
OPTION PRICING BASED ON THE GENERALIZED LAMBDA DISTRIBUTION

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This article proposes the generalized lambda distribution as a tool for modeling nonlognormal security price distributions. Known best as a facile model for generating random variables with a broad range of skewness and kurtosis values, the generalized lambda distribution has potential financial applications, including Monte Carlo simulations, estimations of option-implied state price densities, and almost any situation requiring a flexible density shape. A multivariate version of the generalized lambda distribution is developed to facilitate stochastic modeling of portfolios of correlated primary and derivative securities. © 2001 John Wiley & Sons, Inc. *Jrl Fut Mark* 21:213–236, 2001

INTRODUCTION

Option pricing theory has advanced along many fronts since the pioneering breakthroughs by Black and Scholes (1973) and Merton (1973). At the forefront of these advances has been the development of option pricing solutions within the original Black–Scholes–Merton framework, in which security return volatility is a deterministic function of the un-

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derlying security price and time, that is, $\sigma = \sigma(S, t)$. Within this classic framework, options can be used to form a dynamic, riskless hedge with the underlying security. An absence of arbitrage opportunities in financial markets implies that the resulting option prices are independent of investor risk preferences (Cox & Ross, 1976; Harrison & Kreps, 1979).

The deterministic volatility approach to option valuation has been productive. The famous Black–Scholes (Black & Scholes, 1973) formula represents the special case of a constant volatility diffusion process. Merton (1976) showed that substituting average volatility into the Black–Scholes formula yields the solution where volatility is solely a function of time. Cox and Ross (1976) derived an option pricing formula for the case of a constant elasticity relationship between volatility and the underlying security price. These are well-known solutions. The actual number of known solutions is too large to review in even comprehensive texts. Although large, this number is a fraction of the plausible forms for the deterministic volatility function $\sigma = \sigma(S, t)$, for which no general solution exists.

Flanking the traditional approach of developing specific solutions from exact parametric assumptions, several researchers have recognized the need for general solutions based on limited assumptions. These solutions are intended to apply across varied stochastic process specifications within the Black–Scholes–Merton framework, yet still provide sufficient accuracy to be useful to market participants. In an early effort, Jarrow and Rudd (1982) developed an option pricing formula that supplements the Black–Scholes formula with Edgeworth expansion terms related to the skewness and kurtosis of the underlying security price distribution. More recently, Rubinstein (1994) and Jackwerth and Rubinstein (1996) implemented a nonparametric method to fit a security price distribution to observed option price data. In concert, Buchen and Kelley (1996) and Stutzer (1996) suggested methods of fitting a maximum entropy distribution to historical security returns data and currently observed option prices. Still another approach, well exemplified by Ait-Sahalia and Lo (1998), makes use of growing computing power to extract a price structure consistent with market data using kernel regression techniques.

Broadly described, a parametric option pricing model specifies in advance a stochastic process for a security price from which a future security price distribution can be derived. Evaluating this distribution on one side of a strike price yields the expected payoff to a European option. This approach dates at least to Bachelier (1964) and has advanced to provide a rich inventory of models with varied stochastic process as-

sumptions. Most recent contributions focus on stochastic volatility and interest rate assumptions; a partial list includes Amin and Jarrow (1992), Amin and Ng (1993), Bailey and Stulz (1989), Bakshi and Chen (1997a, 1997b), Bates (1996), Heston (1993), Hull and White (1987), Scott (1997), and Stein and Stein (1991). Bakshi, Cao, and Chen (1997) provided extensive empirical tests based on an extended formula that nests varied models as special cases. They concluded that each nested model is to some degree misspecified by the criterion of internal parameter consistency, but each does in fact yield improved performance relative to a constant interest rate and volatility specification.

In contrast, nonparametric models typically begin with a future security price distribution, often estimated from observed market data. It is well known that a theoretically justified estimate of a future security price distribution, or at least its risk-neutral cousin, can be obtained from European options on the underlying security (Breedon & Litzenberger, 1978; Ross, 1976). Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) took this approach a step further to infer a stochastic process for a security price from its estimated price distribution. In particular, Rubinstein (1994) developed an implied binomial tree algorithm that yields a deterministic-volatility stochastic process from almost any plausible security price distribution.

The aforementioned nonparametric methods share the practical advantage of being data driven, with a minimal reliance on specific parametric assumptions. Notwithstanding, each approach has specific advantages and trade-offs in actual applications. The Jarrow–Rudd method is one of the simplest to implement because it adds only two parameters to the Black–Scholes formula. Moreover, these two parameters are straightforward measures describing the skewness and kurtosis of the underlying security price distribution. Similar simplicity is achieved by the entrenched street method of fitting a volatility smile to option price data. Dumas, Fleming, and Whaley (1998) concluded that fitting a quadratic function to a volatility smile works as well in practical terms as more complex specifications. A quadratic implied volatility function also adds only two parameters to the Black–Scholes formula, in this case describing the shape of the volatility smile. In contrast, other methods cited earlier are parameter-intensive, typically requiring a lengthy vector of parameters that individually provide an incremental description of the underlying price distribution. Nevertheless, parameter intensity is hardly a problem in actual implementations because modern computing power is fully adequate to the task. Rather, the issue of parameter intensity pivots on the

TABLE I
Standard and Poor's 500 Index Call Options Data

<i>Strikes</i>	<i>Call Option Prices</i>		<i>Bid-Ask Price Averages</i>	<i>Butterfly Prices</i>
	<i>Bid</i>	<i>Ask</i>		
0	349.16	349.26	349.210	
250	109.47	109.71	109.590	
275	85.66	86.71	86.185	
300	63.00	64.04	63.520	
325	42.00	42.75	42.375	
330	37.97	38.60	38.285	.130
335	34.01	34.64	34.325	.050
340	30.10	30.73	30.415	.285
345	26.45	27.13	26.790	-.030
350	22.79	23.48	23.135	.120
355	19.32	19.88	19.600	.195
360	15.98	16.54	16.260	.520
365	13.13	13.75	13.440	.215
370	10.52	11.15	10.835	.515
375	8.37	9.12	8.745	.370
380	6.65	7.40	7.025	-.050
385	4.91	5.60	5.255	

Note: Strike, bid, and ask prices are reproduced from Figure 4 in Rubinstein (1994). A butterfly price represents a long butterfly position created by buying a call for each of two outer strikes and writing two calls for the inner strike. Butterfly prices are based on bid-ask price averages and are listed only in cases where adjacent strikes are \$5 apart.

practical trade-off between underfitting a model to data with too few parameters and overfitting with too many.

