THE VALUATION OF OPTIONS WITH RESTRICTIONS ON PREFERENCES AND DISTRIBUTIONS

ANTÓNIO CÂMARA

This article develops a discrete-time, risk-neutral valuation relation (RNVR) for the pricing of contingent claims when preferences in the economy are characterized by decreasing absolute risk aversion and the marginal distribution of the underlying is an inverse coshnormal. The RNVR is applied to obtain closed-form expressions for calls and puts written on nondividend-paying stocks, futures contracts, foreign currencies, and dividend-paying stocks. Such pricing equations contain two parameters, the threshold and rescale parameters, not contained in the Black–Scholes valuation equation. Inverse-coshnormal option values make the approach look interesting. © 2001 John Wiley & Sons, Inc. Jrl Fut Mark 21:1091–1117, 2001

The author is grateful for comments and suggestions from Mark Rubinstein; Richard Stapleton; anonymous referees; the editor, Robert Webb; and participants of the 2000 European Financial Management Association meeting held in Athens.
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Received September 1999; Accepted December 2000

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The Journal of Futures Markets, Vol. 21, No. 12, 1091–1117 (2001)

INTRODUCTION

Executive compensation may include options written on the stocks of the firm. Index options can be used by portfolio managers to limit the downside risk of the portfolio. Importers and exporters negotiate options written on foreign currencies to avoid exchange-rate risk. Producers may hold futures options to hedge against price declines. In general, option market participants might be interested in alternative option-pricing theories.

Option prices obtained in this article have an inverse-coshnormal, risk-neutral density[1] implicit. The literature has discussed three main methods for the risk-neutral pricing of derivative assets. The first method is found in Black and Scholes (1973) and Merton (1973). It is assumed that it is possible to construct and maintain a riskless hedge through continuous trading. This hedging portfolio implies a partial differential equation (PDE). Then, the price of the option is based on the resolution of the PDE under the appropriate boundary conditions.[2]

The second method uses the martingale approach and was initiated by Cox and Ross (1976), Harrison and Kreps (1979), and Harrison and Pliska (1981). These authors showed that a risk-neutral measure exists in an economy where there are no arbitrage opportunities and investors are nonsatiated. The current price of a derivative is given by its expected value under the equivalent martingale measure discounted at the riskless return. It is possible to use this principle to price a contingent claim if it is attainable or marketable, that is, if there is a trading strategy whose payoff replicates the payoff of the contingent claim. This means that the first two methods, when the state space is continuous, assume that markets are dynamically completed by trading.[3]

The third method, initiated by Rubinstein (1976) and Brennan (1979), uses the first-order conditions of a representative agent[4] to obtain a risk-neutral formula. A discrete-time economy with a continuous state space is assumed, which means that markets are dynamically incomplete. Brennan (1979, p. 54) argued that "in addition to the greater generality of

[1]If $\tilde{x}$ is normally distributed, $\tilde{z} = \theta + \lambda \cosh(\tilde{x})$ is an inverse-coshnormal random variable, where θ and λ are the threshold and rescale parameters, respectively, of the inverse-coshnormal density.

[2]To review this area of research, see Wilmott, Dewynne, and Howison (1993), who studied option pricing from a PDE perspective.

[3]Cox and Huang (1989) provided an excellent review article of this investigation field.

[4]The existence of a representative agent is a necessary condition to obtain risk neutral option-pricing formulae in economies not completed by continuous trading, that is, in discrete-time, continuous, state-space economies. Grossman and Zhou (1996) and Zhou (1998) showed that smiles, skews, or sneers in option markets might be generated in economies with representative agents whose utility functions display decreasing absolute risk aversion.

discrete-time models . . . the chief advantage of the discrete-time approach to the pricing of contingent claims is that the restriction of investor preferences eliminates the requirement of the continuous time approach that an instantaneously riskless portfolio may be constructed." This is the method explored in this article.

This article develops a discrete-time, risk-neutral valuation relation (RNVR) for the pricing of contingent claims when preferences in the economy are characterized by decreasing absolute risk aversion (DARA) and aggregate consumption and the underlying variables are bivariate-displaced-lognormal-inverse-coshnormal-distributed. The marginal distribution of the underlying is an inverse-coshnormal. Such characteristics distinguish this article from the existing literature. Previous literature linked either the power utility function to the lognormal family of distributions[5] or the exponential utility function to the normal distribution.[6]

Rubinstein (1976) obtained the Black–Scholes model in an economy where the representative agent has a power utility function and aggregate consumption and the underlying are bivariate lognormal. Turnbull and Milne (1991), under the same assumptions, extended the discrete-time RNVR theory to stochastic interest rates. Camara (1999) used the extended power utility and a bivariate-displaced lognormal for consumption and the underlying to obtain the displaced diffusion model of Rubinstein (1983). Brennan (1979) derived a risk-neutral valuation equation assuming an exponential utility function and that aggregate wealth and the underlying were bivariate-normal-distributed. All these models are special cases of the inverse-coshnormal RNVR from a point of view of both preferences and distributions. Hence, the research developed in this article is motivated by both preferences and option prices (which have a risk-neutral density implicit).

