A Note on Finding the Optimal Allocation Between a Risky Stock and a Risky Bond

JOHN E. ANGUS

The allocation of financial assets among securities with different levels of risk is an essential topic in the study, analysis, and strategic use of derivative securities and markets. In a recent paper, Browne (1999) determined the optimal allocation strategy for dividing investments between a risky stock and a risky bond. In this note, Browne's equation determining the optimal strategy is studied and some methods are described for solving it. In addition, some useful rules-of-thumb, computational methods, and approximation techniques are presented. © 2001 John Wiley & Sons, Inc. Jrl Fut Mark 21:1181–1196, 2001

1. INTRODUCTION

The optimal strategy for the allocation of financial assets among competing securities with different levels of risk is a matter of both practical and academic interest. In fact, optimal portfolio selection methods, along with derivative security pricing, have occupied a large part of the applied and theoretical literature in quantitative finance for the past two decades.

For correspondence, John E. Angus, Department of Mathematics, Claremont Graduate University, 121 East Tenth Street, Claremont, California 91711.

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■ *John E. Angus is a Professor in the Department of Mathematics at Claremont Graduate University in Claremont, California.*

Portfolios are connected with derivative securities and other security markets in many ways. In particular, portfolios may contain both riskless and risky securities, including options and other derivative securities. On the other hand, shares in a portfolio can themselves define securities. Moreover, a share of a portfolio is a security whose value depends on other more basic securities, and can thus be viewed as a derivative security. Hedging portfolios are used as practical and theoretical tools in deriving prices for options and other derivatives. Finally, the strategy of insuring a portfolio against a drop in value by purchasing put options on an appropriate index is well known. Consequently, optimal methods for allocating assets in a portfolio are of fundamental interest either directly or indirectly in all of these contexts.

This article aims at bridging the ever widening gap between theory and implementation by presenting examples and strategies for implementing the solution by Browne (1999) to the optimal allocation problem when the portfolio consists of a risky stock and a risky bond. In this connection, consider a risky stock whose value S is governed by the stochastic differential equation

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t) \tag{1.1}$$

where μ is the local rate of return and $\sigma > 0$ is the volatility. Here, $\{W(t), t \geq 0\}$ is a standard Brownian motion. This is the classical continuous time model for a security (see, e.g., Merton, 1990). A risky bond or money market account may be described as having initially a fixed rate of return $r \geq 0$, but for which at random future times, there are fractional (of random magnitude) reductions in the underlying bond price caused by failed coupon payments from the responsible organization. To avoid unrealizable cases, it is assumed throughout that μ, r, and σ are all finite. If B denotes the value of the bond, then one such model for B is described by the stochastic differential equation

$$dB(t) = rB(t-)\,dt - B(t-)\,dQ(t) \tag{1.2}$$

where the notation $B(t-)$ is shorthand for the left-hand $\lim_{u\to t-} B(u)$. Here, the process $\{Q(t), t \geq 0\}$ is a compound Poisson process, given by

$$Q(t) = \sum_{i=1}^{N(t)} Y_i \tag{1.3}$$

where $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with finite rate $\lambda \geq 0$, and $Y_1, Y_2, \ldots$ is a sequence of independent, identically distributed random variables satisfying $P\{0 \leq Y_1 \leq 1\} = 1$. This model for the bond has been used often in the literature (see, e.g., Jarrow & Turnbull, 1995; Merton, 1971) and is referred to as the model of Poisson bankruptcies. The three objects $\{Y_1, Y_2, \ldots\}$, $\{N(t), t \geq 0\}$, and $\{W(t), t \geq 0\}$ are assumed to be mutually independent. Noting that the mean of the process $Q(t)$ is $\lambda t E(Y_1)$, and introducing the zero-mean martingale

$$\tilde{Q}(t) = Q(t) - \lambda t E(Y_1) \tag{1.4}$$

the model (1.2) can be expressed as

$$dB(t) = (r - \lambda E(Y_1))B(t-)\,dt - B(t-)\,d\tilde{Q}(t) \tag{1.5}$$

It follows that the local rate of return for the risky bond is given by $r - \lambda E(Y_1)$ which may be compared to μ in (1.1). It is well known that the solution to (1.1) is

$$S(t) = S(0)\exp\{(\mu - \sigma^2/2)t + \sigma W(t)\} \tag{1.6}$$

and Browne (1999) presents the solution to (1.2) as

$$B(t) = B(0)\exp(rt)\prod_{i=1}^{N(t)}(1 - Y_i) \tag{1.7}$$

where here and in general a sum running from 1 to 0 and a product running from 1 to 0 will be taken to be 0 and 1, respectively. Examining the structure of (1.7), it is easy to see why the random variables $\{Y_1, Y_2, \ldots\}$ are referred to as "write-downs." Empirical distributions of write-downs have been studied by Franks and Torous (1994), for example.