Table I reproduces a data set originally presented in Rubinstein (1994) comprising 16 Standard and Poor's (S&P) 500 index call option prices and the underlying dividend-adjusted index level. This is an interesting data set because it reveals some nettlesome characteristics of observed option price data. In Table I, the first column lists strike prices, whereas the second and third columns list the corresponding bid and ask call option prices for each strike. The next column reports bid-ask price averages. The last column presents prices of butterfly positions centered on three adjacent strike prices, reported only in instances where adjacent strikes are \$5 apart. These butterfly prices are calculated with bid-ask average prices. Each butterfly price represents a long butterfly position created by buying a call for each of the two outer strikes and writing two calls for the inner strike.

As shown in Table I, butterfly positions centered on 345 and 380 strikes have negative prices. These indicate points of nonconvexity in the observed relationship between option prices and strike prices. Such non-

convexities are common in archival option price data but do not necessarily represent arbitrage violations because they may not reflect the full cost of trading at bid and ask prices.¹ Nonetheless, bid–ask average prices are standard inputs for estimating an implied distribution of future security prices from option price data (Dumas et al., 1998; Jackwerth & Rubinstein, 1996). Consequently, an estimation method that allows overfitting an implied distribution function to option price data might yield unduly small or possibly even negative density values over strikes with negative observed butterfly prices. Indeed, Jackwerth and Rubinstein reported that much of the difficulty encountered in estimating an implied price distribution from option price data stems from the trade-off between overfitting, which tends to produce jagged density estimates, and smoothing procedures designed to avoid overfitting. The jaggedness of the butterfly prices reported in Table I, typical in observed option price data, suggests that this not a trivial task.

This article proposes the generalized lambda distribution (GLD) as a flexible tool for modeling nonlognormal security price distributions. Originally developed as a facile method for generating random variables in Monte Carlo simulations, the GLD was later suggested by Ramberg, Dudewicz, Tadikamalla, and Mykytka (1979) as a supple model for fitting a distribution to data because it permits a broad range of skewness and kurtosis values. Applied to the problem of fitting an implied security price distribution to option price data, it combines elements of the simplicity of Jarrow and Rudd (1982) and the versatility of Buchen and Kelley (1996), Stutzer (1996), and Jackwerth and Rubinstein (1996). Well known among statisticians, the GLD has not yet been advanced as an empirical tool in the finance literature.

The remainder of the article is organized into six sections. The next section describes the GLD and highlights its more salient details. The section thereafter derives option pricing formulas based on the GLD. These formulas are applied in the following section to the problem of recovering a probability distribution implied by option prices. A procedure for exactly calibrating an implied binomial tree to the GLD comes next. Then, the following section develops a multivariate version of the GLD. Finally, there are a summary and a conclusion, in which the major advantages of the GLD in financial applications are reviewed.

¹Other empirical anomalies in option prices have been documented. For example, Bakshi, Cao, and Chen (2000) showed that option prices do not always move with underlying stock prices in the theoretically predicted direction. Bergman, Grundy, and Wiener (1996) provided a general discussion of rational option pricing.

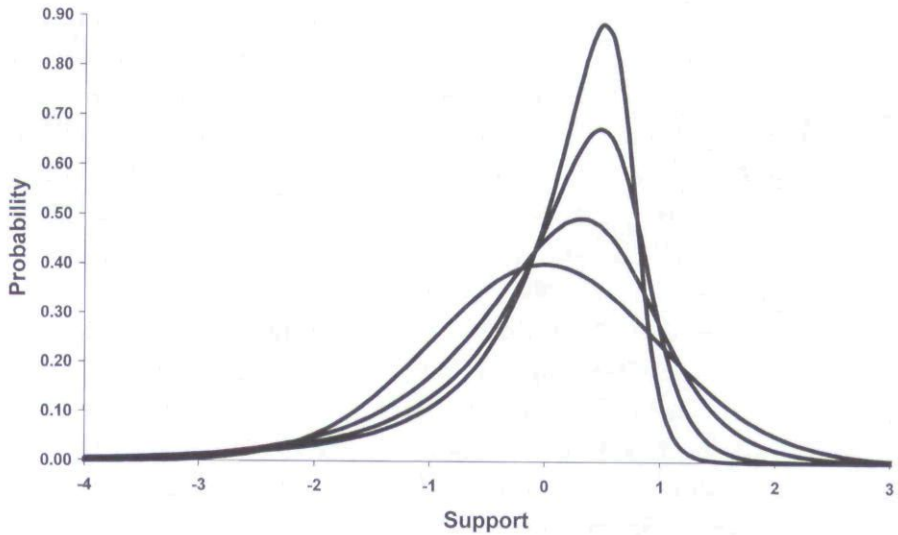


FIGURE 1
Generalized lambda density functions with unit variances.

GLD

The GLD is primarily defined by its percentile function. For this reason, it is frequently applied to algorithms generating random variables in Monte Carlo simulations. The percentile function $s(p)$ defines a GLD with a zero mean and unit variance. The parameters λ_3 and λ_4 determine the shape of the generalized lambda density, and p represents a probability measure:

$$s(p) = \frac{1}{\lambda_2(\lambda_3, \lambda_4)} \left(p^{\lambda_3} - (1 - p)^{\lambda_4} + \frac{1}{\lambda_4 + 1} - \frac{1}{\lambda_3 + 1} \right) \quad (1)$$

The scaling function $\lambda_2 = \lambda_2(\lambda_3, \lambda_4)$, specified in Equation 2, in which $\beta(x,y)$ denotes the complete beta function, assumes that the conditions $\lambda_3 \times \lambda_4 > 0$ and $\min(\lambda_3, \lambda_4) > -1/2$ are satisfied:

$$\begin{aligned} \lambda_2(\lambda_3, \lambda_4) &= \text{sign}(\lambda_3) \times \sqrt{B - A^2} \\ A &= \frac{1}{\lambda_3 + 1} - \frac{1}{\lambda_4 + 1} \\ B &= \frac{1}{2\lambda_3 + 1} + \frac{1}{2\lambda_4 + 1} - 2\beta(\lambda_3 + 1, \lambda_4 + 1) \end{aligned} \quad (2)$$

Ramberg et al. (1979) pointed out that the conditions $\lambda_3 \times \lambda_4 > 0$ and $\min(\lambda_3, \lambda_4) > -1/2$ suffice to ensure that $s(p)$ is a continuous, monoton-

ically increasing function of the probability measure p . In Monte Carlo simulations, p values drawn from a uniform random number generator enter the generalized lambda percentile function $s(p)$ to become random variables with population skewness and kurtosis determined by the shape parameters λ_3 and λ_4 .