Elton and Gruber (1995, p. 218) argued that "while there is general agreement that most investors exhibit decreasing absolute risk aversion (DARA), there is much less agreement concerning relative risk aversion." Blume and Friend (1975, p. 603) reported that "households holding the most diversified portfolios of stocks display evidence of constant proportional risk aversion (CPRA)."[7] According to Blume and Friend (1975, p. 600), "if the net worth is defined more broadly to include homes,

[5]See Rubinstein (1976), Turnbull and Milne (1991), and Camara (1999).

[6]See Brennan (1979).

[7]Mehra and Prescott (1985), who provided several references that document evidence of CPRA, showed that one would need a very large coefficient of relative risk aversion to explain the risk premium on U.S. equities, and they raised doubts about the reasonability of such an assumption.

there is some evidence of increasing relative risk aversion." The results of the research carried out by Blume and Friend (1975, p. 600) also suggest that "if the net worth is defined to exclude housing, there is a slight tendency for decreasing relative risk aversion." Cohn, Lewellen, Lease, and Schlarbaum (1975, p. 606)[8] produced a "study [that] yields strong evidence of decreasing proportional risk aversion (DPRA)." Zhou (1998, p. 1730) presented empirical results that "justify [the] assumption that consumer's utility function exhibits DPRA."

The lognormal price assumption $\Lambda^P(\mu, \sigma)$ underlying the Black–Scholes formulae has been criticized effectively on the basis of empirical studies of actual data from various underlying assumptions. A natural way to extend the model is through the lognormal family of distributions. To add flexibility to the model, one can try to develop a risk-neutral option-pricing equation based on a four-parameter lognormal, $\Lambda^P(\mu, \sigma, \theta, \lambda)$. Unfortunately, the so-called four-parameter lognormal is really only a displaced lognormal, $\Lambda^P(\mu, \sigma, \theta)$; that is, it only has three independent parameters.[9] The actual displaced lognormal is displaced from the origin by its threshold parameter or lower bound θ. The shape and the higher moments (i.e., variance, skewness, and kurtosis) of the two actual densities (i.e., the lognormal and displaced lognormal) are the same.[10] The actual densities are distinguished only through their measures of location.[11] These facts help to motivate alternative specifications for the underlying. This article proposes the inverse-coshnormal distribution, which has four independent parameters and contains all the distributions of the lognormal family as special cases. The inverse-coshnormal distribution, because it has four independent parameters, is able to fit accurately four moments of the actual distribution of the underlying. Examples of this for the Standard & Poor's (S&P) 500 index and the United States Dollar/Great Britain Pound (USD/GBP) exchange rate will be provided later in another article.

The empirical research seems to indicate that when the Black–Scholes equation is used to price index options, it leads to different

[8]Cohn et al. (1975) presented other sources that provide evidence of DPRA.

[9]As remarked by Johnson, Kotz, and Balakrishnan (1994, p. 209), the four-parameter lognormal $\tilde{z}$ is defined by $\tilde{x} = \mu + \sigma \ln(\frac{\tilde{z} - \theta}{\lambda})$, where $\tilde{x}$ is a unit normal variable. This equation can be rewritten as $\tilde{x} = \mu' + \sigma \ln(\tilde{z} - \theta)$ with $\mu' = \mu - \sigma \ln \lambda$, which is only a displaced lognormal. Because neither μ nor μ' affects the option-pricing equation, one concludes that the four-parameter, lognormal option-pricing equation is exactly the same as the displaced-lognormal option-pricing equation.

[10]For a discussion of this and other points concerning the lognormal family of distributions, see Crow and Shimizu (1988).

[11]The location characteristics of $\Lambda^P(\mu, \sigma, \theta)$ are greater by θ than those of $\Lambda^P(\mu, \sigma)$.

deviates[12] than when the formula is used to price stock options, foreign currency options, and futures options.

