# Mean-Variance Efficiency of the Market Portfolio and Futures Trading

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We derived an intertemporal capital asset pricing model in which the mean-variance efficiency of the market portfolio is neither a necessary nor a sufficient condition. We obtained this result by modeling a frictionless, continuously open financial market in which nonredundant futures contracts are available for trade, in addition to cash assets. Introducing such contracts modifies the way investors optimally allocate their wealth. Their portfolios then comprise the riskless asset, a perturbed mean-variance-efficient portfolio of cash assets, and a perturbed mean-variance-efficient portfolio of futures contracts. Furthermore, a $(3 + K)$ mutual fund separation is obtained, with $K$ being the number of economic state variables, in lieu of the usual $(2 + K)$ fund separation. Mean-variance efficiency of the market portfolio is a necessary condition only when cash assets are the sole traded assets. © 2001 John Wiley & Sons, Inc. Jrl Fut Mark 21:329–346, 2001

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## INTRODUCTION

The validity of the various capital asset pricing models (CAPMs) widely used in finance is known to be closely related to mean-variance efficiency. An important distinction, however, has to be made between the standard one-period model of Sharpe (1964) and Lintner (1965) and multiperiod models, such as the intertemporal capital asset pricing models (ICAPMs) of Merton (1973) and Breeden (1979, 1984). The static, one-factor CAPM has long been the dominant paradigm in financial theory, including portfolio allocation, corporate investment decisions, and fund manager performance evaluation. Mounting empirical research, however, reveals that in contrast to a long tradition of results, expected asset returns are not explained by their sensitivity to the market return only but are influenced by other factors as well.[1] This suggests that dynamic or multifactor models could perform better in practical applications.[2]

In the static CAPM, mean-variance efficiency of the market portfolio is a necessary and sufficient condition for an exact linear relationship to exist between the expected return on an asset and its risk measured by its beta. Roll's (1977) critique initiated a flood of research that mainly focused on developing econometric tests of the market portfolio mean-variance efficiency. Kandel and Stambaugh (1995) were the first authors to carefully distinguish the two theoretical implications of the model: market portfolio efficiency and the linear risk–return relationship. They showed that if the market portfolio is not exactly efficient, "either (implication) can hold nearly perfectly while the other fails grossly."[3] Our work does not pursue their line of research and, in particular, does not attempt to extend their result to the Merton–Breeden ICAPM, essentially because the main thrust of their result will obviously hold in a multiperiod context. On the contrary, we make no a priori assumption as to whether the market portfolio is approximately efficient or not.

In the dynamic ICAPM, some economic state variables are assumed to drive the investment opportunity set. In such a setting, investors' optimal demands for risky assets include, in addition to the usual mean-variance components, hedging components. Those hedging terms protect investors against unfavorable shifts in their opportunity set brought about by random changes in the state variables. The market portfolio then need not be mean-variance-efficient (e.g., Fama, 1996). Yet, a special case

[1]See Chen, Roll, and Ross (1986); Jegadeesh and Titman (1993); Fama and French (1996); or Cochrane (1997), among many others.

[2]Fama (1996) provided a thorough analysis of the relationship between ICAPM and multifactor efficiency.

[3]The quote is from their abstract (p. 157).

arises when investors are endowed with time-additive logarithmic utility functions. This case has been thoroughly examined in the literature and is widely used in theoretical finance because it represents a benchmark case for risk aversion.[4] Indeed, logarithmic (Bernoulli) investors are myopic in the sense that they do not hedge against the random fluctuations of their opportunity set. In that case, as in the static CAPM, the market portfolio must be mean-variance-efficient for the (simplified) ICAPM to hold. Our main contribution consists in deriving a new ICAPM in which, independently of whether investors are myopic or not, mean-variance efficiency of the market portfolio is neither a necessary nor a sufficient condition for a linear risk–return relationship.

The economy we consider is more general than that of the usual ICAPM. Although our financial market remains in general incomplete, investors can trade nonredundant futures contracts in addition to primitive risky assets and a riskless asset. Such futures contracts are written neither on financial assets nor on commodities but, for example, on nontradable variables such as the weather, natural catastrophes, the Consumer Price Index, and the Gross National Product. Financial markets and institutions have been under pressure recently to design such contracts (e.g., Athanasoulis, Shiller, & van Wincoop, 1999; Shiller, 1993; Sumner, 1997). Surprisingly, however, little attention has been paid to the pricing of such nonredundant contracts at equilibrium[5] or to the portfolio reallocation effects brought about by the trading of such instruments. We focus on futures contracts and neglect other derivatives for two reasons. First, the nontraded assets or economic aggregates (e.g., Consumer Price Index or Gross Domestic Product) under consideration are not sufficiently volatile to warrant the development of option markets. Second, we are interested in the implications of the marking-to-market mechanism that characterizes futures, as opposed to forward contracts.