Ramberg et al. (1979) showed that the existence of the n th moment of $s(p)$ requires that $\min(\lambda_3, \lambda_4) > -1/n$. Thus, the condition $\min(\lambda_3, \lambda_4) > -1/4$ ensures that fourth and lower moments exist. In this case, coefficients of skewness and kurtosis are well-defined functions of the shape parameters λ_3 and λ_4 . Ramberg et al. provided these expressions for coefficients of skewness and kurtosis, in which A and B are defined in Equation 2:

$$\begin{aligned}\text{Skewness} &= \frac{C - 3AB + 2A^3}{(B - A^2)^{3/2}} \\ \text{Kurtosis} &= \frac{D - 4AC + 6A^2B - 3A^4}{(B - A^2)^2} \\ C &= \frac{1}{3\lambda_3 + 1} - \frac{1}{3\lambda_4 + 1} + 3\beta(2\lambda_4 + 1, \lambda_3 + 1) \\ &\quad - 3\beta(2\lambda_3 + 1, \lambda_4 + 1) \\ D &= \frac{1}{4\lambda_3 + 1} + \frac{1}{4\lambda_4 + 1} - 4\beta(3\lambda_4 + 1, \lambda_3 + 1) \\ &\quad - 4\beta(3\lambda_3 + 1, \lambda_4 + 1) + 6\beta(2\lambda_3 + 1, 2\lambda_4 + 1) \quad (3)\end{aligned}$$

Inverting the first derivative of the percentile function $s(p)$ yields the generalized lambda density function dp/ds :

$$dp/ds = \lambda_2(\lambda_3, \lambda_4)/(\lambda_3 p^{\lambda_3-1} + \lambda_4(1-p)^{\lambda_4-1})$$

The condition $\lambda_3 \times \lambda_4 > 0$ along with the appropriate scaling function ensures that this density function is nonnegative everywhere on the probability measure p .

Figure 1 graphs four generalized lambda densities with zero means, unit variances, and increasing coefficients of kurtosis and negative skewness. The following panel states the skewness and kurtosis values for these densities, along with corresponding values for their shape parameters λ_3 and λ_4 . For any generalized lambda density, interchanging values for the parameters λ_3 and λ_4 reverses the direction of skewness. The equality $\lambda_3 = \lambda_4$ yields a symmetric density function with zero skewness:

Skewness = 0	Skewness = -1	Skewness = -3	Skewness = -6
Kurtosis = 3	Kurtosis = 6	Kurtosis = 30	Kurtosis = 2000
$\lambda_3 = .134912$	$\lambda_3 = .038800$	$\lambda_3 = -.174915$	$\lambda_3 = -.248020$
$\lambda_4 = .134912$	$\lambda_4 = .018738$	$\lambda_4 = -.049470$	$\lambda_4 = -.047853$

An analytic expression for the cumulative distribution function $p(s)$ of the GLD does not exist, except as the inverse of its percentile function $s(p)$. This is not a practical concern because the same is true of the normal distribution and many popular distributions for which numerical methods enable function evaluation. An iterative bisection algorithm is perhaps the simplest method to evaluate the GLD function $p(s)$. That is, to obtain $p(c)$, first compare the value of c with $s(.5)$ and next with $s(.75)$ or $s(.25)$ as $c > s(.5)$ or $c < s(.5)$, respectively. Iterate this bisection procedure until $|s(p) - c| < \epsilon$, with ϵ determining the desired level of precision. Fortran programs for all numerical procedures implemented in this article are available from the author on request.

The generalized lambda percentile function in Equation 1 may be located and scaled to any desired mean and variance without affecting the density shape. In a security market context, setting the mean to s_0e^{rt} and the variance to $(s_0e^{rt})^2(e^{\sigma^2t} - 1)$ yields the percentile function shown in Equation 4, in which s_0 is a current security price, r is an appropriate interest rate, t is the time to a future price realization, and σ^2 is a variance parameter:

$$s(p) = s_0e^{rt} \left(1 + \frac{\sqrt{e^{\sigma^2t} - 1}}{\lambda_2(\lambda_3, \lambda_4)} \left(p^{\lambda_3} - (1 - p)^{\lambda_4} + \frac{1}{\lambda_4 + 1} - \frac{1}{\lambda_3 + 1} \right) \right) \tag{4}$$

The mean and variance of $s(p)$ in Equation 4 have a form identical to that of the lognormal distribution adapted to a security market context (Jarrow & Rudd, 1982).

The percentile function in Equation 4 generates random security prices from distributions with arbitrary means, variances, and coefficients of skewness and kurtosis. Moreover, skewness and kurtosis values can be specified from a wide range of possible values.

There are several potentially interesting alternatives to the GLD that share some of its flexible shape characteristics. Classical examples are the Pearson family and the Johnson and Burr systems of distributions (Stuart & Ord, 1987). A well-known example from the finance literature is the generalized beta distribution of the second kind suggested by Bookstaber

and McDonald (1987). Among these alternatives, only the Burr system possesses a percentile function in closed-form similar to Equation 1 and is, therefore, well suited to Monte Carlo simulation procedures. However, the Burr system has a limited range of negative skewness values (Rodriguez, 1977; Tadikamalla, 1980), which may be an impediment in some applications.

Option pricing formulas consistent with the percentile function $s(p)$ in Equation 4 might usefully supplement financial Monte Carlo simulations and other procedures involving options by allowing arbitrary assumptions regarding the skewness and kurtosis of the underlying security price distribution. Option pricing formulas for the GLD are developed in the next section.