Jackwerth and Rubinstein (1996) and Stutzer (1996) documented that the Black–Scholes model overprices out-of-the-money calls and underprices in-the-money calls in the stock index options market. In general, the inverse-coshnormal, risk-neutral option-pricing formula, when the threshold parameter is negative and the rescale parameter is positive, gives a lower (higher) price for out-of-the-money (in-the-money) call options than the Black–Scholes model. Hull (1997) discussed the relation between the tails of an arbitrary, risk-neutral density (relative to the lognormal, risk-neutral density) and option prices as given by an arbitrary valuation equation (relative to Black–Scholes values). An arbitrary option-pricing equation that gives lower (higher) prices for out-of-the-money (in-the-money) call options than the Black–Scholes model has implicit a risk-neutral density that has a fatter (thinner) left (right) tail than the lognormal, risk-neutral density. It is shown that, for a negative threshold parameter θ and a positive rescale parameter λ, this is the behavior of the inverse-coshnormal, risk-neutral density (when compared to the lognormal, risk-neutral density). Similar patterns to this one for the S&P 500 index options market were reported by Jackwerth and Rubinstein (1996) and Aït-Sahalia and Lo (1998). For the same set of parameters, the inverse-coshnormal option-pricing equation is able to generate implied Black–Scholes volatilities that decrease when the strike price increases. Dumas, Fleming, and Whaley (1998) for the S&P 500 options market reported a sneer with this pattern.

Black (1975) and Gultekin, Rogalski, and Tinic (1982) showed that the Black–Scholes equation underprices out-of-the-money calls and overprices in-the-money calls in the stock options market.[13] Chang, Chang, and Lim (1998) and Zhou (1998) reported similar results for the foreign currency options market and the agricultural futures options market, respectively. In general, the inverse-coshnormal, risk-neutral option-pricing formula, when the threshold parameter is nonnegative

[12]Any null hypothesis to test the Black–Scholes model is a joint hypothesis that (1) the Black–Scholes model is correctly specified and (2) markets are efficient. If the null hypothesis is rejected, it may be the case that (1) the Black–Scholes formulae misprice options in an efficient market, (2) the Black–Scholes prices are the correct ones and the market itself has mispriced them (in which case, there would theoretically be an arbitrage opportunity), and (3) the Black–Scholes equations misprice options and markets are inefficient. If one accepts that the option market is efficient, the statement that the Black–Scholes model misprices in/out-of-the-money options might make sense.

[13]Opposite results in the stock options market were obtained by Macbeth and Merville (1979) for a sample of six stocks during 1976.

and the rescale parameter is positive, gives a higher (lower) price for out-of-the-money (in-the-money) call options than the Black–Scholes model. This is consistent with an inverse-coshnormal, risk-neutral density that has a thinner (fatter) left (right) tail than the lognormal, risk-neutral density. When both θ and λ are positive, the inverse-coshnormal option-pricing model is able to generate Black–Scholes implied volatilities that increase with the exercise price. Chang et al. for currency options and Zhou for commodities futures documented sneers with such a pattern.

The goals of this article are to (1) construct an inverse-coshnormal distribution and discuss some of its properties, (2) establish an RNVR for such a distribution, (3) apply the RNVR to obtain closed-form expressions for several options, and (4) simulate and analyze the model.

PREFERENCES AND DISTRIBUTIONS

This section begins by characterizing preferences and the joint distribution of consumption and the underlying. The marginal distribution of the underlying is an inverse coshnormal. The section concludes comparing the risk-neutral, lognormal density with the risk-neutral, inverse coshnormal density.

Definition 1. The extended power marginal utility function of consumption is given by

$$U'(C) = (C - \tau)^{\varphi} \tag{1}$$

with $\varphi < 0$ and $C > \tau$. This represents increasing, constant, and decreasing proportional risk aversion (denoted IPRA, CPRA, and DPRA, respectively) for $\tau < 0$, $\tau = 0$, and $\tau > 0$, respectively. This utility function represents DARA.

Definition 2. The bivariate-displaced-lognormal-inverse-coshnormal distribution of aggregate consumption, C, and underlying, S, is defined by[14]

$$\left[\ln(C - \tau), \cosh^{-1}\left(\frac{S - \theta}{\lambda}\right)\right] =^{d} N^{P}\left([\mu_c, \mu], \begin{bmatrix} \sigma_c^2 & \rho\sigma\sigma_c \\ \rho\sigma\sigma_c & \sigma^2 \end{bmatrix}\right)$$

with $\tau < C < \infty$, $\theta + \lambda < S < \infty$, where σ_c, σ, μ_c, μ, ρ, θ, λ, and τ are constants such that $-1 < \rho < 1$, $0 < \sigma_c$, $0 < \sigma$, $-\infty < \mu_c < \infty$, $-\infty < \mu < \infty$, $0 < \lambda$, $-\infty < \theta < \infty$, and $-\infty < \tau < \infty$. Here, $=^d$ signifies

[14]This representation is general in the sense that it also covers the singular case (i.e., when the covariance matrix is singular and the density does not exist).

equal in distribution, and $N^P(m, \Sigma)$ denotes an actual normal vector with mean vector m and variance/covariance matrix Σ.