We adopt a partial equilibrium, continuous-time framework a la Breeden (1984), itself a generalization of Merton (1973). Investors are endowed with general Von Neuman–Morgenstern utility functions, and the investment opportunity set is driven by an arbitrary number ($K$) of state variables. Investors have access to a riskless asset, risky cash assets, and futures contracts. In Breeden's model, however, the number of futures contracts is equal to the number $K$ of state variables, which thus makes his financial market complete.[6] In addition, each and every futures

[4]See Merton (1971, 1973); Rubinstein (1976); Breeden (1979); or Cox, Ingersoll, and Ross (1985).
[5]Cox, Ingersoll, and Ross (1981) provided a comprehensive study of the pricing of redundant forward and futures contracts. We develop here a pricing model for nonredundant futures contracts.
[6]Breeden was the first author to introduce futures in an ICAPM. His main objective, however, was

contract is of instantaneous maturity and perfectly correlated with each and every state variable. In contrast, our futures are long lived and not perfectly correlated with the state variables, and the financial market is in general incomplete. The difference proves crucial in that, for instance, futures are held by Breeden's investors as perfect hedges against the fluctuations of the corresponding state variables, whereas they are demanded by our investors for both (imperfect) hedging and speculative purposes.

In such a framework, two interesting questions arise. The first concerns portfolio separation. Our first contribution is to show that a mutual-fund separation result will obtain that involves $(3 + K)$ funds, in lieu of the usual $(2 + K)$ funds. The second issue is whether the market portfolio is necessarily mean-variance-efficient in the modified ICAPM with nonredundant long-lived futures. Our second and main contribution is to prove that this condition is in fact neither necessary nor sufficient for the ICAPM to hold.

Although the martingale approach[7] has become the standard method used to solve a portfolio choice problem in continuous time, it needs major adaptations when futures contracts are included in the investment opportunity set. This leads to heavy notations, in particular if one wants to develop the analysis within an incomplete market framework, as here. Consequently, we use the more traditional stochastic dynamic programming approach pioneered by Merton (1971), as it is easier to implement and enhances the intuition of the results.

The remainder of the article is organized as follows. In the Representative Investor's Wealth Dynamics section, we derive the representative investor's wealth dynamics in an (generally) incomplete market with nonredundant futures contracts. The Optimal Demands section provides the optimal demands for all risky assets and establishes a novel mutual-fund separation theorem. The Market Equilibrium section derives the ICAPM that applies to the primitive (cash) assets and the nonredundant futures. The last section is the conclusion. Most technical derivations and proofs are left to the Appendix.

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very different from ours. He was primarily interested in examining under what conditions an initially incomplete market can be completed through the introduction of instantaneous futures contracts (or commodity options in a separate section) and in characterizing the (Pareto) optimal allocation that ensues. In addition, his was not a single-good but a more general multigood economy, which raised specific questions irrelevant here.

[7]See the seminal articles by Cox and Huang (1989) and He and Pearson (1991) and, for an excellent introduction, the textbooks by Duffie (1996) and Pliska (1997).

## REPRESENTATIVE INVESTOR'S WEALTH DYNAMICS

In this section, the pure exchange economy we consider is described. Of particular interest is the derivation of the representative investor's margin account generated by trading futures contracts. The investor's wealth dynamics is then provided.

### Economy

$N$ primitive (cash) risky assets are traded in the economy. The price $S_i(t)$ of risky asset $i$ ($i = 1 \ldots N$) evolves through time according to the following stochastic differential equation (SDE):

$$dS_i(t) = S_i(t)\mu_{S_i}[t,Y(t)]dt + S_i(t)\sigma_{S_i}[t,Y(t)]'\,dZ(t) \qquad (1)$$

where $Z(t)$ is an $[(N + K) \times 1]$-dimensional Wiener process in $R^{N+K}$, $Y(t)$ is a $(K \times 1)$-dimensional vector of state variables, $\mu_{S_i}[t,Y(t)]$ is a bounded valued function of $t$ and $Y$, $\sigma_{S_i}[t,Y(t)]$ is a bounded $[(K + N) \times 1]$ vector valued function of $t$ and $Y$, and the prime $'$ denotes a transpose. The Wiener process is defined on the usual complete probability space $(\Omega,F,P)$, where $P$ is the physical probability measure.

The dynamics of the $K$ state variables is determined by the following system of SDEs:

$$dY(t) = \mu_Y[t,Y(t)]dt + \Sigma_Y[t,Y(t)]dZ(t) \qquad (2)$$

where $\mu_Y[t,Y(t)]$ is a bounded $(K \times 1)$ vector valued function of $t$ and $Y$ and $\Sigma_Y[t,Y(t)]$ is a bounded $[K \times (N + K)]$ matrix valued function of $t$ and $Y$.

For convenience, we often write the dynamics of the $N$ asset prices in the following compact form:

$$dS(t) = I_S\mu_S[t,Y(t)]dt + I_S\Sigma_S[t,Y(t)]dZ(t) \qquad (3)$$

where $I_S$ is an $(N \times N)$ diagonal matrix valued function of $S(t)$ whose $i$th diagonal element is $S_i(t)$, $\mu_S[t,Y(t)]$ is an $(N \times 1)$-dimensional vector whose $i$th component is $\mu_{S_i}[t,Y(t)]$, and $\Sigma_S[t,Y(t)]$ is an $[N \times (N + K)]$ matrix valued function whose $i$th element is $\sigma_{S_i}[t,Y(t)]$.