OPTION PRICING FORMULAS BASED ON THE GLD

Basic Formulas

Pricing formulas for European call and put options based on the GLD are readily derived from the security price percentile function in Equation 4. With $p(s)$ representing a security price distribution on an option expiration date, Equation 5 states the expected expiration-date payoff to a European call option with strike price k , wherein the change of variable $s(p) = s$ is applied in moving from the left side to the right side of the equation:

$$\int_k^\infty (s - k)dp(s) = \int_{p(k)}^1 (s(p) - k)dp \quad (5)$$

$p(k)$ denotes the probability that a call option expires out of the money. Substituting Equation 4 into the right side of Equation 5, completing the integration, and discounting the expected payoff yield this call price formula:

$$\begin{aligned} \text{Call} &= s_0 G_1 - e^{-rt} k G_2 \\ G_1 &= 1 - p(k) + \frac{\sqrt{e^{\sigma^2 t} - 1}}{\lambda_2(\lambda_3, \lambda_4)} \\ &\quad \left(\frac{p(k) - p(k)^{\lambda_3+1}}{\lambda_3 + 1} + \frac{1 - p(k) - (1 - p(k))^{\lambda_4+1}}{\lambda_4 + 1} \right) \\ G_2 &= 1 - p(k) \end{aligned} \quad (6a)$$

Applying put-call parity results in this put price formula:

$$\text{Put} = e^{-rt}k(1 - G_2) - s_0(1 - G_1) \quad (6b)$$

In actual applications, the first four moments of a target security price distribution are matched to those of the generalized lambda percentile function in Equation 4. Option prices are then calculated with the corresponding formulas in Equation 6a or 6b. Intuition suggests that the first four moments of a security price distribution will account for most of the stochastic influences affecting the underlying security price.

Comparisons with Black–Scholes Formula Values

A pertinent concern in many applications is how well the aforementioned GLD option pricing formulas might perform in applications where the Black–Scholes (Black & Scholes, 1973) formula yields exact results. In the Black–Scholes model, future security prices are lognormally distributed with mean $s_0 e^{rt}$ and variance $(s_0 e^{rt})^2 (e^{\sigma^2 t} - 1)$. These match the mean and variance of the generalized lambda percentile function in Equation 4. To adapt the GLD to approximate the Black–Scholes model, skewness and kurtosis are matched to those of the target lognormal distribution.

Expressions for lognormal skewness and kurtosis are as follows, in which $q^2 = (e^{\sigma^2 t} - 1)$ (Jarrow & Rudd, 1982):

$$\text{Skewness} = 3q + q^3 \quad \text{Kurtosis} = 3 + 16q^2 + 15q^4 + 6q^6 + q^8$$

For example, the volatility and time parameters $\sigma = .4$ and $t = .167$ imply lognormal skewness and kurtosis values of .4975 and 3.4433, respectively. Identical skewness and kurtosis values for the GLD are obtained with the parameter values $\lambda_3 = .054989$ and $\lambda_4 = .012446$ in Equation 4. The IMSL Fortran routine BCPOL efficiently computes values for the parameters λ_3 and λ_4 from given skewness and kurtosis values and is similar to the algorithm used by Ramberg et al. (1979) to produce their reference tables.

Table II reports option price and price derivative values based on the Black–Scholes formula and the GLD call price formulas in Equation 6a, with appropriately matched mean, variance, skewness, and kurtosis values. All formulas share common parameter values for the security price, $s_0 = 50$; the volatility, $\sigma = .4$; the time, $t = .167$, and the interest rate, $r = .05$.

Table II reveals that call price formulas derived from the GLD yield option price and price derivative values that differ from corresponding

TABLE II
Comparison of Option Price and Price Derivative Values

Strike	B-S	GLD	B-S	GLD
		<i>Call Price</i>		
40	10.588	10.587	.933	.938
45	6.480	6.450	.782	.784
50	3.451	3.434	.553	.546
55	1.599	1.612	.326	.322
60	.652	.671	.163	.164
		<i>Gamma</i>		
40	.016	.014	-.095	-.101
45	.036	.038	-.169	-.178
50	.048	.049	-.207	-.212
55	.044	.043	-.182	-.178
60	.030	.029	-.122	-.117
		<i>Rho</i>		
40	.060	.060	.029	.029
45	.053	.055	.063	.064
50	.039	.040	.081	.082
55	.023	.024	.072	.071
60	.012	.013	.048	.048
			<i>Vega</i>	

Note: B-S indicates Black-Scholes formula values. GLD indicates values based on the generalized lambda distribution formula in Equation (6a). Delta, theta, rho, and vega values are based on derivatives with respect to security price, time, interest rate, and volatility, respectively. Gamma values are based on a second derivative with respect to the security price. GLD derivative values were computed numerically. Theta values were calibrated to reflect a 1-week time decay. Common parameter values are as follows: security price, $s_0 = 50$; volatility, $\sigma = .4$; time, $t = .167$; and interest rate, $r = .05$.

Black-Scholes formula values by at most a few cents. These deviations are similar in magnitude to what is obtainable with a binomial tree approximation of the Black-Scholes formula (Breen, 1991; Cox & Rubinstein, 1985) and are small relative to the price ticks of exchange traded options.

RECOVERING A PROBABILITY DISTRIBUTION IMPLIED BY OPTION PRICES

Option pricing formulas based on the GLD adapt well to the problem of recovering a probability distribution from observed option prices. The effectiveness of such a recovery procedure is demonstrated here and compared with several alternative procedures with the option price data listed in Table I. For these price data, option maturity is 164 days, the interest rate is 9%, and the dividend-adjusted index level is 349.21.

Fitting a GLD to the call option price data in Table I involves estimating the three parameters σ , λ_3 , and λ_4 by minimizing the sum of

TABLE III
Standard and Poor 500 Index Observed and Fitted Call Prices

Strikes	Bid-Ask Average	Fitted Call Option Values				
		Rubinstein	Volatility Smile	Log-normal	Normal	GLD
250	109.590	109.71	109.59	109.20	109.11	109.84
275	86.185	86.70	86.22	85.83	85.91	86.47
300	63.520	63.90	63.53	63.72	63.82	63.69
325	42.375	42.09	42.13	42.50	42.46	42.14
330	38.285	37.99	38.09	38.36	38.31	38.07
335	34.325	34.01	34.17	34.30	34.26	34.11
340	30.415	30.16	30.36	30.36	30.32	30.28
345	26.790	26.45	26.70	26.56	26.54	26.59
350	23.135	22.90	23.20	22.95	22.95	23.07
355	19.600	19.55	19.87	19.56	19.59	19.73
360	16.260	16.45	16.74	16.43	16.47	16.61
365	13.440	13.65	13.84	13.58	13.62	13.73
370	10.835	11.15	11.18	11.03	11.06	11.11
375	8.745	8.96	8.79	8.79	8.80	8.78
380	7.025	7.07	6.69	6.85	6.83	6.75
385	5.255	5.45	4.89	5.21	5.15	5.04
RMSD		.27	.24	.19	.21	.23