Lemma 1 (the marginal inverse-coshnormal distribution). Suppose that consumption and the underlying variable have a displaced-lognormal-inverse-coshnormal distribution. Then, the underlying variable S has a marginal inverse-coshnormal distribution $\nabla^P(\mu, \sigma, \theta, \lambda)$. If there is a density, it is given by

$$f(S) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{(S-\theta)^2 - \lambda^2}} \exp\left[-\frac{1}{2}\left(\frac{\cosh^{-1}(\frac{S-\theta}{\lambda}) - \mu}{\sigma}\right)^2\right] \tag{2}$$

for $S > \theta + \lambda$, where μ, σ, θ, and λ are constants such that $-\infty < \mu < \infty$, $-\infty < \theta < \infty$, $0 < \sigma$, and $0 < \lambda$.[15]

Proof. The marginal density of S is by definition $f(S) = \int_{-\infty}^{\infty} f(C, S)\, dC$, where $f(C, S)$ denotes the joint density when it exists. Solving this equation yields the desired result.[16]

Because the inverse coshnormal has four independent parameters, it fits accurately the first four moments of the actual data,[17] whereas the lognormal underlying the Black–Scholes formulae will probably only fit accurately two moments of the data. Figure 1 display an inverse coshnormal fit to monthly S&P 500 index returns statistics derived by Corrado and Su (1997). The actual data (for the period 1986–1995) has a mean of 0.118, a variance of 0.0222, a skewness of −1.19, and a kurtosis of 9.27. Figure 2 displays an inverse coshnormal fit to the daily spot exchange-rate ($/pound) statistics calculated by Sarwar and Krehbiel (2000). The data (for the period 1993–1995) have a mean of 1.54, a variance of 0.002916, a skewness of 0.77, and a kurtosis of 4.86. Although this article assumes an actual inverse coshnormal, the distribution that affects option prices is the risk-neutral inverse coshnormal.

As Cox and Ross (1976) first pointed out, if an RNVR exists, the following equality is obtained:

$$E^Q[S|V] = Vr$$

[15]If negative skewness is desired with $\lambda > 0$, then $\cosh^{-1}(\frac{\theta - S}{\lambda})$ replaces $\cosh^{-1}(\frac{S-\theta}{\lambda})$. Alternatively, λ might assume any value on the real line with the original definition of the density. In this case, the density has positive (negative) skewness if the actual λ is positive (negative).

[16]A detailed proof of the marginal and conditional distributions derived in this article is available on request.

[17]The appendix, in the proof of lemma 2, derives the first four moments of an actual inverse-coshnormal distribution.

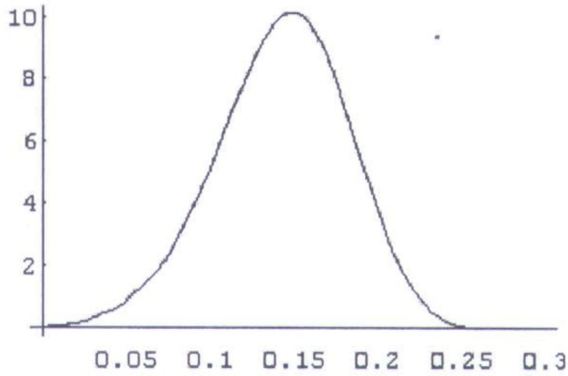

FIGURE 1
Inverse-coshnormal density function. This density accurately fits the first four moments of monthly S&P 500 index return statistics (1986–1995): $E^P(S) = 0.118$; $\mathrm{Var}^P(S) = 0.222$; $\mathrm{Ske}^P(S) = -1.19$; and $\mathrm{Kur}^P(S) = 9.27$. The statistics were calculated and reported by Corrado and Su (1997).

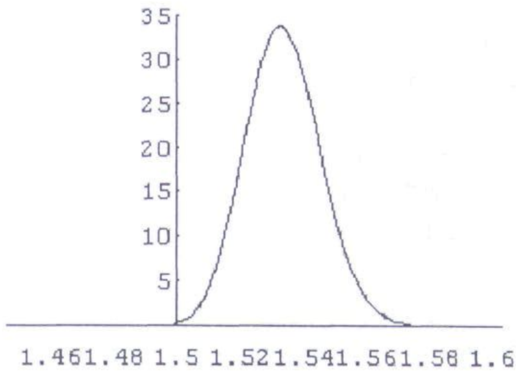

FIGURE 2
Inverse-coshnormal density function. This density accurately fits the first four moments of daily spot exchange rate ($/pound) statistics (1993–1995): $E^P(S) = 1.54$; $\mathrm{Var}^P(S) = 0.0029$; $\mathrm{Ske}^P(S) = 0.77$; and $\mathrm{Kur}^P(S) = 4.86$. The statistics were calculated and repoted by Sarwar and Krehbiel (2000).

where V is the current value of the underlying, r is one plus the interest rate, and the conditional expectation is taken with respect to the risk-neutral measure Q.

The next proposition follows from imposing the previous risk-neutral restriction to an underlying inverse coshnormal distributed.