Now assume that there is an investment strategy $\{f(t), t \geq 0\}$ with the interpretation that at time t, the fraction $f(t) \in [0, 1]$ of wealth is invested in the risky stock, and the remainder in the risky bond. It is assumed that $\{f(t), t \geq 0\}$ is an $\Im_t$ admissible adapted control process, where $\Im_t$ is the appropriately augmented natural filtration generated by $\{W(s), Q(s), 0 \leq s \leq t\}$, $t \geq 0$. Note that the bond value changes in jump-discontinuities and therefore, so may the wealth of the investor using strategy f. Denote the wealth of the investor at time t by $X^f(t)$ and consider the stochastic differential of $X^f(t)$. Now for a small (infinitesimal) $dt > 0$ the wealth will change from $X^f(t-)$ to $X^f(t-) + dX^f(t)$ from the time t to the time $t + dt$. Just prior to time t, there is $f(t-)X^f(t-)$ of the wealth in stock, and $(1 - f(t-))X^f(t-)$ of the wealth in risky bond. From time t to time $t + dt$, the value of the stock changes by the fraction $dS(t)/S(t)$ and the value of the bond changes by the fraction $dB(t)/B(t-)$, so that after substituting (1.1) and (1.2) the wealth process satisfies the stochastic differential equation

$$dX^f(t) = X^f(t-)[r + f(t-)(\mu - r)]\,dt + f(t-)X^f(t-)\sigma\,dW(t) - (1 - f(t-))X^f(t-)\,dQ(t) \tag{1.8}$$

Browne (1999) gives the solution to (1.8) as

$$\ln X^f(t) = \ln X^f(0) + \int_0^t \left(r + (\mu - r)f(u) - \frac{1}{2}f^2(u)\sigma^2\right) du + \sigma\int_0^t f(u)\,dW(u) + \sum_{i=1}^{N(t)} \ln(1 - (1 - f(\tau_i-))Y_i) \tag{1.9}$$

where $\{\tau_i, i = 1,2,\ldots\}$ is the sequence of jump times in $\{N(t), t \geq 0\}$, and shows that the optimum strategy $\{f(t), t \geq 0\}$ is actually given by a constant $f(t) \equiv f^*$ for all $t \geq 0$ where f^* satisfies

$$f^* = \arg\sup_{0\leq f\leq 1} \{r + (\mu - r)f - f^2\sigma^2/2 + \lambda E \ln(1 - (1 - f)Y_1))\} \tag{1.10}$$

The remarkable fact about this solution, pointed out by Browne (1999), is that it is the solution to three different optimization problems:

- Choose an admissible investment policy to minimize the expected time to reach a given level b of wealth;
- Choose an admissible investment policy to maximize the expected logarithm of terminal wealth at a fixed terminal time T;
- Choose an admissible investment policy to maximize the actual (long run) growth rate of wealth.

In the next sections, properties of the optimum fraction f^* are investigated, and approximate methods for finding it are examined.

2. THE OPTIMUM STRATEGY

From (1.10), the function to optimize over the interval $0 \leq f \leq 1$ is

$$C(f) = r + (\mu - r)f - \frac{1}{2}f^2\sigma^2 + \lambda E[\ln(1 - (1 - f)Y_1)] \tag{2.1}$$

