0.0000	0.2957	0.2957	0.0000
	95.50	0.0018	0.0015	−0.0003	0.1077	0.0998	−0.0079
	96.00	0.0000	0.0000	0.0000	0.0250	0.0199	−0.0051
0.5	94.00	1.0000	1.0000	0.0000	1.0932	1.1078	0.0146
	94.50	0.5078	0.5092	0.0014	0.7172	0.7243	0.0071
	95.00	0.1381	0.1381	0.0000	0.4138	0.4138	0.0000
	95.50	0.0110	0.0098	−0.0012	0.2074	0.1951	−0.0123
	96.00	0.0001	0.0001	0.0000	0.0841	0.0698	−0.0143
1.0	94.00	1.0000	1.0000	0.0000	1.2073	1.2317	0.0244
	94.50	0.5303	0.5331	0.0028	0.8621	0.8754	0.0133
	95.00	0.1917	0.1917	0.0000	0.5733	0.5733	0.0000
	95.50	0.0378	0.0341	−0.0037	0.3536	0.3369	−0.0167
	96.00	0.0029	0.0019	−0.0010	0.1961	0.1695	−0.0266

Note: The options are American call. τ is the life of the option, and X is the exercise price. Δ is the difference between the option price from Black's model and that from the HJM model. For the HJM model, $\kappa = 0.01$, $n = 50$, and $m = 5$. For Black's model, $n = 400$. The current Eurodollar futures price is 95. The initial term structure is flat at 5%.

In summary, when the volatility of the spot rate is proportional to the spot rate, the difference between the HJM model and Black's model is insignificant. In this sense, Black's model is much stronger than it seems to be in pricing the Eurodollar futures options. When the spot-rate structure is in the same class considered by the CIR model, Black's model fails to get results similar to the HJM model in pricing the Eurodollar futures options when the maturity of the option is larger or equal to half a year, and the volatility for Black's model is about 30%.

HJM MODELS WITH VARIOUS VOLATILITY STRUCTURES

In this section, simulation tests are carried out to examine the differences among the HJM models with various volatility structures.

Assume the initial term structure is flat at the interest rate implied by the current futures price. Let $\kappa = 0.01$ for the nonconstant HJM models. We first calculate the option prices with different exercise prices when $\gamma = 1$ and $\sigma = 0.1$. Then we use the at-the-money option price to calculate the implied volatilities for the constant-volatility HJM model and the HJM model with $\gamma = 0.5$. A comparison of different HJM models in pricing the Eurodollar futures options with different times to maturity and exercise prices for both the American and European styles is shown in Table VI. The differences among the HJM models with different volatility structures increase with the increasing time to maturity. Yet even when the time to maturity is as long as 1 year, the differences among the HJM models are within 0.6 basis point for both American and European options. Therefore, there is no significant difference for different volatility structures in pricing the Eurodollar futures options when the forward-rate volatility is low.

Flesaker (1993b) selected 0.02 as the volatility level for the constant-volatility HJM model. This volatility level is much higher than the volatility level we have used. Thus, we use 0.02 as the volatility level for the constant HJM model to repeat the testing process. This time we first calculate the option prices with different exercise prices for the constant-volatility HJM model and then use the at-the-money price to calculate the implied volatilities for the HJM models when $\gamma = 1$ and $\gamma = 0.5$, respectively. The results are shown in Table VII. As can be seen, the difference between the HJM models with nonconstant volatility and the HJM model with constant volatility increases with the increasing time to maturity. Even for the short-term options, which have 0.25 year to maturity, the differences among these three HJM

TABLE VI

Comparison of the Different HJM Models with a Lower Volatility Level of the Forward Rate