Nonredundant futures are also available for trade in this economy. These are long-lived assets; that is, they do not have instantaneous maturity. They are written on nontradable but observable variables that are variously affected by the state variables, such as Gross National Product and the like. The net supply of these instruments is zero. We assume, without loss of generality, that the maturity dates of all futures contracts extend beyond the representative investor's horizon date. The settlement price of the $j$th futures contract solves the following SDE:

$$dF_j(t) = F_j(t)\mu_{F_j}[t,Y(t)]dt + F_j(t)\sigma_{F_j}[t,Y(t)]'\ dZ(t) \tag{4}$$

where $\mu_{F_j}[t,Y(t)]$ is an $[(N + K) \times 1]$ bounded vector valued function of $t$ and $Y$, and $\sigma_{F_j}[t,Y(t)]$ is a bounded $[(N + K) \times 1]$ vector valued function of $t$ and $Y$.

We assume that there exist at most $H \leq K$ nonredundant futures contracts. Together with the primitive assets, they form the basis of the financial market. Assets in the basis have linearly independent cash flows. Therefore, in the extreme case where $H$ is equal to $K$, the financial market is complete in the sense of Harrison and Kreps (1979). In general, however, the market is incomplete ($H < K$). Regardless of whether the market is complete or not, futures are not redundant instruments, and their degree of correlation with the primitive assets is otherwise arbitrary. The vector process of the futures prices may be written as follows:

$$dF(t) = I_F\mu_F[t,Y(t)]dt + I_F\Sigma_F[t,Y(t)]dZ(t) \tag{5}$$

where $I_F$ is an $(H \times H)$ diagonal matrix valued function of $F(t)$ whose $j$th diagonal element is $F_j(t)$, $\mu_F[t,Y(t)]$ is an $(H \times 1)$-dimensional vector whose $j$th component is $\mu_{F_j}[t,Y(t)]$, and $\Sigma_F[t,Y(t)]$ is an $[H \times (N + K)]$ matrix valued function whose $j$th element is $\sigma_{F_j}[t,Y(t)]$.

The variance–covariance matrixes $\Sigma_S\Sigma_S'$ and $\Sigma_F\Sigma_F'$ and the variance–covariance matrix of all assets, that is, $\Sigma\Sigma'$ where $\Sigma' \equiv [\Sigma_S\ \Sigma_F]$, are assumed to be positive definite.

Finally, we turn to the investor's margin account generated by his or her trading on futures contracts. Margins can be lent or borrowed at an instantaneously riskless interest rate $r(t)$. The diffusion process followed by $r(t)$ is completely general and need not be made explicit. If futures are marked to market continuously rather than daily, the value of the margin account at time $t$ is written as follows:

$$X(t) = \int_0^t \exp\left[\int_s^t r(u)du\right]\Theta(s)'dF(s) \tag{6}$$

where $\Theta(t)$ is the $(H \times 1)$ vector of the number of futures contracts held (as opposed to traded) by the investor at time $t$.

We can now derive the dynamics of the representative investor's wealth.

## Representative Investor's Wealth

Let $\alpha$ be the $(N \times 1)$ vector of the proportions of the representative agent's wealth invested in the primitive assets, and let $C$ be his or her instantaneous consumption rate. Dropping for convenience the explicit time dependence of all processes, we write his or her budget constraint as follows:

$$dW = W\alpha' I_S^{-1} dS + dX - Cdt$$

Changes in wealth are due to consumption, fluctuations in cash asset prices, and variations in the margin account value. Using Equations 3, 5, and 6 and applying Itô's lemma yield the following wealth dynamics:

$$dW = [W\alpha'\mu_S + rX + \Theta' I_F \mu_F - C]dt + [W\alpha'\Sigma_S + \Theta' I_F \Sigma_F]dZ$$

Using the definition of wealth $W \equiv W\alpha' 1_N + X$ to eliminate $X$, we can rewrite the wealth dynamics as follows:

$$\begin{aligned} dW = {} & [W\alpha'(\mu_S - r1_N) + Wr + \Theta' I_F \mu_F - C]dt \\ & + [W\alpha'\Sigma_S + \Theta' I_F \Sigma_F]dZ \end{aligned}$$

To further simplify the notations, we denote by $\theta$ the $(H \times 1)$ vector of the ratios of the futures nominal positions (not their values) to the investor's wealth, that is, $\theta \equiv (1/W)\Theta' I_F$. Consequently, the investor's wealth dynamics is finally given by:

$$\begin{aligned} dW = {} & [W\alpha'(\mu_S - r1_N) + Wr + W\theta' \mu_F - C]dt \\ & + [W\alpha'\Sigma_S + W\theta'\Sigma_F]dZ \end{aligned} \tag{7}$$

Alternatively, Equation 7 could be expressed in a way that emphasizes the relative changes in wealth by using rates of return rather than returns and defining $c \equiv C/W$ as the percentage consumption of wealth:

$$\frac{dW}{W} = [\alpha'(\mu_S - r1_N) + r + \theta'\mu_F - c]dt + [\alpha'\Sigma_S + \theta'\Sigma_F]dZ$$

We turn now to the derivation of the representative individual's optimal demands. Market equilibrium is characterized in the Market Equilibrium section.

## OPTIMAL DEMANDS

We derive first the optimal demands for all risky assets and then a new mutual-fund separation theorem.

### Demand for Risky Assets

The representative investor is endowed with a Von Neuman–Morgenstern utility function and maximizes the expected utility of his consumption flow under a budget constraint; that is, he or she solves

$$E_t\left[\int_t^{\tau} U[s,C(s)]ds\right] \tag{8}$$

subject to equation (7)
and to positive consumption P-a.s.