Note: Bid-ask averages are based on bid and ask call option prices reported in Figure 4 of Rubinstein (1994). Fitted call option values in the third column are reproduced directly from Rubinstein. Fitted values in the fourth through seventh columns were computed with the volatility smile specified in Equation 8, lognormal and normal density expansion formulas represented by Equation 9, and the generalized lambda distribution (GLD) call price formula in Equation 6a. Root mean squared deviations (RMSDs) in the third through seventh columns are based on deviations from corresponding bid-ask averages in the second column.

squares stated in Equation 7. In Equation 7, $C(k_j, \sigma, \lambda_3, \lambda_4)$ represents the GLD call price formula from Equation 6a with the arguments s_0 , t , and r suppressed. $C(k_j)$ denotes an observed call price for the strike price k_j :

$$\min_{\sigma, \lambda_3, \lambda_4} \sum_j [C(k_j, \sigma, \lambda_3, \lambda_4) - C(k_j)]^2 \tag{7}$$

Minimizing this sum of squares with the GLD call price formula from Equation 6a yields the volatility and shape parameters $\sigma = .168$, $\lambda_3 = -.065042$, and $\lambda_4 = -.020756$. These estimates for the parameters λ_3 and λ_4 imply skewness and kurtosis values of -1.58 and -8.55 , respectively.

Table III compares fitted option prices obtained via this least-squares procedure with fitted values obtained by Rubinstein (1994). Rubinstein's algorithm produces an implied security price distribution satisfying the restriction that all fitted option prices lie within observed bid-ask spreads.

This algorithm was later abandoned in Jackwerth and Rubinstein (1996), as it did not yield solutions with some data sets. However, the fitted prices in the third column of Table III represent a solution for this data set. For the reader's convenience, strike prices and bid-ask average prices from Table I are listed again in the first and second columns, respectively, of Table III.

Table III also presents fitted option prices obtained with a volatility smile technique and two density expansion models. The volatility smile technique is based on the quadratic regression equation stated in Equation 8, in which $IV(k)$ denotes a fitted implied volatility for an option with strike price k . Using this function adds just two parameters to the Black-Scholes model. Dumas et al. (1998) demonstrated the effectiveness of this parsimonious specification in modeling an implied volatility function:

$$IV(k) = .482 - .212 k/s_0 - .096(k/s_0)^2 \quad R^2 = .995 \quad (8)$$

Summing coefficients in this regression yields an at-the-money ($k = s_0$) volatility of $\sigma = .173$. The high regression R^2 value indicates that the volatility smile fits a parabolic shape quite well. The fourth column of Table III reports fitted option prices for each strike computed by the substitution of fitted volatility values $IV(k)$ from Equation 8 into the Black-Scholes formula.

The two density expansion methods are based on Equation 9, in which C_X represents an expanded formula composed by a Black-Scholes formula price, denoted by C_{BS} , and skewness and kurtosis adjustment terms derived from the density function $a(S_t)$ for the expiration date security price. The coefficients c_3 and c_4 are measures related to skewness and kurtosis deviations, respectively, from a normal or lognormal distribution depending on the expansion employed:

$$C_X = C_{BS} + \frac{c_3}{3!} \frac{da(K)}{dS_t} + \frac{c_4}{4!} \frac{d^2a(K)}{dS_t^2} \quad (9)$$

In the Jarrow and Rudd (1982) model, Equation 9 represents an Edgeworth expansion of the lognormal density. Alternatively, Equation 9 can represent an expansion of the normal density (Rubinstein, 1998). Corrado and Su (1996, 1997a, 1997b) showed that both expansions are effective in flattening empirical volatility smiles implied by S&P 500 index option prices.

With least-squares procedures used to simultaneously estimate implied volatilities and coefficients of skewness and kurtosis, the lognormal expansion model produces an implied volatility of $\sigma = .161$ and skewness

and kurtosis values of -1.253 and 3.188 , respectively. The normal expansion model produces an implied volatility of $\sigma = .178$ and skewness and kurtosis values of -1.648 and 4.934 , respectively. The fifth and sixth columns of Table III report fitted call option prices for each strike with the lognormal and normal density expansion models, respectively.

Beneath each column of fitted option prices in the third through seventh columns of Table III are reported root mean squared deviations (RMSDs) between fitted prices listed in that column and corresponding bid-ask average prices listed in the second column. These RMSDs of fitted prices from bid-ask average prices represent a goodness-of-fit measure. The RMSDs obtained from Rubinstein's fitted prices and the fitted volatility smile are \$.27 and \$.25, respectively. RMSDs based on the lognormal and normal density expansion models are \$.19 and \$.21, respectively. In comparison, the GLD call price formula in Equation 6a produces call prices resulting in an RMSD of \$.23. Differences in magnitude among these RMSDs appear insignificant relative to the average bid-ask spread of \$.78 for these data.

Figure 2 presents probability densities inferred from the Rubinstein data with the three recovery methods described earlier. In Figure 2, the smooth but distinguishably bimodal density is based on an expansion of the lognormal density. The bimodal density shape reported in Figure 4 of Rubinstein (1994) is similar, although less markedly bimodal. Not shown here, an expansion of the normal density yields a recovered density shape that is also distinguishably bimodal, although slightly less so than that produced by the lognormal expansion.

Figure 2 also plots two unimodal densities with nearly identical shapes. These are the densities associated with the volatility smile in Equation 8 and the GLD call price formula in Equation 6a. Little visible difference exists between these two densities, except that the mode of the generalized lambda density lies slightly to the left of the mode of the volatility smile density.