τ		X	$\gamma = 1$	$\gamma = 0.5$	$\gamma = 0$	Δ_1	Δ_2	Δ_3
0.25	*A*	94.00	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
		94.50	0.5008	0.5005	0.5003	−0.0005	−0.0002	−0.0003
		95.00	0.0985	0.0985	0.0985	0.0000	0.0000	0.0000
		95.50	0.0015	0.0017	0.0021	0.0006	0.0004	0.0002
		96.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	E	94.00	0.9875	0.9875	0.9876	0.0001	0.0001	0.0000
		94.50	0.4965	0.4961	0.4958	−0.0007	−0.0003	−0.0004
		95.00	0.0983	0.0983	0.0983	0.0000	0.0000	0.0000
		95.50	0.0015	0.0017	0.0021	0.0006	0.0004	0.0002
		96.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.5	*A*	94.00	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
		94.50	0.5090	0.5077	0.5065	−0.0025	−0.0012	−0.0013
		95.00	0.1378	0.1378	0.1378	0.0000	0.0000	0.0000
		95.50	0.0098	0.0108	0.0121	0.0023	0.0003	0.0010
		96.00	0.0001	0.0001	0.0002	0.0001	0.0001	0.0000
	E	94.00	0.9757	0.9757	0.9757	0.0000	0.0000	0.0000
		94.50	0.5020	0.5010	0.4999	−0.0026	−0.0011	−0.0010
		95.00	0.1371	0.1371	0.1371	0.0000	0.0000	0.0000
		95.50	0.0098	0.0108	0.0121	0.0023	0.0003	0.0010
		96.00	0.0001	0.0001	0.0002	0.0001	0.0001	0.0000
1.0	*A*	94.00	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
		94.50	0.5323	0.5296	0.5263	−0.0060	−0.0033	−0.0027
		95.00	0.1905	0.1905	0.1905	0.0000	0.0000	0.0000
		95.50	0.0341	0.0370	0.0391	0.0050	0.0021	0.0029
		96.00	0.0018	0.0027	0.0039	0.0021	0.0012	0.0009
	E	94.00	0.9580	0.9563	0.9560	−0.0020	−0.0003	0.0017
		94.50	0.5203	0.5182	0.5151	−0.0052	−0.0031	0.0021
		95.00	0.1885	0.1885	0.1885	0.0000	0.0000	0.0000
		95.50	0.0339	0.0369	0.0388	0.0049	0.0019	0.0030
		96.00	0.0018	0.0027	0.0038	0.0020	0.0011	0.0009

Note: The initial term structure is flat at 5%. The current Eurodollar futures price is 95. τ is the life of the option, and X is the exercise price. γ is the spot-rate volatility elasticity with respect to the spot rate. For $\gamma = 1$, $\sigma = 0.1$, $\kappa = 0.01$, $n = 50$, and $m = 5$. For $\gamma = 0.5$, $\kappa = 0.01$, $n = 50$, and $m = 5$. For $\gamma = 0$, $\kappa = 0$, and $n = 300$. *A* represents American options, and *E* represents European options. Δ_1 is the difference between Columns 6 and 4, Δ_2 is the difference between Columns 6 and 5, and Δ_3 is the difference between Columns 5 and 4.

models can be larger than 1 basis point. Therefore, we conclude that when the forward rate is quite volatile and depends on the interest rate, the HJM models with different volatility structures are significantly different from one another in pricing the Eurodollar futures options.

The differences between the constant HJM model and the nonconstant HJM models are presented in Table VIII. In the table, σ_1 is the implied volatility for the constant HJM model with the option

TABLE VII

Comparison of the Different HJM Models with a Higher Volatility Level of the Forward Rate