Note that no a priori assumptions about the relative sizes of the parameters σ, μ, r have been made, and the distribution of Y_1 could be any arbitrary distribution concentrated on [0, 1]. The first step is to determine conditions under which the optimum value $C(f)$ occurs for f in the interior of the interval [0, 1]. Applying corollary 2 to the Lebesgue Dominated Convergence Theorem in Rao (1973, p. 136), to justify interchanging the expectation and derivative, the derivative of the function C for f in (0, 1) is

$$C'(f) = \lambda E\left(\frac{Y_1}{1-(1-f)Y_1}\right) + \mu - r - f\sigma^2 \tag{2.2}$$

and an $f \in (0, 1)$ yielding a maximum must satisfy $C'(f) = 0$. Write

$$g(f) = \lambda E\left(\frac{Y_1}{1-(1-f)Y_1}\right) + \mu - r \tag{2.3}$$

and agree to define

$$\begin{aligned} E\left(\frac{Y_1}{1-Y_1}\right) &\equiv \lim_{f\to 0+} E\left(\frac{Y_1}{1-(1-f)Y_1}\right) \\ E\left(\frac{Y_1^2}{1-Y_1}\right) &\equiv \lim_{f\to 0+} E\left(\frac{Y_1^2}{1-(1-f)Y_1}\right) \end{aligned} \tag{2.4}$$

In this fashion, the left-hand members are always well defined, being either finite positive numbers, or $+\infty$ in case $(1-y)^{-1}$ is not integrable near $y = 1$ with respect to the measure $P\{Y_1 \in dy\}$. Note that

$$g(1-) = \lambda E(Y_1) + \mu - r \tag{2.5}$$

and

$$g(0+) = \lambda E\left(\frac{Y_1}{1-Y_1}\right) + \mu - r \tag{2.6}$$

so that $g(0+) \geq g(1-)$ with strict inequality unless $\lambda = 0$ or $E(Y_1) = 0$ ($E(Y_1) = 0$ being the same as $P\{Y_1 = 0\} = 1$ since Y_1 can take values only in [0, 1]). Either of the two conditions $\lambda = 0$ or $E(Y_1) = 0$ is equivalent to the bond being risk free. Next, once again applying corollary 2 to the Lebesgue Dominated Convergence Theorem in Rao (1973, p. 136), notice that in (0, 1), the derivative of g is

$$g'(f) = -\lambda E\left(\left(\frac{Y_1}{1-(1-f)Y_1}\right)^2\right) \tag{2.7}$$

so that $g'(f) \leq 0$ in (0, 1) with strict inequality unless $\lambda = 0$ or $E(Y_1) = 0$. Therefore, the function $y = g(f)$ will cross the linear function $y = f\sigma^2$

exactly once in (0, 1) provided $g(0+) > 0$ and $g(1-) < \sigma^2$, that is, provided

$$-\lambda E\left(\frac{Y_1^2}{1 - Y_1}\right) < \mu - (r - \lambda E(Y_1)) < \sigma^2 \tag{2.8}$$

Note that the quantity in the middle of (2.8) is the difference between the stock's and the bond's local rates of return; see (1.1) and (1.5). What has also been shown is that if $g(1-) \geq \sigma^2$ ($g(0+) \leq 0$), the maximum of $C(f)$ will occur at some $f \geq 1$ (some $f \leq 0$) so that $f^* = 1$ ($f^* = 0$) will be optimal. Expression (2.8) thus gives a rule for determining when there will be a nontrivial strategy (i.e., an optimum strategy $f^* \in (0, 1)$). In fact, the following lemma has been demonstrated:

Lemma 2.1. The maximum value of $C(f)$ in the interval [0, 1] occurs at $f = f^* = 1$ (i.e., never invest in the bond) when

$$\mu - (r - \lambda E(Y_1)) \geq \sigma^2 \tag{2.9}$$

The maximum value of $C(f)$ in the interval [0, 1] occurs at $f = f^* = 0$ (i.e., never invest in the stock) when

$$(r - \lambda E(Y_1)) - \mu \geq \lambda E\left(\frac{Y_1^2}{1 - Y_1}\right) \tag{2.10}$$

and when the write-down distribution satisfies

$$\lim_{f \to 0+} E\left(\frac{1}{1 - (1 - f)Y_1}\right) = +\infty \tag{2.11}$$

and $\lambda > 0$, then $f = 0$ is never optimal. Finally, there is a maximum value of $C(f)$ in the interval (0, 1) when (2.8) holds.