$U$ is a well-behaved utility function, and $\tau$ ($\geq t$) stands for the investor's horizon. Let $J[t,W(t),Y(t)]$ be his indirect utility of wealth function defined by $J(\cdot) \equiv \max E_t\ [\int_t^{\tau} U[s,C(s)]ds]$. Standard computations shown in the Appendix lead to the following.

*Proposition 1*

Under our set of assumptions, the representative investor's optimal demand for risky assets is expressed as follows:

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\theta}\end{pmatrix} = (\Sigma\Sigma')^{-1}\begin{pmatrix}\mu_S - r1_N\\ \mu_F\end{pmatrix}\left(-\frac{J_W}{WJ_{WW}}\right) + (\Sigma\Sigma')^{-1}\Sigma\Sigma_Y'\left(-\frac{J_{WY}}{WJ_{WW}}\right) \tag{9}$$

The demand for (both primitive and derivative) risky assets thus retains its two familiar components. The first term on the right hand side (RHS) of Equation 9 is the mean-variance *speculative component*, and the second term is the dynamic Merton–Breeden *hedging component*. The economic interpretation of these two terms is provided later. However, although Equation 9 resembles the framework in Breeden (1984),[8] it is actually different in two essential ways.[9] First, in our framework the demands for both cash assets ($\hat{\alpha}$) and futures ($\hat{\theta}$) involve a speculative term

[8]Actually, Breeden essentially discussed two cases, one of incomplete markets without futures and one of complete markets where each and every futures contract of instantaneous maturity is perfectly correlated with each and every state variable. We compare our results to the ones obtained in the second case because we do have futures contracts.

[9]See his Equation 5 relative to the demand for primitive assets combined with his Equation 6 relative to that for instantaneous futures, or his Equation 24.

and a hedging term, whereas in Breeden the demand for futures is governed by hedging motives only and the demand for cash assets is governed by speculative motives only. Second, both the speculative and hedging components depend on the full $[(N + H) \times (N + H)]$ covariance matrix $\Sigma\Sigma'$ of all risky assets. In contrast, Breeden's speculative term depends on the $[N \times N]$ covariance matrix $\Sigma_S\Sigma_S'$ of primitive assets only, and the hedging term depends on the $[N \times K]$ matrix $\Sigma_S\Sigma_Y'$ of covariances of primitive assets with the state variables only.

The fundamental reason for this striking difference is that in Breeden's framework, where the market is completed by instantaneous maturity futures contracts, each and every contract is assumed to be perfectly correlated with each and every state variable. The investor's first–best optimum strategy comprises a perfect hedge against unfavorable fluctuations of each state variable, and this perfect hedge is easily obtained by one trading only futures. In contrast, in our more general (in this respect) setting futures contracts have finite rather than infinitesimal maturities, and their variance–covariance matrix is completely arbitrary. In particular, no futures contract has to be perfectly correlated with a state variable. This is true even though a given futures contract may happen to be more positively correlated with a particular state variable than any other primitive asset, as is likely. In particular, if the market is incomplete, perfect hedges are not feasible, and the investor's first–best optimum cannot be reached. Therefore, except for the excess return terms present in the mean-variance speculative component of Equation 9, $(\mu_S - r1_N)$ and $\mu_F$, respectively, the optimal demands for primitive and derivative instruments are formally identical and depend in the same way on the whole structure of correlations. This is true whether the market is incomplete ($H < K$) or just complete ($H = K$). Now, one may wonder what would happen to our investor's optimal demands if instantaneous futures perfectly correlated with the state variables were introduced. Then, the demand for cash assets would include the speculative part only, as in Breeden's model, but, unlike it, the demand for futures would comprise both a (perfect) hedge term and a speculative one.

For expository reasons, we analyze the second term on the RHS of Equation 9 first. The latter is composed of information-based elements whose purpose is to hedge wealth against unfavorable changes in the $K$ economic state variables. The rational investor constructs his portfolio comprising all available risky assets so as to protect himself against decreases in wealth due to shifts in the investment opportunity set brought about by these changes. This component, present in both demands for primitive and derivative assets, is preference-dependent because $J_{WY}$ (≡

$\partial J_W/\partial Y$) is the cross-partial derivative that represents the effect of the state variable $Y$ on the investor's wealth marginal value. The terms $-J_{WY}/WJ_{WW}$ reflect that dependence, each one of them being interpreted as a coefficient of relative risk tolerance with respect to the relevant state variable along the wealth's optimal path. In the particular case of logarithmic utility, the (Bernoulli) investor is myopic to changes in his or her investment opportunity set (all cross-partial derivatives $J_{WY}$ vanish), and this Merton–Breeden hedge component then disappears.