Proposition 1 (the inverse-coshnormal risk-neutral density). If $f(S)$ is the inverse-coshnormal actual density of the underlying stochastic variable with parameters μ, σ, θ, and λ, that is, $f(S)$ is $\nabla^P(\mu, \sigma, \theta, \lambda)$, then $g(S)$ is its inverse-coshnormal, risk-neutral density with scale parameter $\cosh^{-1}[\frac{Vr - \theta}{\lambda \exp(\sigma^2/2)}]$, shape parameter σ, threshold parameter θ, and rescale parameter λ; that is, $g(S)$ is $\nabla^Q(\cosh^{-1}[\frac{Vr - \theta}{\lambda \exp(\sigma^2/2)}], \sigma, \theta, \lambda)$. The limiting properties of the inverse-coshnormal distribution are now investigated.

Lemma 2 (the moments of the inverse-coshnormal, risk-neutral distribution). The mean, variance, skewness,[18] and kurtosis of the inverse-coshnormal, risk-neutral distribution are given by

$$E^Q(S_\nabla) = Vr$$

$$\mathrm{Var}^Q(S_\nabla) = (Vr - \theta)^2(\omega - 1) - 1/2\lambda^2(\omega^2 - 1)$$

$$\mathrm{Ske}^Q(S_\nabla) = \pm\left[\frac{(\omega + 2)(\omega - 1)^{1/2}}{[1 - 1/2\lambda^2 \frac{\omega + 1}{(Vr - \theta)^2}]^{3/2}} - 3/4\lambda^2 \frac{(Vr - \theta)(\omega + 1)^2(\omega - 1)^{1/2}}{[(Vr - \theta)^2 - 1/2\lambda^2(\omega + 1)]^{3/2}}\right]$$

$$\mathrm{Kur}^Q(S_\nabla) = \frac{\omega^4 + 2\omega^3 + 3\omega^2 - 3}{[1 - 1/2\lambda^2 \frac{\omega + 1}{(Vr - \theta)^2}]^2} + \lambda^2 \frac{(Vr - \theta)^2(\omega - 3\omega^2 + 3\omega^4 - \omega^7)}{[(Vr - \theta)^2(\omega - 1) - 1/2\lambda^2(\omega^2 - 1)]^2} + \lambda^4 \frac{1/8\omega^8 - 1/2\omega^2 + 3/8}{[(Vr - \theta)^2(\omega - 1) - 1/2\lambda^2(\omega^2 - 1)]^2}$$

where $\omega = \exp(\sigma^2)$ and S_∇ denotes that S has an inverse-coshnormal distribution.

Proof. See the appendix.

The mean of any risk-neutral distribution is Vr. The limits of the variance, skewness, and kurtosis of the inverse-coshnormal distribution when λ approaches zero from the right are, respectively, the

[18]If the density is positively (negatively) skewed, the skewness has the sign + (−).

variance, skewness, and kurtosis of the displaced-lognormal, risk-neutral density $\Lambda^Q(\ln(Vr-\theta)-\frac{1}{2}\sigma^2,\sigma,\theta)$. If, additionally, $\theta=0$, the moments of the lognormal, risk-neutral density implicit in the Black–Scholes model is obtained, as one can see in the next proposition.

Proposition 2 (the limits of the risk-neutral variance, skewness, and kurtosis). Suppose that there is a positively skewed, inverse-coshnormal, risk-neutral distribution.

i. Let $\lambda\to 0$. Then,

$$\begin{aligned}\lim_{\lambda\to 0}\mathrm{Var}^Q(S_\nabla) &= \mathrm{Var}^Q(S_\Lambda)\\ \lim_{\lambda\to 0}\mathrm{Ske}^Q(S_\nabla) &= \mathrm{Ske}^Q(S_\Lambda)\\ \lim_{\lambda\to 0}\mathrm{Kur}^Q(S_\nabla) &= \mathrm{Kur}^Q(S_\Lambda)\end{aligned}$$

ii. Let both $\lambda\to 0$ and $\theta=0$. Then,

$$\begin{aligned}\lim_{\lambda\to 0}\mathrm{Var}^Q(S_\nabla) &= \mathrm{Var}^Q(S_{\Lambda_{\theta=0}})\\ \lim_{\lambda\to 0}\mathrm{Ske}^Q(S_\nabla) &= \mathrm{Ske}^Q(S_{\Lambda_{\theta=0}})\\ \lim_{\lambda\to 0}\mathrm{Kur}^Q(S_\nabla) &= \mathrm{Kur}^Q(S_{\Lambda_{\theta=0}})\end{aligned}$$

where S_Λ and $S_{\Lambda_{\theta=0}}$ denote that S has a displaced-lognormal distribution and a lognormal distribution, respectively.

Proof. It follows from using standard limit properties, the moments of the risk-neutral lognormal, and the results of lemma 2.

It is interesting to investigate whether it is possible to derive stronger results than the convergence of the variance, skewness, and kurtosis. Such results are presented in the next proposition.