τ		X	$\gamma = 1$	$\gamma = 0.5$	$\gamma = 0$	Δ_1	Δ_2	Δ_3
0.25	Call	94.00	1.1014	1.0894	1.0764	−0.0250	−0.0130	−0.0120
		94.50	0.7148	0.7057	0.6947	−0.0201	−0.0110	−0.0091
		95.00	0.3986	0.3986	0.3986	0.0000	0.0000	0.0000
		95.50	0.1830	0.1923	0.1995	0.0165	0.0072	0.0093
		96.00	0.0614	0.0738	0.0850	0.0236	0.0112	0.0124
	Put	94.00	0.1114	0.0993	0.0861	−0.0253	−0.0132	−0.0121
		94.50	0.2210	0.2119	0.2009	−0.0201	−0.0110	−0.0091
		95.00	0.4004	0.4004	0.4004	0.0000	0.0000	0.0000
		95.50	0.6806	0.6898	0.6970	0.0164	0.0072	0.0092
		96.00	1.0561	1.0683	1.0792	0.0231	0.0109	0.0122
0.5	Call	94.00	1.2273	1.2060	1.1820	−0.0453	−0.0240	−0.0213
		94.50	0.8658	0.8531	0.8376	−0.0282	−0.0155	−0.0127
		95.00	0.5581	0.5581	0.5581	0.0000	0.0000	0.0000
		95.50	0.3241	0.3370	0.3469	0.0228	0.0099	0.0129
		96.00	0.1584	0.1804	0.1995	0.0411	0.0191	0.0220
	Put	94.00	0.2470	0.2256	0.2014	−0.0456	−0.0242	−0.0214
		94.50	0.3773	0.3646	0.3491	−0.0282	−0.0155	−0.0127
		95.00	0.5611	0.5611	0.5611	0.0000	0.0000	0.0000
		95.50	0.8193	0.8322	0.8418	0.0225	0.0096	0.0129
		96.00	1.1472	1.1691	1.1878	0.0406	0.0187	0.0129
1.0	Call	94.00	1.4244	1.3897	1.3525	−0.0719	−0.0372	−0.0347
		94.50	1.0805	1.0614	1.0401	−0.0404	−0.0213	−0.0191
		95.00	0.7748	0.7748	0.7748	0.0000	0.0000	0.0000
		95.50	0.5221	0.5414	0.5582	0.0361	0.0168	0.0193
		96.00	0.3189	0.3547	0.3874	0.0685	0.0327	0.0358
	Put	94.00	0.4608	0.4263	0.3893	0.0715	−0.0370	−0.0345
		94.50	0.6009	0.5820	0.5610	0.0399	−0.0210	−0.0189
		95.00	0.7796	0.7796	0.7796	0.0000	0.0000	0.0000
		95.50	1.0119	1.0310	1.0480	0.0361	0.0170	0.0191
		96.00	1.2969	1.3320	1.3645	0.0676	0.0325	0.0351

Note: The initial term structure is flat at 5%. The current Eurodollar futures price is 95. The options are American style. τ is the life of the option, and X is the exercise price. γ is the spot-rate volatility elasticity with respect to the spot rate. Δ_1 is the difference between Columns 6 and 4, Δ_2 is the difference between Columns 6 and 5, and Δ_3 is the difference between Columns 5 and 4. For $\gamma = 1$ and $\gamma = 0.5$, $\kappa = 0.01$, $n = 50$, and $m = 5$. For $\gamma = 0$, $\sigma = 0.02$, $\kappa = 0$, and $n = 300$.

prices of the HJM model when $\gamma = 0.5$ as inputs. Similarly, σ_2 is the implied volatility for the constant HJM model with the option prices of the HJM model when $\gamma = 1$ as inputs. σ_1 and σ_2 are calculated with the put-option prices for the options with an exercise price below 95 and with the call-option prices for the options with an exercise price of 95 or greater. The relationship of the implied volatility for options with half a year to maturity with respect to the exercise price is shown in Figure 1.

TABLE VIII
Implied Volatility

	$\tau = 0.25$		$\tau = 0.5$		$\tau = 1$	
X	σ_1	σ_2	σ_1	σ_2	σ_1	σ_2
94.00	0.02197	0.02107	0.02199	0.02106	0.02205	0.02109
94.50	0.02115	0.02063	0.02105	0.02058	0.02106	0.02056
95.00	0.02000	0.02000	0.02000	0.02000	0.02000	0.02000
95.50	0.01906	0.01959	0.01912	0.01962	0.01903	0.01955
96.00	0.01796	0.01909	0.01805	0.01909	0.01798	0.01904

Note: σ_1 is the implied volatility for the constant volatility HJM model with the theoretical price from the HJM model when $\gamma = 1$ as input, and σ_2 is the implied volatility from the HJM model when $\gamma = 0.5$. σ_1 and σ_2 are calculated with the option prices of the constant HJM model as inputs. τ is the life of the option.