These imply that for the stock to dominate completely, its local rate of return must exceed the bond's local rate of return by an amount equal to or greater than the square of the stock's volatility, but the analogous condition for the bond to be completely dominant is more stringent. Note that by analogy with the volatility term in the stochastic differential equation for the stock, the squared volatility for the bond is $\lambda E(Y_1^2) = \lim_{t \to 0} \operatorname{var} \frac{\tilde{Q}^2(t)}{t}$. Thus for the bond to dominate completely, its local rate of return must exceed the local rate of return for the stock by an amount that represents an inflation of the squared volatility of the bond.

3. SPECIAL CASES

3.1 Risk Free Bond ($\lambda E(Y_1) = 0$)

In this case, (2.9) and (2.10) tell us that the optimum strategy is $f^* = 0$ when $r \geq \mu$, and $f^* = 1$ when $\mu \geq r + \sigma^2$. Thus, for the bond to dominate completely, the risk free rate of return must simply exceed the local rate of return for the stock, but for the stock to dominate, the local rate of return for the stock must exceed the risk free rate of return by at least an amount given by the square of the stock's volatility. When neither of these cases hold, (2.11) reduces to a quadratic that is easily maximized to yield an optimum strategy of $f^* = (\mu - r)/\sigma^2$. These may be summarized into one formula as

$$f^* = \max\{0, \min\{(\mu - r)/\sigma^2, 1\}\} \tag{3.1}$$

(This corrects expression 22 in Browne [1999].)

3.2. Constant Write-Down ($P\{Y_1 = a\} = 1$ where $a \in (0, 1]$, $\lambda > 0$)

Assume first the special condition that $a \neq 1$. In this case, (2.9) and (2.10) tell us that the optimum strategy is $f^* = 0$ when $r - \lambda a \geq \mu + \lambda a^2/(1 - a)$, and $f^* = 1$ when $\mu \geq r - \lambda a + \sigma^2$. Notice that the quantity λa^2 is the squared volatility of the bond in this case. Thus, for the bond to dominate completely in this case, its local rate of return must exceed the local rate of return for the stock by at least an amount given by the square of its volatility inflated by the factor $1/(1 - a)$, and for the stock to dominate, the local rate of return for the stock must exceed the local rate of return for the bond by at least an amount given by the square of the stock's volatility.

When neither of these conditions are met, i.e., when

$$-\frac{\lambda a^2}{1 - a} < \mu - (r - \lambda a) < \sigma^2 \tag{3.2}$$

then the optimum strategy is given by maximizing the quadratic that is determined by (2.1), yielding

$$f^* = \frac{a(\mu - r) - (1 - a)\sigma^2 + \sqrt{[a(\mu - r) - (1 - a)\sigma^2]^2 + 4\sigma^2 a(a\lambda + (1 - a)(\mu - r))}}{2\sigma^2 a} \tag{3.3}$$

Now, the case $a = 1$ works similarly, except by the convention (2.4), the left side of (3.2) is $-\infty$ since $\lambda > 0$ so that the bond can never dominate completely (i.e., $f = 0$ is never optimal). This makes sense because the first write-down, which is sure to happen eventually since $\lambda > 0$, will bankrupt the bond. When

$$\mu - (r - \lambda) < \sigma^2 \tag{3.4}$$

is satisfied, the problem (1.10) then becomes one of finding the root in (0, 1) of $\lambda/f + \mu - r - f\sigma^2 = 0$, which is easily seen to be

$$f^* = \frac{\mu - r + \sqrt{(\mu - r)^2 + 4\lambda\sigma^2}}{2\sigma^2} \tag{3.5}$$

If (3.4) is violated, then the optimum strategy is $f^* = 1$. Thus, the rule of thumb persists: Never invest in the bond unless the difference between the local rate of return for the stock and the local rate of return for the bond is less than the squared volatility of the stock.

3.3. Two-Point Write-Down Distribution: ($P\{Y_1 = a\} = p = 1 - P\{Y_1 = 0\}$, $a \in (0, 1]$, $p \in (0, 1)$, $\lambda > 0$)

By exploiting properties of the homogeneous Poisson process in (1.2) and (1.5), this case actually reduces to the constant write-down case above with λ replaced by $p\lambda$. This is true because with probability $(1 - p)$, there is no actual write-down when the Poisson event occurs. The remaining events in the Poisson process at which write-downs of value a occur constitute a Poisson process with rate $p\lambda$, the so-called thinned process. Thus, this case may be viewed as constant write-downs of size a occurring at the events of a Poisson process with rate $p\lambda$, and the results of the previous section apply.