Proposition 3 (the limit of the inverse-coshnormal, risk-neutral density). Suppose that there is a positively skewed, inverse-coshnormal, risk-neutral density.

i. Let $\lambda\to 0$. Then,

$$\lim_{\lambda\to 0}\nabla^Q\left(\cosh^{-1}\left[\frac{Vr-\theta}{\lambda\exp(\frac{\sigma^2}{2})}\right],\sigma,\theta,\lambda\right)=\Lambda^Q\left[\ln(Vr-\theta)-\frac{1}{2}\sigma^2,\sigma,\theta\right]$$

ii. Let both $\lambda\to 0$ and $\theta=0$. Then,

$$\lim_{\lambda\to 0}\nabla^Q\left(\cosh^{-1}\left[\frac{Vr-\theta}{\lambda\exp(\frac{\sigma^2}{2})}\right],\sigma,\theta,\lambda\right)=\Lambda^Q_{\theta=0}\left(\ln V+\ln r-\frac{1}{2}\sigma^2,\sigma\right)$$

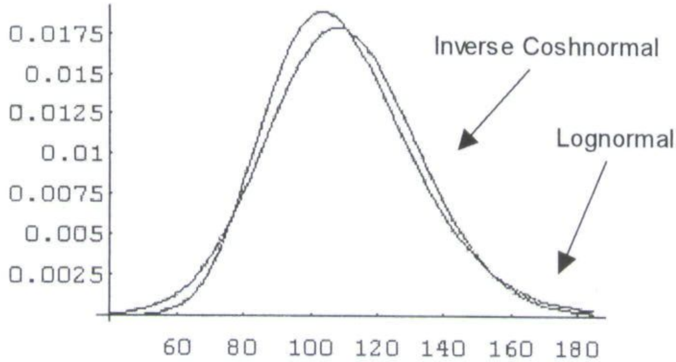

FIGURE 3

This figure displays two risk-neutral densities with the same mean $E^Q(S) = 110$ and variance $\mathrm{Var}^Q(S) = 493.1$: an inverse coshnormal $g(S) \sim \nabla^Q\,(4.38, 0.056, -290, 10)$ and a lognormal $g(S) \sim \Lambda^Q_{\theta=0}\,(4.68, 0.2)$. The picture "is consistent with the S&P 500 index distribution having a fatter left tail and a thinner right tail than a lognormal distribution has" (Stutzer, 1996, p. 1634).

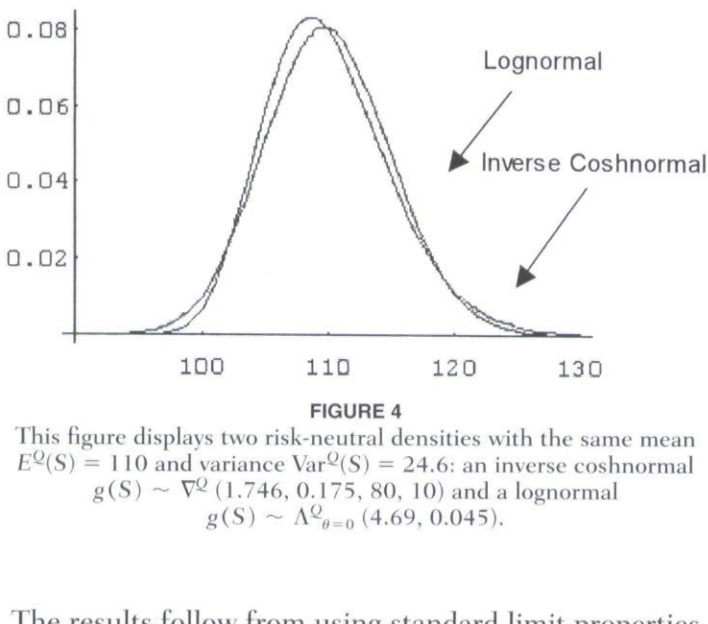

FIGURE 4

This figure displays two risk-neutral densities with the same mean $E^Q(S) = 110$ and variance $\mathrm{Var}^Q(S) = 24.6$: an inverse coshnormal $g(S) \sim \nabla^Q\,(1.746, 0.175, 80, 10)$ and a lognormal $g(S) \sim \Lambda^Q{}_{\theta=0}\,(4.69, 0.045)$.

Proof. The results follow from using standard limit properties.