CALIBRATING AN IMPLIED BINOMIAL TREE TO THE GLD

An implied binomial tree infers a stochastic process from a security price distribution and is widely used to calculate prices for options on futures and other American-style options from nonstandard security price distributions. Rubinstein (1994) pointed out that his proposed implied binomial tree algorithm can be used with almost any plausible distribution to obtain a discrete time stochastic process with a valid limiting diffusion

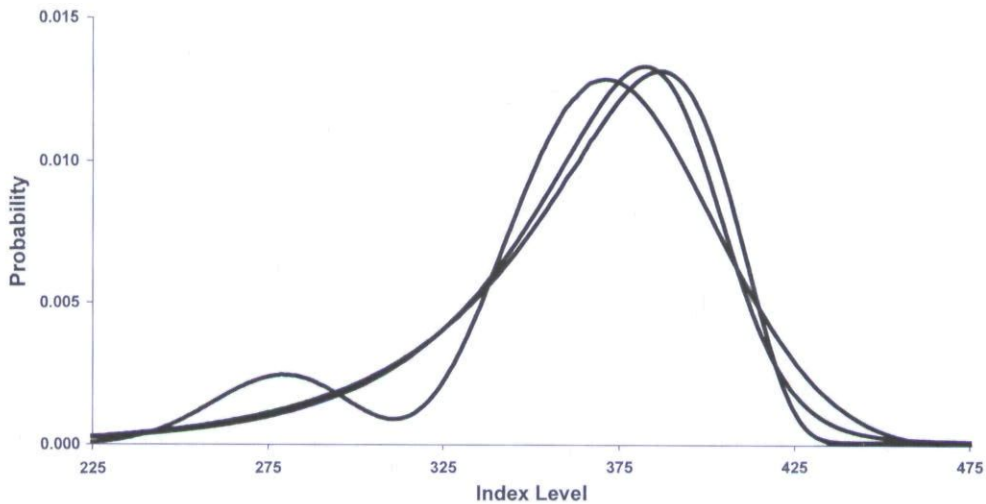


FIGURE 2
Option-implied probability density functions.

process. Construction of an implied binomial tree begins by a nodal value and a nodal probability being assigned to each of its terminal nodes. An ordered sequence of N nodal values and probabilities uniquely defines an implied tree with $N - 1$ steps. This section describes a method to precisely calibrate the nodal values and probabilities of an implied tree to the generalized lambda percentile function in Equation 4.

The method is straightforward. To calibrate an implied binomial tree with $N - 1$ steps leading to N terminal nodes, first form an ascending sequence of $N + 1$ strike prices from $k_0 \equiv 0$ through $k_N \equiv \infty$. The N intervals $k_{j+1} - k_j > 0, j = 0, 1, 2 \dots N - 1$, need not be equally spaced, but in practice at least a subset will normally be selected to match observed equally spaced strike prices. Precise calibration to the generalized lambda percentile function in Equation 4 occurs when European option prices calculated on the tree for each of the first N strikes exactly match prices calculated with the corresponding GLD option pricing formula. This requires satisfying the N equalities stated next, in which $C(k_j, N)$ denotes a call price calculated on an implied tree with N terminal nodes and $C(k_j, \sigma, \lambda_3, \lambda_4)$ represents a call price calculated from Equation 6a:

$$C(k_j, \sigma, \lambda_3, \lambda_4) = C(k_j, N) \quad \forall j = 0, 1, 2 \dots N - 1$$

The procedure for determining nodal values and probabilities needed to satisfy these equalities is described next.

Following the notation used earlier, I denote the probability of a security price realization less than or equal to the strike price k_j with $p(k_j)$.

The probability of a security price realization in the interval $k_{j+1} - k_j$ is, therefore, $p(k_{j+1}) - p(k_j)$, where $p(k_{N+1}) \equiv 1$ and $p(k_0) = 0$. The probability $p(k_{j+1}) - p(k_j)$ is assigned to node j .

In terms of the generalized lambda percentile function in Equation 4, expected values of security price realizations within the interval $k_{j+1} - k_j$ are specified in Equation 10:

$$E_p(s|k_j < s \leq k_{j+1}) = \frac{\int_{p(k_j)}^{p(k_{j+1})} s(p) dp}{p(k_{j+1}) - p(k_j)} = s_0 e^{rt} \left(1 + \frac{\sqrt{e^{\sigma^2 t}} - 1}{\lambda_2(\lambda_3, \lambda_4)} \right) \left(\frac{1}{p(k_{j+1}) - p(k_j)} \times \left(\frac{p(k_{j+1})^{\lambda_3+1} - p(k_j)^{\lambda_3+1}}{\lambda_3 + 1} - \frac{[1 - p(k_{j+1})]^{\lambda_4+1} - [1 - p(k_j)]^{\lambda_4+1}}{\lambda_4 + 1} \right) \right) \quad (10)$$

The expected value $E_p(s|k_j < s \leq k_{j+1})$ is assigned to node j for $j = 0, 1, 2 \dots N-1$.

These nodal values $E_p(s|k_j < s \leq k_{j+1})$ and probabilities $p(k_{j+1}) - p(k_j)$ precisely calibrate an implied binomial tree to the percentile function in Equation 4. This is demonstrated by the summation of expected values of all call option payoffs above the j th strike price to yield this call price equality:

$$\begin{aligned} C(k_j, N) &= e^{-rt} \sum_{i=j}^N (E_p(s|k_i < s \leq k_{i+1}) - k_j) [p(k_{i+1}) - p(k_i)] \\ &= e^{-rt} (E_p(s|k_j < s \leq k_{N+1}) - k_j) [1 - p(k_j)] \\ &= e^{-rt} E_p \max(0, s(p) - k_j) \\ &= C(k_j, \sigma, \lambda_3, \lambda_4) \end{aligned} \quad (11)$$

In particular, the strike $k_0 = 0$ yields the initial security price $C(k_0, N) = s_0$.

This calibration method might be implemented with other distributions. The principal advantage of the GLD is its adaptability to Monte Carlo applications. It also allows assigning any of a broad range of skewness and kurtosis values while a unimodal, bell-shaped density is maintained. This contrasts with density expansion methods that yield a unimodal density within only a limited range of skewness and kurtosis values (Stuart & Ord, 1987). For example, Rubinstein (1998) showed that an expansion of the normal density must limit the absolute value of skewness to less than one to maintain a unimodal form. The next section provides a multivariate extension of the GLD.

MULTIVARIATE GLD

This section begins by developing a multivariate version of the GLD that facilitates stochastic modeling of multiple, correlated security prices. Immediately afterward, the multivariate GLD is extended to a multi-index security returns model.

Bradley and Fleisher (1994) suggested obtaining correlated generalized lambda variables via a transformation from normal variables. Their method proceeds by first generating correlated normal variables with a Cholesky decomposition of the correlation matrix $\mathbf{R}$. Then, a univariate p value is computed for each generated normal variable. These univariate p values are transformed into generalized lambda variables as, for example, described earlier in this article. The correlation matrix $\mathbf{G}$ represents the resulting correlation structure among these generalized lambda variables.