3.4. Uniform (0, 1) Write-Downs ($\lambda > 0$)

When Y_1 is uniform on (0, 1), then the right side of (2.10) is positive infinity, so $f = 0$ is never optimum. When $\mu - (r - \lambda/2) \geq \sigma^2$, then $f^* = 1$ by (2.9), and when

$$\mu - (r - \lambda/2) < \sigma^2 \tag{3.6}$$

the optimum strategy is the solution in $f \in (0, 1)$ to

$$\mu - r - \sigma^2 f - \lambda\left(\frac{1 - f + \ln(f)}{(1 - f)^2}\right) = 0 \tag{3.7}$$

Note that the root to (3.7) depends only on

$$k_1 = \frac{\mu - r}{\sigma^2} \tag{3.8}$$

and

$$k_2 = \frac{\lambda}{\sigma^2} \tag{3.9}$$

so that condition (3.6) is equivalent to

$$k_1 + k_2/2 - 1 < 0 \tag{3.10}$$

and there is no lower bound required on k_1 to get a unique solution to (3.7). Thus when (3.10) is violated, then $f^* = 1$.

A closed form solution to (3.7) does not exist, but whenever (3.10) holds, a Newton iteration to solve for the root would lead to the recursion:

$$f_{n+1} = f_n + \frac{\left[k_1 - f_n - k_2\left(\frac{1}{1 - f_n} + \frac{\ln(f_n)}{(1 - f_n)^2}\right)\right]}{\left[1 + \frac{k_2}{(1 - f_n)^2}\left(1 + \frac{1}{f_n} + \frac{2\ln(f_n)}{(1 - f_n)}\right)\right]} \tag{3.11}$$

With initial value $f_0 = 1/2$, this recursion converges rapidly to the solution f^* when (3.10) holds as long as k_1/k_2 is not too large in the negative direction. In fact, when $k_1/k_2 \ll 0$, the optimum f^* gets close to 0, and (3.7) has a solution that is approximately the solution to the equation

$$k_1 - k_2(1 + \ln(f)) = 0 \tag{3.12}$$

which is (3.7) with the relations $f \approx 0$ and $1/(1 - f) \approx 1/(1 - f)^2 \approx 1$ substituted. The solution to (3.12) is

$$f = \exp\left(\frac{k_1}{k_2} - 1\right) \tag{3.13}$$

and this is approximately the solution to (3.7) when $k_1/k_2 \ll 0$ and (3.10) holds. The value (3.13) can be used to initialize the Newton iteration when $k_1/k_2 \ll 0$.

Using $f_0 = 1/2$ when $\exp(k_1/k_2 - 1) > 1/2$, and $f_0 = \exp(k_1/k_2 - 1)$ when $\exp(k_1/k_2 - 1) \leq 1/2$ in the iteration scheme (3.11) when (3.10) holds, the iteration converges robustly. A MATLAB script for this algorithm is shown in Figure 1.

This script was used to generate the plots in Figures 2 and 3 showing the optimum proportion of stock as a function of k_1 for the case of uniform write downs for $k_2 = 0.25(0.25)2.5$.

3.5. Beta Distributed Write-Downs ($\lambda > 0$)

The uniform case can be generalized somewhat by taking the write-down distribution to be modeled by the two-parameter beta distribution with density

$$p(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1} 1_{(0,1)} \tag{3.14}$$

This allows more flexibility in modeling the write-down distribution, having the ability to reflect a mode at or shifted toward 0 or 1, or even a bimodal distribution with modes at both 0 and 1. Here, $\alpha > 0$, and $\beta > 0$ are parameters ($\alpha = \beta = 1$ yields the uniform case). It follows that

$$E\left(\frac{Y_1^2}{1 - Y_1}\right) = \frac{\alpha}{\alpha + \beta} \frac{\alpha + 1}{\beta - 1} \tag{3.15}$$

provided $\beta > 1$, and the expectation is $+\infty$ otherwise. The mean write-down is

$$EY_1 = \frac{\alpha}{\alpha + \beta} \tag{3.16}$$

so in terms of k_1 and k_2 defined previously, it follows from Lemma 1 that the optimum value of $C(f)$ in the interval $[0, 1]$ is strictly in the interval $(0, 1)$ when