This proposition shows that the inverse-coshnormal, risk-neutral density is an extension of the lognormal, risk-neutral density. Next, it is be seen that the inverse-coshnormal density has the features of the state-price density observed empirically. Figures 3 and 4 display two risk-neutral densities with the same mean and standard deviation: an inverse coshnormal and a lognormal. The inverse coshnormal has a thinner or fatter right tail and a fatter or thinner left tail than the lognormal, meaning less or more skewness than the one implied by the Black–Scholes equation.[19] Jackwerth and Rubinstein (1996), Stutzer (1996), and Aït-Sahalia and Lo (1998) documented that the distribution implied by S&P 500 option prices has a fatter left tail and a thinner right tail than the lognormal, indicating an implicit density by option market prices with less skewness than that predicted by the Black–Scholes model. Zhou (1998) presented results for agricultural options that are consistent with an implied density by option market prices with a thinner left tail and a fatter right tail than the lognormal. Similar results for currencies are obtained by Chang et al. (1998), indicating an implicit density by option market prices with more skewness than the predicted by the Black–Scholes model.

GENERAL VALUATION MODEL

This section assumes that the aggregation problem is already solved, which is a sufficient condition for prices to be derived in a representative agent economy.[20] The representative individual, who has a time-additive separable utility function of consumption, optimizes the expected utility function of consumption both at the beginning and end of the economy:

$$\max_{C_0, C_1} E^P\{U(C_0, C_1)\}$$

where

$$C_1 = (W_0 - C_0)r + \sum_k \sum_j n_{kj}[C_k(S_{j1}) - P_{kj}r]$$

and E^P (.) is the expectations operator under the true probability measure; C_0 is the aggregate consumption at the beginning of the period; C_1 is the aggregate consumption at the end of the period; $U(C_0)$ is the utility for beginning-of-period consumption of the representative agent; $U(C_1)$ is the utility for end-of-period consumption of the representative agent; W_0 is the initial wealth of the representative agent; r is the riskless return, or one plus the interest rate;[21] n_{kj} is the demand for units of claims ($k = 1, \ldots, K$ for all j); $C_k(S_{j1})$ is the end-of-period payoff associated

[19]An analysis similar to this could be done with the skewness premium derived by Bates (1991).

[20]The aggregation problem is solved when prices in the economy are determined independently of the distribution of initial wealth. See, for example, Huang and Litzenberger (1988) for a review of the literature. Rubinstein (1974) derived sufficient conditions that were subsequently shown by Brennan and Kraus (1978) to also be necessary. These conditions are that all investors have identical cautiousness and beliefs or that investors have exponential utility functions.

[21]The interest rate is constant as in the Black–Scholes economy. Hence, the price of a bond that pays \$1 at the end-of-period is $B = \frac{1}{r}$.

with the contingent claim k as a function of the underlying variable j; P_{kj} is the current price of the claim $C_k(S_{j1})$; and S_{j1} is the end-of-period value of an arbitrary underlying variable j.[22] Maximizing the expected utility of consumption of the representative agent yields the current price of individual claims in terms of their end-of-period payoffs:[23]

$$P = r^{-1}E^P[C(S)Z(S)] \tag{3}$$

where

$$Z(S) = \frac{E^P[U'(C)|S]}{E^P[U'(C)]} \tag{4}$$

defines the pricing kernel.[24] In particular, the underlying risks S can themselves be priced with a particular application of the general formula:

$$V = r^{-1}E^P[SZ(S)] \tag{5}$$

where V is the current value of a nondividend-paying underlying.[25]

The remaining results of this article are derived under the following assumption.

Fundamental assumption (DARA and bivariate-displaced-lognormal-inverse-coshnormal distribution). The representative agent has an extended power utility function of consumption and, in particular, an extended power marginal utility function of consumption given by Equation 1, and aggregate consumption and the random payoff of the underlying variable are bivariate-displaced-lognormal-inverse-coshnormal-distributed as in Definition 2.

[22]Later, the arbitrary underlying variable S is further specified to derive equations for option prices written on stocks, futures, and foreign currencies.

[23]The subscripts associated with the claims are dropped henceforth.

[24]This equation is standard in the preference-based option-pricing literature. See, for example, Brennan (1979, pp. 57, 59, Equation 14). Brennan (1979, p. 57) defined $Z(S)$ as the "conditional expected relative marginal utility function" or the "conditional marginal utility function." The equation is obtained with the law of iterated expectations in the price of the claim.

[25]Equation 5, if r is an interest rate compounded continuously, is $V = e^{-rt}E^P[SZ(S)]$, and a similar adjustment should be done in Equation 3. There are alternative versions of Equation 5 depending on what the underlying is. If the underlying is a futures contract, Equation 5 is given by $F = E^P[SZ(S)]$, where F is the futures price. If the underlying is the spot exchange rate (i.e., the current price in dollars of one unit of foreign currency), Equation 5 is given by $V = e^{-(r-r_f)t}E^P[SZ(S)]$, where r_f is the risk-free interest rate in the foreign currency. If the underlying is a stock that pays a known dividend, where D is the present value of the dividends discounted at the riskless rate, Equation 5 is $V = D + e^{-rt}E^P[SZ(S)]$. The maturity of the bank accounts and the maturity of the option contracts should coincide with the end of the economy.