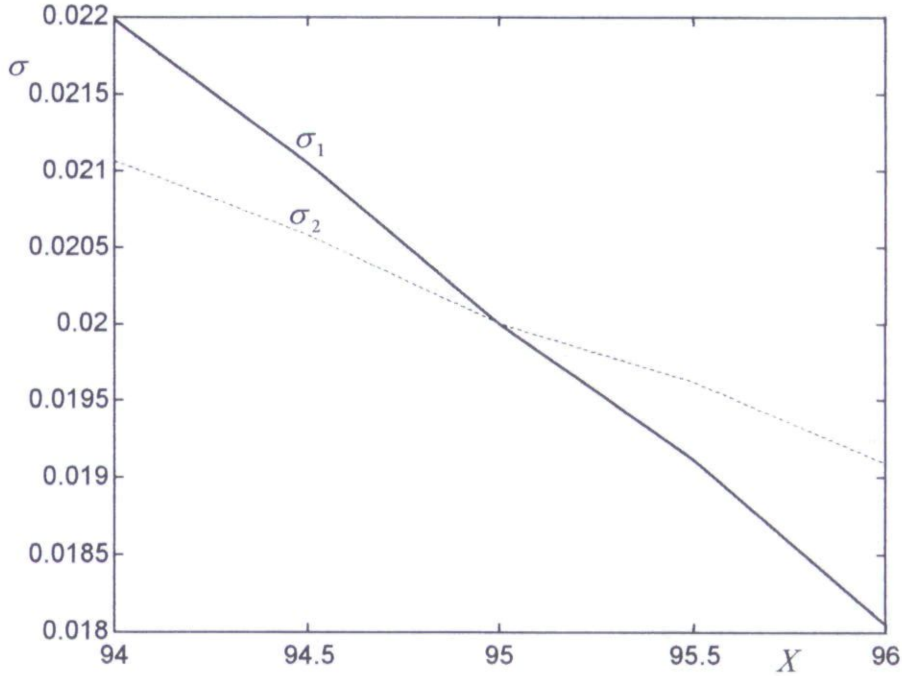

FIGURE 1
Volatility smile. σ_1 is the implied volatility for the constant-volatility HJM model with the theoretical price from the HJM model when $\gamma = 0.5$ as input; σ_2 is the implied volatility from the HJM model when $\gamma = 1$.

The strong relationship between the implied volatility and the exercise price really highlights the importance of correctly estimating the parameter γ. When the true volatility structure is such that $\gamma = 1$ or $\gamma = 0.5$, the constant HJM model will lead to incorrect results when the forward rate is very volatile. This is because the constant HJM model uses just a single volatility number to represent the volatilities at different exercise prices, which, in fact, are significantly different.

In summary, the differences among the HJM models with various volatility structures are insignificant when the volatility of the forward rate is relatively small. When the volatility of the forward rate is very high, the HJM models with different volatility structures are significantly different from one another.

EUROPEAN OPTION VERSUS AMERICAN OPTION

Because of their simplicity, European option models are frequently used for theoretical and empirical research. Unfortunately, the options traded on Eurodollar futures in the Chicago Mercantile Exchange are American-style. If the differences between American options and European options are trivial under certain conditions, we can use the European option models to price American options. It is very obvious that for deeply in-the-money options, the American and European styles are quite different. This is because the early exercise property makes the American options mainly determined by the intrinsic value, whereas the European options are mainly determined by the whole stochastic process of their lifetime. For options deeply out-of-the-money, the probability of early exercise is very small, and the difference between the European and American is trivial.

The HJM models could be used to examine the difference between European and American options. Because the differences between the European and American options deeply in the money or out of the money are already clear, here we focus on the options in the money or out of the money within 50 basis points. The call-option prices for both American and European options with different times to maturity and different exercise prices at various volatility parameter levels are presented in Table IX.

The volatility parameter σ for different HJM models is selected to make the value of σr^{γ} at the initial point comparable for different HJM models. The conclusions about the difference between the American and European styles from the three different HJM models are the same. Consider the options with the exercise price 94.5. From Table IX, we can see that the difference between the American options and the European options increases with the increasing time to maturity. When the maturity is less than or equal to half a year, the difference between the American and European options is within 0.65 basis point for $\gamma = 1$, within 0.75 for $\gamma = 0.5$, and within 0.65 for $\gamma = 0$. Even the highest one, 0.75, is within 1 basis point, which is not significant. When the maturity is 1 year, the difference is about 1 basis point, which is significant. The differences between the American and European options are related to the exercise prices.