$$-k_2 \frac{\alpha}{\beta - 1} < k_1 < 1 - k_2 \frac{\alpha}{\alpha + \beta} \tag{3.17}$$

when $\beta > 1$(when $0 < \beta \leq 1$, the left side of the inequality is $-\infty$); the optimum value of $C(f)$ is $f^* = 1$ when $k_1 \geq 1 - k_2\alpha/(\alpha + \beta)$, and it is $f^* = 0$ when $k_1 \leq -k_2\alpha/(\beta - 1)$ when $\beta > 1$. When $0 < \beta \leq 1, f = 0$ is never optimal.

Unlike the uniform case, the expression $E[\ln(1 - (1 - f)Y_1)]$ that appears in (2.1) cannot be computed in closed form for general $\alpha > 0$ and $\beta > 0$, and so numerical integration or a discrete approximation to the distribution must be employed to calculate $E(\frac{Y_1}{1-(1-f)Y_1})$ for varying $f \in (0, 1)$ in order to solve (2.2). Fortunately, a secant iteration procedure, instead of Newton iteration, avoids the need to compute a second derivative of $C(f)$ (see (4.13)).

4. APPROXIMATION METHODS

Other than some of the aforementioned cases, closed form solutions to finding the optimum strategy seem rare. In fact, the situation is likely to be even more complex than the case of uniform write-downs illustrated above. Still, conditions (2.9) and (2.10) are useful for determining where to look for an optimum solution. When an optimum value f^* is to be found in (0, 1), approximations to (2.2) based on series expansions can yield some useful approximations and have the advantage that it is only necessary to know a finite number of moments of the write-down distribution. These are illustrated below for the case that $P\{0 < Y_1 < 1\} = 1$.

First, when the optimum is in (0, 1), the condition that yields $C'(f) = 0$ can be expressed as

$$\frac{\lambda}{\sigma^2} E\left(\frac{Y_1}{1 - (1 - f)Y_1}\right) + \frac{\mu - r}{\sigma^2} - f = 0 \tag{4.1}$$

so in general the optimum depends only on the independent parameters defined previously:

$$k_1 = \frac{\mu - r}{\sigma^2}, \qquad k_2 = \frac{\lambda}{\sigma^2} \tag{4.2}$$

As a first attempt, an expansion about 0 of the expression inside the expectation above converts (4.1) to

$$k_1 - f + k_2 \sum_{n=0}^{\infty} (1 - f)^n E(Y_1^{n+1}) = 0 \tag{4.3}$$

Here, the interchange of expectation and infinite sum is justified by Fubini's theorem because the terms in the infinite sum are all nonnegative as long as $0 \leq f \leq 1$. The first-, second-, and third-order approximations to the optimum f^* are obtained from (4.3) by taking only the first 1, 2, and 3 terms in the infinite sum and solving the resulting equation for a root in f. These yield, in turn,

$$f_1^* = k_1 + k_2 EY_1 \tag{4.4}$$

$$f_2^* = \frac{k_1 + k_2(EY_1 + E(Y_1^2))}{1 + k_2 E(Y_1^2)} \tag{4.5}$$

and

$$f_3^* = 1 - \frac{\left[-(1 + k_2E(Y_1^2)) + \sqrt{(1 + k_2E(Y_1^2))^2 - 4k_2E(Y_1^3)(k_1 + k_2EY_1 - 1)}\right]}{2k_2E(Y_1^3)} \tag{4.6}$$

as the first-, second-, and third-order approximations to the optimum f^*, respectively. Note that each of these need not lie in (0, 1), so in each case, the approximation must be amended to be 0 where the approximation is negative, and 1 where the approximation exceeds 1.