Bradley and Fleisher (1994) pointed out that the generalized lambda correlations and normal correlations composing the matrices $\mathbf{G}$ and $\mathbf{R}$, respectively, will differ somewhat. Obtaining a precise target configuration for the matrix $\mathbf{G}$ requires making incremental adjustments to the off-diagonal elements of $\mathbf{R}$ until the resulting correlation structure in $\mathbf{G}$ converges to the target configuration. Bradley and Fleisher suggested a simulation algorithm to compute the generalized lambda correlations in $\mathbf{G}$ at each adjustment. However, the accuracy of the algorithm depends on the size of the simulation and, therefore, may be computationally burdensome. A direct approach involves computing generalized lambda correlations via the following integral, where the left-side expression $\gamma(\lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24})$ denotes a correlation between two generalized lambda variables with the indicated shape parameters:

$$\gamma(\lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s[N(z_1); \lambda_{13}, \lambda_{14}] s[N(z_2); \lambda_{23}, \lambda_{24}] n(z_1, z_2; \rho) dz_1 dz_2$$

In the right-side integrand, $s[N(z); \lambda_3, \lambda_4]$ is a unit-variance generalized lambda variable (a la Equation 1) generated by the normal probability $N(z)$ and shape parameters λ_3 and λ_4 . $n(z_1, z_2; \rho)$ is a bivariate standard normal density with the correlation parameter ρ . The IMSL Fortran routine TWODAQ performs this numerical integration quickly and efficiently.

The following panel compares target generalized lambda correlations with the normal correlations required to obtain them for the four generalized lambda densities presented in Figure 1. The first two columns

state skewness and kurtosis values for these densities. As shown, the required normal correlations are somewhat larger in absolute value than the generalized lambda correlations they produce:

Skewness	Kurtosis	Target Generalized-Lambda Correlations			Required Normal Correlations		
		γ			ρ		
0	3						
-1	6	.1			.104		
-3	30	.3	-.5		.340	-.583	
-6	2000	.5	-.3	-.1	.793	-.377	-.141

Development of the multivariate generalized lambda system in the context of a multi-index model proceeds as follows: Equation 12 defines the security return y_j as a linear function of the security-specific return x_j and the N orthogonal index returns and factor weights f_F and $\beta_{F,j}$, respectively:²

$$y_j = x_j + \sum_{F=1}^N \beta_{F,j} f_F \tag{12}$$

Exact expressions for the variance, skewness, and kurtosis of the security return y_j are stated as

$$\begin{aligned} \text{Var}(y_j) &= \text{Var}(x_j) + \sum_{F=1}^N \beta_{F,j}^2 \text{Var}(f_F) \\ \text{Skew}(y_j) &= \left(\text{Skew}(x_j) \text{Var}^{3/2}(x_j) + \sum_{F=1}^N \beta_{F,j}^3 \text{Skew}(f_F) \text{Var}^{3/2}(f_F) \right) / \text{Var}^{3/2}(y_j) \\ \text{Kurt}(y_j) &= \left(\text{Kurt}(x_j) \text{Var}^2(x_j) + 6 \text{Var}(x_j) \sum_{F=1}^N \beta_{F,j}^2 \text{Var}(f_F) \right. \\ &\quad \left. + \sum_{F=1}^N \beta_{F,j}^4 \text{Kurt}(f_F) \text{Var}^2(f_F) \right. \\ &\quad \left. + 6 \sum_{F=1}^{N-1} \sum_{H=F+1}^N \beta_{F,j}^2 \beta_{H,j}^2 \text{Var}(f_F) \text{Var}(f_H) \right) / \text{Var}^2(y_j) \end{aligned} \tag{13}$$

This multivariate system lends itself to modeling the stochastic properties of portfolios. Portfolio variance, skewness, and kurtosis based on the weights $w_j, j = 1, 2 \dots P$, for a portfolio of P securities are stated by these expressions:

²Elton and Gruber (1987) reviewed multi-index models and provided an extensive bibliography.

$$\begin{aligned}
 \text{Var}\left(\sum_{j=1}^P w_j y_j\right) &= \sum_{j=1}^P w_j^2 \text{Var}(x_j) + \sum_{F=1}^N \text{Var}(f_F) \left(\sum_{j=1}^P w_j \beta_{F,j}\right)^2 \\
 \text{Skew}\left(\sum_{j=1}^N w_j y_j\right) &= \\
 &= \frac{\sum_{j=1}^P w_j^3 \text{Skew}(x_j) \text{Var}^{3/2}(x_j) + \sum_{F=1}^N \text{Skew}(f_F) \text{Var}^{3/2}(f_F) \left(\sum_{j=1}^P w_j \beta_{F,j}\right)^3}{\text{Var}^{3/2}\left(\sum_{j=1}^N w_j y_j\right)} \\
 \text{Kurt}\left(\sum_{j=1}^N w_j y_j\right) &= \\
 &= \frac{\left(\sum_{j=1}^P w_j^4 \text{Kurt}(x_j) \text{Var}^2(x_j) + 6 \sum_{j=1}^{P-1} \sum_{k=i+1}^P w_k^2 w_j^2 \text{Var}(x_k) \text{Var}(x_j) \right. \\
 &\quad + 3 \sum_{j=1}^P \sum_{F=1}^N w_j^2 \left(\sum_{k=1}^P w_k \beta_{F,k}\right)^2 \text{Var}(x_j) \text{Var}(f_F) \\
 &\quad + \sum_{F=1}^N \left(\sum_{j=1}^P w_j \beta_{F,j}\right)^4 \text{Kurt}(f_F) \text{Var}^2(f_F) \\
 &\quad \left. + 6 \sum_{F=1}^{N-1} \sum_{H=F+1}^N \left(\sum_{j=1}^P w_j \beta_{F,j}\right)^2 \text{Var}(f_F) \left(\sum_{j=1}^P w_j \beta_{H,j}\right)^2 \text{Var}(f_H) \right)}{\text{Var}^2\left(\sum_{j=1}^N w_j y_j\right)} \quad (14)
 \end{aligned}$$

These portfolio variance, skewness, and kurtosis expressions might be used in conjunction with the GLD option pricing formulas derived earlier to calculate accurate option values for security market indexes or other security portfolios.

SUMMARY AND CONCLUSION

This article introduced the GLD as a flexible tool for modeling nonlog-normal security price distributions. Major advantages of the GLD include the flexibility to assign almost any combination of skewness and kurtosis values, the ease of random number generation due to the simple form of its percentile function, and the ready construction of a multivariate distribution in which each marginal distribution may be assigned unique

skewness and kurtosis values. These advantages suggest several financial applications, including estimation of option-implied state price densities and risk analyses based on Monte Carlo simulations, wherein varied distributional assumptions may be applied to portfolios of primary and derivative securities.