TABLE IX
European versus American Options

					$\gamma = 1$					
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
94.5	0.1	0.5008	0.4965	0.0043	0.5090	0.5025	0.0065	0.5323	0.5210	0.0113
	0.3	0.6197	0.6177	0.0020	0.7244	0.7203	0.0041	0.8685	0.8592	0.0093
					$\gamma = 0.5$					
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
94.5	0.02	0.5000	0.4949	0.0051	0.5036	0.4961	0.0075	0.5184	0.5060	0.0124
	0.08	0.6624	0.6604	0.0020	0.7906	0.7865	0.0041	0.9612	0.9525	0.0087
					$\gamma = 0$					
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
94.5	0.005	0.5004	0.4960	0.0044	0.5071	0.5006	0.0065	0.5284	0.5175	0.0109
	0.02	0.6947	0.6930	0.0017	0.8376	0.8346	0.0030	1.0401	1.0360	0.0041
					$\gamma = 1$					
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
95	0.1	0.0985	0.0983	0.0002	0.1378	0.1371	0.0007	0.1905	0.1885	0.0020
	0.3	0.2949	0.2942	0.0007	0.4115	0.4097	0.0018	0.5665	0.5619	0.0046
					$\gamma = 0.5$					
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
95	0.02	0.0884	0.0882	0.0002	0.1236	0.1230	0.0006	0.1710	0.1692	0.0018
	0.08	0.3526	0.3518	0.0008	0.4917	0.4898	0.0019	0.6763	0.6715	0.0048
					$\gamma = 0$					
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
95	0.005	0.0998	0.0996	0.0002	0.1398	0.1392	0.0006	0.1940	0.1923	0.0017
	0.02	0.3986	0.3980	0.0006	0.5581	0.5568	0.0013	0.7748	0.7730	0.0018

TABLE IX
(Continued)

		$\gamma = 1$								
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
95.5	0.1	0.0015	0.0015	0.0000	0.0098	0.0098	0.0000	0.0341	0.0339	0.0002
	0.3	0.1013	0.1012	0.0001	0.1958	0.1953	0.0005	0.3342	0.3325	0.0017
		$\gamma = 0.5$								
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
95.5	0.02	0.0008	0.0008	0.0000	0.0068	0.0068	0.0000	0.0256	0.0255	0.0001
	0.08	0.1513	0.1510	0.0003	0.2774	0.2767	0.0007	0.4462	0.4439	0.0023
		$\gamma = 0$								
		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
X	σ	A	E	Δ	A	E	Δ	A	E	Δ
95.5	0.005	0.0020	0.0022	0.0000	0.0128	0.0128	0.0000	0.0413	0.0412	0.0001
	0.02	0.1995	0.1993	0.0002	0.3469	0.3463	0.0006	0.5582	0.5576	0.0006

Note: The initial term structure is flat at 5%. The current Eurodollar futures price is 95. When $\gamma = 1$ and $\gamma = 0.5$, $m = 5$, $\kappa = 0.01$, and $n = 50$. When $\gamma = 0$, $\kappa = 0$ and $n = 300$. *A* denotes American call options, and *E* denotes European call options. Δ is the difference between the American option price and the European option price. σ is the volatility parameter, *X* is the exercise price, and τ is the life of the option. γ is the spot-rate volatility elasticity with respect to the spot rate.

On the whole, as can be seen in Table IX, except for the option that is 1 year to maturity and has 50 basis points in the money, the difference between the European and American styles is insignificant for all other options. Therefore, we conclude that the differences between the European-style and American-style options with maturities less than or equal to 0.5 year are insignificant. However, the differences are significant for options with a maturity equal to 1 year.