The first term on the RHS of Equation 9 is the speculative component, the product of the investor's relative risk-tolerance coefficient $(-J_W/WJ_{WW})$ by a mean-variance term. It has been interpreted since Merton (1973) as the demand for risky assets by a single-period mean-variance utility maximizer or, alternatively, by a multiperiod logarithmic investor. It is dubbed speculative in the sense that an individual exhibiting infinite risk aversion would set it equal to zero. To gain further insight and make comparisons with known results, we rewrite this component as follows (neglecting the risk-tolerance coefficient):

$$(\Sigma\Sigma')^{-1}\begin{pmatrix}\mu_S - r1_N\\ \mu_F\end{pmatrix} = \begin{bmatrix} A & -(\Sigma_S\Sigma_S')^{-1}\,\Sigma_S\Sigma_F'B \\ -(\Sigma_F\Sigma_F')^{-1}\,\Sigma_F\Sigma_S'A & B\end{bmatrix}\begin{bmatrix}\lambda_S\\ \lambda_F\end{bmatrix} \tag{10}$$

where

$$\lambda_S \equiv (\Sigma_S\Sigma_S')^{-1}\,(\mu_S - 1_N r)$$

$$\lambda_F = (\Sigma_F\Sigma_F')^{-1}\mu_F$$

and

$$A = [\Sigma_S\Sigma_S' - \Sigma_S\Sigma_F'(\Sigma_F\Sigma_F')^{-1}\,\Sigma_F\Sigma_S']^{-1}\,(\Sigma_S\Sigma_S')$$

$$B = [\Sigma_F\Sigma_F' - \Sigma_F\Sigma_S'(\Sigma_S\Sigma_S')^{-1}\,\Sigma_S\Sigma_F']^{-1}\,(\Sigma_F\Sigma_F')$$

Two basic sets of market prices of risk thus are identified, $\lambda_S$ and $\lambda_F$. The first one is traditional and concerns cash assets only, whereas the second involves futures contracts only. The numerator of $\lambda_F$ is not in excess return form (the riskless interest rate $r$ is not subtracted from $\mu_F$) because, unlike cash assets, there is no actual investment with futures contracts; that is, no cost of carry is involved.

Given Equation 10, we can rewrite the speculative part of Equation 9 as follows:

$$\begin{pmatrix} \hat{\alpha}_{\mathrm{Spec}} \\ \hat{\theta}_{\mathrm{Spec}} \end{pmatrix} \equiv \left(-\frac{\mathrm{J_W}}{\mathrm{WJ_{WW}}}\right) \begin{bmatrix} \mathrm{A}\lambda_{\mathrm{S}} - (\Sigma_{\mathrm{S}}\Sigma_{\mathrm{S}}')^{-1}\,\Sigma_{\mathrm{S}}\Sigma_{\mathrm{F}}'\mathrm{B}\lambda_{\mathrm{F}} \\ \mathrm{B}\lambda_{\mathrm{F}} - (\Sigma_{\mathrm{F}}\Sigma_{\mathrm{F}}')^{-1}\,\Sigma_{\mathrm{F}}\Sigma_{\mathrm{S}}'\,\mathrm{A}\lambda_{\mathrm{S}} \end{bmatrix} \tag{11}$$

Consider the two portfolios $A\lambda_S$ and $B\lambda_F$. It is useful to examine first the particular case in which primitive assets and futures contracts are not correlated $[\Sigma_S\Sigma_F' = 0_{N\times H}]$. In that case, $A$ and $B$ are identity matrices, so Equation 11 greatly simplifies:

$$\begin{pmatrix} \hat{\alpha}_{\mathrm{spec}} \\ \hat{\theta}_{\mathrm{spec}} \end{pmatrix} = \left(-\frac{J_W}{WJ_{WW}}\right) \begin{bmatrix} \lambda_S \\ \lambda_F \end{bmatrix}$$

The speculative part of the investor's portfolio then consists of two basic, mean-variance-efficient, well-separated portfolios: the first one is the traditional efficient portfolio containing primitive assets only, and the second is the efficient portfolio containing futures contracts only. Note, however, that we are neither in Merton's (1973) world nor in Breeden's (1984), where the speculative part of the investor's portfolio consists of our first (mean-variance-efficient) portfolio of cash assets only. Again, the difference is due to the spanning property of all the traded assets in our setting.

More generally, however, in the presence of (imperfect) correlation between primitive assets and futures contracts $[\Sigma_S\Sigma_F' \neq 0_{N\times H}]$, the speculative part of the demands for risky assets is given by Equation 11 where matrices $A$ and $B$ are no longer identity matrices but are completely general. It immediately follows that neither $A\lambda_S$ nor $B\lambda_F$ are mean-variance-efficient. Furthermore, in the demand for cash assets, an extra term is subtracted from $A\lambda_S$ that depends on $B\lambda_F$ and thus accounts for cross-hedging (with futures) effects. Similarly, in the demand for futures, an extra term is subtracted from $\mathrm{B}\lambda_F$ that depends on $\mathrm{A}\lambda_S$ and takes cross-hedging (with primitives) into account. Consequently, the two portfolios $\alpha_{\mathrm{spec}}$ and $\theta_{\mathrm{spec}}$ are severely perturbed and lose their separation property. They both depend on the whole structure of correlation between primitive and derivative assets and on the expected excess returns on all traded assets, and both are mean-variance-inefficient.

## Mutual-Fund Separation Theorem

This framework involving nonredundant futures contracts generates an interesting mutual-fund separation theorem. Merton's (1973) traditional separation result states that, at equilibrium, all investors divide their investment among the riskless asset, the market portfolio, and the $K$ funds that are best correlated with the $K$ state variables. Therefore, a $(K + 2)$ fund separation is obtained. Here we can state the following.