The GLD is readily applied to the problem of recovering an implied security price distribution from market-observed option prices. An example of such a recovery procedure revealed that although estimates based on the GLD are comparable to those obtained with other methods, generalized lambda estimates ensure a unimodal, bell-shaped density. Complementing the recovery of an implied distribution, a method to calibrate exactly an implied binomial tree to a generalized lambda density was demonstrated.

Finally, a multivariate version of the GLD was developed that allows an arbitrary correlation structure among generalized lambda variables. This development was extended to a multi-index security returns model, and skewness and kurtosis expressions for linear combinations of correlated generalized lambda variables were provided. This multivariate system would be particularly useful in Monte Carlo simulations based on multiple correlated securities where the experimenter wishes to assign arbitrary skewness and kurtosis values to the target security price distributions.

APPENDIX: CLAMPING THE GLD

For negative shape parameters, that is, $\lambda_3, \lambda_4 < 0$, the range of support for the GLD spans the real line, thereby assigning positive probabilities to negative security price values. Jackwerth and Rubinstein (1996) imposed clamping to force a price density to zero in regions where a price realization is disallowed. This imposition is readily applied to the GLD. The resulting security price percentile function that disallows negative price realizations is this generalization of Equation 1:

$$\begin{aligned}
 s(p) = \frac{s_0 e^{rt}}{1 - p(0)} & \left(1 + \frac{\sqrt{e^{\sigma^2 t}} - 1}{\lambda_2 [p(0); \lambda_3, \lambda_4]} \left(p^{\lambda_3} - (1 - p)^{\lambda_4} \right. \right. \\
 & \left. \left. + \frac{[1 - p(0)]^{\lambda_4 + 1}}{\lambda_4 + 1} - \frac{1 - p(0)^{\lambda_3 + 1}}{\lambda_3 + 1} \right) \right) \quad (A1)
 \end{aligned}$$

The scaling function $\lambda_2[p(0); \lambda_3, \lambda_4]$ defined in Equation A2, in which $\beta_p(x, y)$ denotes the incomplete beta function, correctly realigns the variance of $s(p)$ in Equation A1 on $(s_0 e^{rt})^2 (e^{\sigma^2 t} - 1)$:

$$\begin{aligned}\lambda_2[p(0); \lambda_3, \lambda_4] &= \text{sign}(\lambda_3) \times \sqrt{B - A^2} \\ A &= \frac{1 - p(0)^{\lambda_3+1}}{\lambda_3 + 1} - \frac{[1 - p(0)]^{\lambda_4+1}}{\lambda_4 + 1} \\ B &= \frac{1 - p(0)^{2\lambda_3+1}}{2\lambda_3 + 1} + \frac{[1 - p(0)]^{2\lambda_4+1}}{2\lambda_4 + 1} \\ &\quad - 2(\beta(\lambda_3 + 1, \lambda_4 + 1) - \beta_{p(0)}(\lambda_3 + 1, \lambda_4 + 1)) \quad (\text{A2})\end{aligned}$$

The resulting call price formula is the following generalization of Equation 6a:

$$\text{Call} = s_0 G_1 - e^{-rt} k G_2$$

$$\begin{aligned}G_1 &= [1 - p(k)]/[1 - p(0)] \\ &\quad + \frac{\sqrt{e^{\sigma^2 t} - 1}}{\lambda_2[1 - p(0)]} \left(\frac{p(k) - p(0) + p(0)^{\lambda_3+1} - p(k)^{\lambda_3+1}}{\lambda_3 + 1} \right. \\ &\quad \left. - \frac{p(k) - p(0) + [1 - p(k)]^{\lambda_4+1} - [1 - p(0)]^{\lambda_4+1}}{\lambda_4 + 1} \right) \quad (\text{A3})\end{aligned}$$

$$G_2 = [1 - p(k)]/[1 - p(0)]$$

Put-call parity yields the corresponding put price formula:

$$\text{Put} = e^{-rt} k(1 - G_2) - s_0(1 - G_1) \quad (\text{A4})$$

In Equations A3 and A4, the scaling parameter is stated simply as λ_2 but is taken from Equation A2. With $p(0) = 0$, Equations A1, A3, and A4 all collapse to Equations 4, 6a, and 6b, respectively, and are, therefore, preferred as a more general implementation.

An implementation of clamping might also use these generalizations of the coefficients of skewness and kurtosis from Equation 3, in which A and B are defined in Equation A2:

$$\begin{aligned}\text{Skewness} &= \frac{C - 3AB + 2A^3}{(B - A^2)^{3/2}} \quad \text{Kurtosis} = \frac{D - 4AC + 6A^2B - 3A^4}{(B - A^2)^2} \\ C &= \frac{1 - p(0)^{3\lambda_3+1}}{3\lambda_3 + 1} - \frac{[1 - p(0)]^{3\lambda_4+1}}{3\lambda_4 + 1} + 3[\beta(2\lambda_4 + 1, \lambda_3 + 1) \\ &\quad - \beta_{p(0)}(2\lambda_4 + 1, \lambda_3 + 1)] - 3[\beta(2\lambda_3 + 1, \lambda_4 + 1) - \beta_{p(0)}(2\lambda_3 + 1, \lambda_4 + 1)] \\ D &= \frac{1 - p(0)^{4\lambda_3+1}}{4\lambda_3 + 1} + \frac{[1 - p(0)]^{4\lambda_4+1}}{4\lambda_4 + 1} - 4[\beta(3\lambda_4 + 1, \lambda_3 + 1) \\ &\quad - \beta_{p(0)}(3\lambda_4 + 1, \lambda_3 + 1)] - 4[\beta(3\lambda_3 + 1, \lambda_4 + 1) - \beta_{p(0)}(3\lambda_3 + 1, \lambda_4 + 1)] \\ &\quad + 6[\beta(2\lambda_3 + 1, 2\lambda_4 + 1) - \beta_{p(0)}(2\lambda_3 + 1, 2\lambda_4 + 1)]\end{aligned}$$

With plausible parameter values, the probability $p(0)$ is typically trivial. For example, the GLD was applied to fit an implied security price distribution to the option price data listed in Table I. Despite the strong negative skewness exhibited by this fitted distribution, the derived probability of a negative price realization is only $p(0) = 8.4E - 5$. Extending the maturity from the original 164 days to 365 days, while all other parameter values are retained, yields the probability $p(0) = 9.0E - 4$.

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