Another approach is to expand the expression inside the expectation in (4.1) into a Taylor series about $E(Y_1)$. Doing this and interchanging expectation and infinite sum leads to

$$E\left(\frac{Y_1}{1-(1-f)Y_1}\right) = \frac{EY_1}{1-(1-f)EY_1} + \sum_{n=1}^{\infty}\frac{(1-f)^{n-1}}{(1-(1-f)EY_1)^{n+1}}E((Y_1 - EY_1)^n) \tag{4.7}$$

valid for $0 < f < 1$. Again, the interchange of expectation and infinite sum are justified by Fubini's theorem since the Taylor series converges absolutely as long as $0 < f < 1$. By truncating the infinite series at $n = K$, one may attempt to approximate the solution to (4.1) by instead solving an easier polynomial root-finding problem, namely

$$k_1 - f + k_2\left(\frac{EY_1}{1-(1-f)EY_1} + \sum_{n=1}^{K}\frac{(1-f)^{n-1}}{(1-(1-f)EY_1)^{n+1}}E((Y_1 - EY_1)^n)\right) = 0 \tag{4.8}$$

Since $E(Y_1 - EY_1) = 0$, taking the truncation at $K = 1$ leads to solving a quadratic, giving the approximation

$$f_{K=1}^* = 1 - \left[\frac{1-(k_1-1)EY_1 - \sqrt{(1-(k_1-1)EY_1)^2 + 4EY_1(k_1 - 1 + k_2EY_1)}}{2EY_1}\right] \tag{4.9}$$

which coincides with (3.3) with a replaced by EY_1. For $K > 1$ solving (4.8) can be recast into finding a specific root to a polynomial of degree at most $K + 2$ for which iterative methods (e.g., Newton's method or the secant method) work very well. In both cases ($K = 1$ and $K > 1$), the approximations must be amended to be 0 where they are negative, and 1 where they are greater than 1).

As an illustration of how these approximation methods perform, the uniform write-down case is convenient since it is readily solved exactly via Newton iteration. Also, since the uniform distribution is symmetric about its mean, the odd-order central moments in (4.8) vanish making the approximation method based on that expression particularly easy to study. In the following, (4.6) and (4.8) with $K = 8$ are specialized to the uniform write-down case.

In the uniform (0, 1) case for the write-downs, $E(Y_1) = 1/2, E(Y_1^2) = 1/3$, and $E(Y_1^3) = 1/4$, so that (4.6) reduces to

$$f_3^* = 1 + \frac{2\left((1 + k_2/3) - \sqrt{(1 + k_2/3)^2 - k_2(k_1 + k_2/2 - 1)}\right)}{k_2} \tag{4.10}$$

Substituting the even-order central moments of the uniform, namely $E(Y_1 - 1/2)^{2k} = \frac{1}{2^{2k}(2k+1)}$, the expression (4.8) becomes

$$k_2\left(\frac{\rho}{2} + \rho^3\frac{(1-f)}{12} + \rho^5\frac{(1-f)^3}{80} + \rho^7\frac{(1-f)^5}{448} + \rho^9\frac{(1-f)^7}{2304}\right) + k_1 - f = 0 \tag{4.11}$$

where

$$\rho = \frac{2}{2 - (1 - f)} \tag{4.12}$$

Expressions (4.11) and (4.12) are easily solved using the secant method. Denoting the left-hand side of (4.11) by $h(f)$, this iteration procedure is

$$f_{n+1} = f_n - \frac{h(f_n)(f_n - f_{n-1})}{h(f_n) - h(f_{n-1})} \tag{4.13}$$

Using initial values $f_1 = 1/2$ when $\exp(k_1/k_2 - 1) > 1/2, f_1 = \exp(k_1/k_2 - 1)$ when $\exp(k_1/k_2 - 1) \leq 1/2$, and $f_0 = 0.8f_1$, this iteration scheme converges robustly for the uniform case.

Caution is needed in applying all of these approximations as they all tend to break down to some degree for this problem as k_1/k_2 gets small or becomes more and more negative. This is illustrated in the plot of these approximations in comparison to the exact values as shown Figure 4. In Figure 4, the first-, second-, and third-order approximations refer to

(4.4), (4.5), and (4.6) respectively, while the $K = 1$ (central) and $K = 8$ (central) refer to approximations based on (4.9) and (4.11), respectively. There it can be seen that (4.10) breaks down much earlier than the approximation based on solving (4.11), which remains fairly accurate throughout the plot. When (4.11) does begin to break down, this is caused by the fact that when $k_1/k_2 \ll 0$, the optimum f gets close to 0, and as $f \to 0+$,

$$E\left(\frac{Y_1}{1 - (1 - f)Y_1}\right) \to +\infty \tag{4.14}$$

and the finite approximations to the infinite series for this expression begin to break down in this regime. It is also noteworthy that the approximation (4.9), which is really a first-order approximation, outperforms the first- and second-order approximations (4.4) and (4.5). This, and the quality of the approximation from (4.11), is due to the more effective procedure of centering the series expansion at the mean of the write-down distribution. The quality of this approximation is very good, considering the diffuseness of the uniform distribution.