*Proposition 2*

Under our set of assumptions, when nonredundant futures contracts are traded, investors are indifferent to the reduction of their investment opportunity set from the riskless asset, $N$ primitive assets, and $H$ futures contracts to $(K + 3)$ mutual funds.

One fund is added to Merton's list. It contains futures contracts only and constitutes the second part of the speculative component of the investor's demand for risky assets. To characterize the latter fund in a simple way, we restrict ourselves to the case of Bernoulli (myopic) investors. Extension to other types of utility functions is straightforward.

Because $J_{WY}$ is equal to zero and the relative risk-tolerance coefficient is equal to one for a Bernoulli investor, the demand for risky assets (Equation 9) simplifies to

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \end{pmatrix} = \begin{bmatrix} A & -(\Sigma_S\Sigma_S')^{-1}\ \Sigma_S\Sigma_F'B \\ -(\Sigma_F\Sigma_F')^{-1}\ \Sigma_F\Sigma_S'A & B \end{bmatrix} \begin{bmatrix} \lambda_S \\ \lambda_F \end{bmatrix}$$

Summing over all investors yields

$$\begin{pmatrix} \alpha_M \\ 0_H \end{pmatrix} = \begin{bmatrix} A & -(\Sigma_S\Sigma_S')^{-1}\ \Sigma_S\Sigma_F'B \\ -(\Sigma_F\Sigma_F')^{-1}\ \Sigma_F\Sigma_S'A & B \end{bmatrix} \begin{bmatrix} \lambda_S \\ \lambda_F \end{bmatrix} \tag{12}$$

When $B$ is eliminated with the second row of Equation 12, the first row becomes

$$\alpha_M = (\mathrm{I}_N - (\Sigma_S\Sigma_S')^{-1}\ \Sigma_S\Sigma_F'(\Sigma_F\Sigma_F')^{-1}\ \Sigma_F\Sigma_S')A\lambda_S \tag{13}$$

where $\alpha_M$ is the vector of weights in the market portfolio.

Equation 13 clearly implies that the market portfolio differs from the usual one in a very important way. It is not proportional to a mean-variance-efficient portfolio of cash assets. It immediately follows that it need not be mean-variance-efficient. The only case in which the market portfolio is mean-variance-efficient is when cash asset prices and futures prices are not correlated. Now, recall that this result was derived for Bernoulli investors. When individuals are not myopic and hedge against random shifts in the state variables, it is already known that the market portfolio need not be efficient.[10] Our finding thus extends this result to even logarithmic investors who do not hedge against state variables. Therefore, in our economy and in contrast with previous work, the nonnecessarily efficient nature of the market portfolio is not due to Merton–Breeden hedging activity.

[10]See Fama (1996).

We can now restate the optimal demand for risky assets as follows:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \end{pmatrix} = \begin{bmatrix} I_N & 0 \\ -(\Sigma_F\Sigma_F')^{-1}\ \Sigma_F\Sigma_S' & B\ -\ (\Sigma_F\Sigma_F')^{-1}\ \Sigma_F\Sigma_S'(\Sigma_S\Sigma_S')^{-1}\ \Sigma_S\Sigma_F'B \end{bmatrix} \begin{bmatrix} \alpha_M \\ \lambda_F \end{bmatrix} \tag{14}$$

We find as usual that optimal investment in the risky cash assets is proportional to the market portfolio. However, in our framework it is not necessarily mean-variance-efficient. Furthermore, the portfolio of futures contracts in which part of the wealth is invested is not necessarily mean-variance-efficient either.

## MARKET EQUILIBRIUM

In this section, we derive the ICAPM for the cash assets and the nonredundant futures contracts, and then we interpret and discuss the results. The class of utility functions is no longer restricted to the logarithmic function.

To characterize market equilibrium, we impose that the usual market clearing conditions hold. Net positions in the futures contracts are set equal to zero ($\hat{\theta} = 0_H$) as well as the sum of all margin accounts. Total wealth thus is invested in the primitive assets only ($\hat{\alpha}' 1_N = 1$). Combining these conditions and the optimality conditions yields the following.

*Proposition 3*

Under our set of assumptions, at equilibrium the expected returns on cash assets satisfy the following relationship:

$$\mu_{S_i} = r + \beta_{i,M}(\mu_M - r) + \sum_{k=1}^{K} \beta_{i,Y_k}\ (\mu_{Y_k} - r) \tag{15}$$

The expected *returns* on futures contracts satisfy[11]

$$\mu_{F_j} = \beta_{j,M}\ (\mu_M - r) + \sum_{k=1}^{K} \beta_{j,Y_k}\ (\mu_{Y_k} - r) \tag{16}$$

In Equations 15 and 16, $\beta_{l,U}$ stands for the sensitivity of the return on asset $l$ to the excess return on the market portfolio ($U = M$) or to the

[11]The word *return* is italicized because there is, of course, no actual investment in futures contracts, either bought or sold.

(adjusted for the riskless rate) drift of a state variable process ($U = Y_k$). The proof is left to the Appendix.

The excess return on a cash asset is expressed in the usual ICAPM form. This result is striking because we have found the market portfolio to be not necessarily efficient. In a dynamic model where the risk generated by the state variables is priced by the market, a multibeta ICAPM is obtained whether the market portfolio is efficient or not. Moreover, we have carried this result over to Bernoulli investors who do not price the risks associated with the state variables. In short, the market portfolio may not be mean-variance-efficient because of the presence of nonredundant futures contracts in investors' portfolios, but the traditional ICAPM for cash assets still holds for myopic investors. As for the result of Equation 16, it does not really come as a surprise because the marking-to-market mechanism makes futures contracts similar to cash assets. Therefore, except for the obvious absence of the riskless rate of interest, the ICAPM must equally hold for futures.