DIFFERENCE BETWEEN FUTURES PRICE AND FORWARD PRICE AND ITS EFFECTS ON OPTIONS PRICES

A substantial amount of academic research has been carried out on the differences between the forward price and the futures price when the interest rates are stochastic. Recently, Flesaker (1993a) used the HJM model with constant volatility and got closed-form equations for both the Eurodollar futures and forward prices. His research has shown that for contract maturities up to around 6 months, Eurodollar futures prices are not significantly different from the corresponding forward price. For longer maturities, however, the magnitude of the difference may be quite significant and increases with the maturity and the level of interest-rate volatility.

In a risk-neutral world, the futures are treated as an asset paying continuously compounding dividends at the same rate as the risk-free rate. Therefore, the expected rate of return on the futures equals zero in the risk-neutral world. Recently, Amin and Morton (1994) observed that the futures price for a continuously marked-to-market futures contract follows a martingale because opening a futures position requires no investment. If the futures price at date t for a contract that matures at date T is $efp_T(t)$, then

$$efp_T(t) = E_t[efp_T(T)] \tag{23}$$

$efp_T(T)$ is the futures price at maturity, which equals the forward price and also the spot price at maturity date T. It follows that we can use the lattice method to calculate the futures price by going backward without the discount factor.

Our findings so far follow from Equations 20 and 21, which means that we have been using the forward price to substitute for the futures price. We are aware of the fact that such substitutions may lead to some errors. In this section, we are going to examine the differences between the futures price and the forward price and their effects on the Eurodollar futures options.

First, we calculate the futures price based on Equation 23 and then calculate the options price. Consider the case when $\gamma = 1$. Assume that the initial term structure is flat at the interest rate implied by the forward price. Because the futures price equals the forward price on the maturity date, as long as we know the initial term structure, we know the futures price on the maturity date. Thus, using Equation 23 and going backward, we can get the lattice of the futures price. The initial futures price on the lattice should be completely consistent with the current futures price. In practice, we only know the current futures price, not the forward rate. Therefore, we can go inversely. We use the current futures price to calculate the implied forward rate. Then we use this initial term structure as inputs to calculate the option prices, which are based on the futures prices.

Second, we use the initial term structure previously obtained as inputs to calculate the forward price and the option prices based on Equations 20 and 21, which is what we have done in previous sections.

First, let $\sigma = 0.1$. With the methods described previously, when the current forward price is fixed at 95, the current Eurodollar futures prices is 94.9998 for 0.25 year to maturity, 94.9994 for 0.5 year to maturity, and 94.9982 for 1 year to maturity. The difference between the forward price and the futures price is 0.02 basis point for 0.25 year to maturity, 0.06 basis point for 0.5 year to maturity, and 0.18 basis point for 1 year to maturity. Thus, the difference increases with the increasing time to maturity.

The option prices based on both forward prices and futures prices with different exercise prices and different times to maturity are presented in Table X. Column 2 lists the option prices based on the forward price, whereas Column 1 prices are based on the futures price. For the deeply in-the-money options, the prices of options are determined by the intrinsic value. Therefore, the difference is just the difference of the forward price and futures price. The differences for all other options, which are not deeply in the money, are within 0.04 basis points for options even with 1 year to maturity.

TABLE X

Effects on the Option Prices from the Difference between the Forward Price and Futures Price

		$\tau = 0.25$			$\tau = 0.5$			$\tau = 1$		
σ	X	*1*	2	Δ	*1*	2	Δ	*1*	2	Δ
0.1	93.00	1.9998	2.0000	0.0002	1.9994	2.0000	0.0006	2.0000	1.9982	0.0018
	93.50	1.4998	1.5000	0.0002	1.4994	1.5000	0.0006	1.5000	1.4982	0.0018
	94.00	0.9998	1.0000	0.0002	0.9994	1.0000	0.0006	1.0000	0.9982	0.0018
	94.50	0.5006	0.5008	0.0002	0.5087	0.5090	0.0003	0.5323	0.5319	0.0004
	95.00	0.0985	0.0985	0.0000	0.1377	0.1378	0.0001	0.1905	0.1904	0.0001
	95.50	0.0015	0.0015	0.0000	0.0098	0.0098	0.0000	0.0341	0.0341	0.0000
	96.00	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0018	0.0018	0.0000
0.3	93.00	1.9979	2.0000	0.0021	2.0056	2.0088	0.0024	2.0494	2.0548	0.0054
	93.50	1.5043	1.5055	0.0012	1.5386	1.5403	0.0017	1.6181	1.6213	0.0032
	94.00	1.0359	1.0365	0.0006	1.1031	1.1040	0.0009	1.2216	1.2234	0.0018
	94.50	0.6195	0.6197	0.0002	0.7240	0.7244	0.0004	0.8676	0.8685	0.0009
	95.00	0.2948	0.2949	0.0001	0.4113	0.4115	0.0002	0.5660	0.5665	0.0005
	95.50	0.1013	0.1013	0.0000	0.1958	0.1958	0.0000	0.3341	0.3342	0.0001
	96.00	0.0199	0.0199	0.0000	0.0700	0.0700	0.0000	0.1685	0.1686	0.0001