To the extent that the uniform distribution represents a rather diffuse special case of the beta distribution, it might be expected that the approximation procedure based on solving (4.8) with $K = 8$ will also work well for all unimodal beta distributed write-downs having smaller variance than the uniform. In fact, since the approximation is based on an expansion involving central moments of the write-down distribution, it can be anticipated that to cause this approximation procedure to perform at its worst, one should take a write-down distribution with the highest possible variance. To accomplish this, first note that for any random variable Y satisfying $P\{0 \le Y \le 1\} = 1$, $\mathrm{var}(Y) = E(Y - EY)^2 \le E(Y - 1/2)^2 \le 1/4$, the first inequality following because the quantity $E(Y - a)^2$ is minimized at $a = EY$, and the second inequality following because the absolute difference between any number in $[0, 1]$ and $1/2$ is always less than or equal to $1/2$. Taking Y to have a beta distribution with $\alpha = \beta = \delta$ yields a symmetric bimodal distribution with modes at 0 and 1, and letting $\delta \to 0+$, the variance of this beta distribution increases to $1/4$, and the distribution converges to the discrete distribution that places probability $1/2$ at each of the two points in $\{0, 1\}$. This limiting symmetric Bernoulli distribution achieves a variance exactly equal to the maximum possible value of $1/4$. Applying the results of Section 3.3, when the write-downs satisfy

$$P\{Y_1 = 1\} = P\{Y_1 = 0\} = 1/2 \tag{4.15}$$

and (3.10) is satisfied, the exact optimum stock proportion is found by

$$f^* = \frac{k_1 + \sqrt{k_1^2 + 2k_2}}{2} \tag{4.16}$$

When (3.10) fails, then $f^* = 1$. For this write-down distribution, the odd central moments vanish by symmetry, and the even central moments are $E(Y_1 - 1/2)^{2k} = (1/2)^{2k}$, which translates (4.8) with $K = 8$ into

$$k_2\left(\frac{\rho}{2} + \rho^3\frac{(1-f)}{4} + \rho^5\frac{(1-f)^3}{16} + \rho^7\frac{(1-f)^5}{64} + \rho^9\frac{(1-f)^7}{256}\right) + k_1 - f = 0 \tag{4.17}$$

with ρ defined as in (4.12). To examine the approximation based on solving (4.8) for this extreme case, (4.17) was solved via the secant method and compared to the exact solution (4.16). As always, a solution to (4.17) exceeding 1 was replaced by 1, and a solution less than 0 was assigned the value 0. Figure 5 is a graph of the results for the case $k_2 = 0.5$. This example is typical of the quality of the approximation for this extreme case over the range of $k_2 = 0.25(0.25)2.5$. As in Figure 4, it can be seen that the approximation tends to break down for $k_1/k_2 \ll 0$, but here the divergence is more pronounced than in Figure 4. Still, the approximation is quite good with the maximum absolute error on the order of 0.054, although the relative error there is quite high since this occurs in the region of small f^*. Of course, the approximation can be improved further by taking the truncation K to be larger. This, and the results for the uniform case, suggests that larger values of K may only be needed in the approximation based on (4.8) when the write-down distribution is much more diffuse than the uniform, and only then when dealing with the regime of small f^*.

5. CONCLUSIONS

Asset allocation in portfolios is a topic that is relevant to derivative securities and markets in many ways. Indeed, portfolios are themselves derivatives, and dynamically hedged portfolios of stocks and bonds are widely used for replicating (hence pricing) derivative securities. In this article, optimal asset allocation in a portfolio of stock and risky bond has been examined. The equation defining the optimal allocation in such a portfolio has been analyzed and some useful rules developed for determining when the solution is non-trivial. Several special cases for which exact solutions exist have been analyzed. Finally, approximations and iterative approaches to solving for the optimal strategy have been presented and illustrated.

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