## CONCLUDING REMARKS

When an incomplete financial market of cash assets is completed (partially or not) by the introduction of nonredundant futures contracts, mean-variance efficiency of the market portfolio is not required. This result holds regardless of whether individuals exhibit myopic behaviors or not. Because derivative assets have enriched the opportunity set of direct investors and fund managers alike, this finding is important for the practical aspects of asset allocation, risk control and management, and performance measurement. Moreover, this result may have important implications for empirically testing the ICAPM because the linear relationship between expected return and beta continues to hold for risky assets despite the possible inefficiency of the market portfolio.

Our results have been derived within a frictionless market framework. In particular, the optimal demands and equilibrium relationships are very tractable, so that the model should be applicable to the analysis of a variety of theoretical and empirical issues. The introduction of some market frictions such as transaction costs, portfolio composition restrictions, and default or counterparty risk may lead to a further understanding of the peculiarities of futures markets.

## APPENDIX

### Proof of Proposition 1

We follow the traditional stochastic dynamic programming popularized in the financial literature by Merton (1971). Given Equation 8 in the text, let $J[t,W(t),Y(t)]$ be the indirect utility of the wealth function (i.e., the value function):

$$J[t,W(t),Y(t)] \equiv \max E_t\left[\int_t^{\tau} U[s,C(s)]ds\right]$$

Let $L(t)$ be the differential generator of $J(.)$. We assume that $J(.)$ is an increasing and strictly concave function of $W$.[12]

If $\psi \equiv LJ + U$, the Hamilton–Jacobi–Bellman equation is

$$\begin{aligned}\text{Max } \psi\ (c,\ \alpha,\ \theta) = \text{Max}\Big[U &+ J_W\mu_W + J_Y\mu_Y + \frac{1}{2}J_{WW}\sigma_W'\sigma_W \\ &+ \frac{1}{2}J_{YY}\Sigma_Y\Sigma_Y' + J_{WY}\Sigma_Y\sigma_W\Big] = 0\end{aligned}$$

with obvious notations for partial derivatives of the value function $J(.)$ with respect to each of its arguments. Note that $\mu_W$ is the drift of the wealth process given by Equation 7 in the text.

The necessary and sufficient conditions for optimality are derived from this equation by its first derivatives being computed with respect to the control variables C, $\alpha$, and $\theta$:

$$\psi_C = U_C - J_W \leq 0 \tag{A1}$$

$$\hat{C}\psi_C = 0 \tag{A2}$$

$$\begin{aligned}\psi_\alpha = (\mu_S - r1_N)WJ_W &+ \Sigma_S\Sigma_Y'WJ_{WY} \\ &+ [\Sigma_S\Sigma_S'\hat{\alpha} + \Sigma_S\Sigma_F'\hat{\theta}]W^2J_{WW} = 0\end{aligned} \tag{A3}$$

$$\psi_\theta = \mu_F WJ_W + \Sigma_F\Sigma_Y'WJ_{WY} + [\Sigma_F\Sigma_S'\hat{\alpha} + \Sigma_F\Sigma_F'\hat{\theta}]W^2J_{WW} = 0 \tag{A4}$$

^ above a variable denotes an optimal value. The first two (Kuhn and Tucker) equations concern consumption and follow from the nonnegativity constraint imposed on the optimal consumption path.

Combining Equations A3 and A4 leads to Equation 9 in the text.

## Proof of Proposition 3

Using A3 and A4, one can easily show that at equilibrium

$$\begin{pmatrix}\mu_S - 1_N r \\ \mu_F\end{pmatrix} = V_{SF,W}\left(-\frac{WJ_{WW}}{J_W}\right) + V_{SF,Y}\left(-\frac{J_{WY}}{J_W}\right) \tag{A5}$$

where

[12]Merton (1973) and Cox et al. (1985) provided necessary conditions for $J$ to possess these properties.

$$V_{SF,W} = \begin{pmatrix} \Sigma_S \Sigma_S' \hat{\alpha} \\ \Sigma_F \Sigma_S' \hat{\alpha} \end{pmatrix}$$

and

$$V_{SF,Y} = \begin{pmatrix} \Sigma_S \Sigma_Y' \\ \Sigma_F \Sigma_Y' \end{pmatrix}$$

Premultiplying A5 by $(\hat{\alpha}' \ 0_H')$yields the equilibrium risk–return trade-off for total wealth (equal to the market portfolio value):

$$\mu_W - r = V_{W,W} \left( \frac{-WJ_{WW}}{J_W} \right) + V_{W,Y}' \left( -\frac{J_{WY}}{J_W} \right) \tag{A6}$$

$\mu_W$ is the expected return on total wealth, $V_{W,W}$ is the variance of total wealth, and $V_{W,}Y$ is the $(K \times 1)$ vector of covariances between total wealth and the state variables.