Note: σ is the volatility parameter, X is the exercise price, and τ is the life of the option. Δ is the difference between Columns 2 and 1. $\gamma = 1$, $\kappa = 0.01$, $m = 5$, and $n = 50$. The forward price is fixed at 95. When $\sigma = 0.1$, the futures price is 94.9998 for 0.25 year to maturity, 94.9994 for 0.5 year to maturity, and 94.9982 for 1 year to maturity. When $\sigma = 0.3$, the futures price is 94.9979 for 0.25 year to maturity, 94.9944 for 0.5 year to maturity, and 94.9830 for 1 year to maturity.

Next, let σ increase from 0.1 to 0.3 and the forward price still be fixed at 95. The futures price is 94.9979 for 0.25 year to maturity, 94.9944 for 0.5 year to maturity, and 94.9830 for 1 year to maturity. The difference between the forward price and the futures price increases with the increasing time to maturity. For the options with 1 year to maturity, the difference between the forward price and futures price goes up to 1.70 basis points. However, the difference between the forward price and the futures price has a very limited effect on pricing the options. For options with 1 year to maturity, the biggest value of Δ is 0.54 basis point, which is not significant. So even for the volatility parameter as high as 0.3, the difference between the option price based on Equations 20 and 21 and that based on Equation 23 is insignificant, which verifies the methodology we used in the previous sections.

CONCLUSION

Without severe restrictions on the term structure, the evolution of the term structure is not Markovian under the martingale with respect to finite state variables. The restrictions set by the RS algorithm have partly removed the non-Markovian property and allowed the evolution of the term structure to be described by two state variables. Quite interestingly, the restrictions are imposed on the volatility structure of the forward rate, not on the volatility structure of the spot rate, which keeps most of the generality of the original HJM model. The HJM model, combined with the RS algorithm, could be very powerful in pricing interest-rate derivatives in practice. The simulation tests in this article on the Eurodollar futures options with the HJM model indeed provide evidence for this capacity.

As long as we get the implied volatility for Black's model with respect to each maturity time, the results from Black's model do not deviate significantly from the HJM model when the volatility of spot rate is proportional to the spot rate. In this sense, Black's model is much stronger than one might think in pricing Eurodollar futures options. However, when the volatility structure of the spot rate follows the process considered by the CIR model (Cox et al., 1985), the difference between the HJM model and Black's model is significant.

The results show that the constant HJM model gives very close results to the nonconstant HJM model when the term structure is not quite volatile. If the volatility of the forward rate is very high, it fails to provide results close to the nonconstant HJM model. This highlights the importance of the estimates of the γ parameter when the term structure is very volatile.

The results also show that the difference between the European-style and American-style options at the money or out of the money within 50 basis points is insignificant when the life options are less than 1 year. However, the difference could be significant for the options with 1 year to maturity. Also, the difference depends on the exercise price. When the volatility of the spot rate is proportional to the spot rate, the difference between the forward price and futures price is negligible if the volatility parameter is at 10%. In this case, the difference between the option price based on Equations 20 and 21 and that based on Equation 23 is also negligible. When the volatility parameter goes up to 30%, the difference between the forward price and futures price could be larger than 1 basis point, but its effects on the price of the options are still trivial.

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