Also, premultiplying A5 by $\Sigma_Y(\Sigma'\Sigma)^{-1}\Sigma'$ gives

$$\begin{aligned} \mu_Y - 1_H r &\equiv \Sigma_Y(\Sigma'\Sigma)^{-1}\Sigma' \begin{pmatrix} \mu_S - 1_N r \\ \mu_F \end{pmatrix} \\ &= V_{Y,W} \left( -\frac{WJ_{WW}}{J_W} \right) + V_{Y,Y} \left( -\frac{J_{WY}}{J_W} \right) \end{aligned} \tag{A7}$$

where $V_{Y,W}$ $(=V_{W,Y})$ is a $(K \times 1)$ covariance vector and $V_{Y,Y}$ is a $(K \times K)$ variance–covariance matrix.

Combining A6 and A7 yields

$$\begin{pmatrix} \mu_W - r \\ \mu_Y - 1_H r \end{pmatrix} = \begin{pmatrix} V_{W,W} & V_{W,Y}' \\ V_{Y,W} & V_{Y,Y} \end{pmatrix} \begin{pmatrix} -\dfrac{WJ_{WW}}{J_W} \\ -\dfrac{J_{WY}}{J_W} \end{pmatrix}$$

Defining

$$\Sigma_{WY} \equiv \begin{pmatrix} V_{W,W} & V_{W,Y} \\ V_{Y,W} & V_{Y,Y} \end{pmatrix}$$

$$\begin{pmatrix} -\dfrac{WJ_{WW}}{J_W} \\ -\dfrac{J_{WY}}{J_W} \end{pmatrix} = (\Sigma_{WY})^{-1} \begin{pmatrix} \mu_W - r \\ \mu_Y - 1_H r \end{pmatrix} \tag{A8}$$

Substituting A8 into A5 and defining $V_{SF,WY} \equiv (V_{SF,W}\, V_{SF,Y})$ yield

$$\begin{pmatrix} \mu_s - 1_N r \\ \mu_F \end{pmatrix} = V_{SF,WY}(V_{WY})^{-1} \begin{pmatrix} \mu_W - r \\ \mu_Y - 1_H r \end{pmatrix} \tag{A9}$$

The results of Equations 15 and 16 in the text follow from A9 because the value of primitive assets represents total wealth at equilibrium so that $W$ is the value of the market portfolio $M$.

## BIBLIOGRAPHY

Athanasoulis, S., Shiller, R., & van Wincoop, E. (1999). Macro markets and financial security. FRBNY Economic Policy Review, 5, 21–39.

Breeden, D. T. (1979). An intertemporal asset pricing model with stochastic consumption and investment opportunities. Journal of Financial Economics, 7, 265–296.

Breeden, D. T. (1984). Futures markets and commodity options hedging and optimality in incomplete markets. Journal of Economic Theory, 32, 275–300.

Chen, N. F., Roll, R., & Ross, S. (1986). Economic forces and the stock market. Journal of Business, 59, 383–403.

Cochrane, J. H. (1997). Where is the market going? Uncertain facts and novel theories. Economic Perspectives, XXI, 6.

Cox, J., & Huang, C. F. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory, 49, 33–83.

Cox, J., Ingersoll, J., & Ross, S. (1981). The relation between futures and futures prices. Journal of Financial Economics, 9, 321–346.

Cox, J., Ingersoll, J., & Ross, S. (1985). An intertemporal general equilibrium model of asset prices. Econometrica, 51, 363–383.

Duffie, D. (1996). Dynamic asset pricing theory. Princeton, NJ: Princeton University Press.

Fama, E. (1996). Multifactor portfolio efficiency and multifactor asset pricing. Journal of Financial and Quantitative Analysis, 31, 441–465.

Fama, E., & French, K. (1996). Multifactor explanations of asset pricing anomalies. Journal of Finance, 51, 55–84.

Harrison, M., & Kreps, D. (1979). Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, 20, 381–408.

He, H., & N. Pearson (1991). Consumption and portfolio policies with incomplete markets and short sale constraints. Journal of Economic Theory, 54, 259–304.

Jegadeesh, N., & Titman, S. (1993). Returns to buying winners and selling losers: Implications for stock market efficiency. Journal of Finance, 48, 65–91.

Kandel, S., & Stambaugh, R. (1995). Portfolio inefficiency and the cross-section of expected returns. Journal of Finance, 50, 157–184.

Lintner, J. (1965). The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. Review of Economics and Statistics, 47, 13–37.

Merton, R. C. (1971). Optimum portfolio and consumption rules in a continuous time model. Journal of Economic Theory, 3, 373–413.

Merton, R. C. (1973). An intertemporal capital asset pricing model. Econometrica, 41, 867–887.

Pliska, S. (1997). Introduction to mathematical finance. Oxford and Cambridge: Blackwell.

Roll, R. (1977). A critique of the asset pricing theory's tests: Part I. On past and potential testability of the theory. Journal of Financial Economics, 4, 129–176.

Rubinstein, M. (1976). The strong case for the generalized logarithmic utility model as the premier model of financial markets. Journal of Finance, 31, 551–571.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. Journal of Finance, 19, 425–442.

Shiller, R. (1993). Macro markets: Creating institutions for managing society's largest economic risks [Clarendon Lectures in Economics]. Oxford: Clarendon.

Sumner, S. (1997). Can monetary stabilization policy be improved by CPI futures targeting? Journal of Money, Credit, and Banking, 29, 542–545.