The RNVR is derived in two steps. The first step evaluates the pricing kernel (Equation 4).

Lemma 3 (the form of the pricing kernel). Suppose that the fundamental assumption holds. Then, the pricing kernel (Equation 4) can be written as[26]

$$Z(S) = \exp\left[\rho\varphi\frac{\sigma_c}{\sigma}\left[\cosh^{-1}\left(\frac{S-\theta}{\lambda}\right)-\mu\right]-\frac{1}{2}\varphi^2\sigma_c^2\rho^2\right] \tag{6}$$

Proof. See the appendix.

Proposition 4 (sufficient condition for the discrete-time RNVR). Suppose that the fundamental assumption holds. Then, there is a risk-neutral valuation relationship:

$$E^P[C(S)Z(S)] = E^Q[C(S)]$$

where $E^Q(.)$ is the expectations operator with respect to the inverse-coshnormal, risk-neutral density.

Proof. Assume that the fundamental assumption holds. Then, by Lemma 1 the marginal distribution of S is an inverse coshnormal with the density given by Equation 2.

The current value of the underlying stochastic variable (Equation 5), which has a marginal inverse coshnormal distribution as in Equation 2, after the pricing kernel (Equation 6) is used and the resulting expression is simplified is given by

$$\begin{aligned} Vr = &\int_{-\infty}^{\infty} S\frac{1}{\sqrt{2\pi}\sigma\sqrt{(S-\theta)^2-\lambda^2}} \\ &\times \exp\left[-\frac{1}{2\sigma^2}\left[\cosh^{-1}\left(\frac{S-\theta}{\lambda}\right)-(\mu+\varphi\sigma_w\rho\sigma)\right]^2\right]dS \end{aligned} \tag{7}$$

The underlying equilibrium of the risk-averse economy is obtained when Equation 7 is evaluated. It follows directly that

$$\varphi\sigma_w\rho\sigma+\mu = \cosh^{-1}\left[\frac{Vr-\theta}{\exp(\frac{1}{2}\sigma^2)}\right] \tag{8}$$

The current value of the contingent claim (Equation 3), because the underlying variable has a marginal inverse-coshnormal distribution as in

[26]Brennan (1979, Equations 17 and 31) derived equations of this type for the pricing kernel under the assumptions of a bivariate normal and a bivariate lognormal. The pricing kernel has an exponential form and, therefore, is always positive.

Equation 2, after the pricing kernel (Equation 6) is used and the resulting expression is simplified, can be written as

$$Pr = \int_{-\infty}^{\infty} C(S)\frac{1}{\sqrt{2\pi}\sigma\sqrt{(S-\theta)^2-\lambda^2}} \times \exp\left[-\frac{1}{2\sigma^2}\left[\cosh^{-1}\left(\frac{S-\theta}{\lambda}\right)-(\varphi\sigma_u\rho\sigma+\mu)\right]^2\right]dS \quad (9)$$

Using the equilibrium relation (Equation 8) to eliminate preference parameters from the valuation equation (Equation 9) yields

$$Pr = \int_{-\infty}^{\infty} C(S)\frac{1}{\sqrt{2\pi}\sigma\sqrt{(S-\theta)^2-\lambda^2}} \times \exp\left[-\frac{1}{2\sigma^2}\left[\cosh^{-1}\left(\frac{S-\theta}{\lambda}\right)-\cosh^{-1}\left[\frac{Vr-\theta}{\lambda\exp(\frac{1}{2}\sigma^2)}\right]\right]^2\right]dS \quad (10)$$

which is an RNVR.

This RNVR is consistent with the result given by Proposition 1 because it has an inverse-coshnormal, risk-neutral density implicit. The next section applies the RNVR to obtain closed-form expressions for calls and puts written on nondividend-paying stocks, futures contracts, foreign currencies, and dividend-paying stocks.

APPLICATIONS

The RNVR derived in the last section is general in the sense that it can be used to price any derivative. Two applications with particular interest, on the basis of continuously compounded interest rates, are given in the next corollary.

Corollary 1 (call and put option prices on a nondividend-paying stock). Suppose that the fundamental assumption holds. Then,

i. The price of an European call option written on a nondividend-paying stock is given by the formula

$$\begin{aligned} P_c &= \frac{\lambda}{2}e^{(\frac{\sigma^2}{2}-r)t}(\gamma_1+\sqrt{\gamma_1^2-1})N(\gamma_2+\sigma\sqrt{t}) \\ &+ \frac{\lambda}{2}e^{(\frac{\sigma^2}{2}-r)t}(\gamma_1+\sqrt{\gamma_1^2-1})^{-1}N(\gamma_2-\sigma\sqrt{t}) \\ &- (K-\theta)e^{-rt}N(\gamma_2) \end{aligned} \quad (11)$$

ii. The price of an European put option written on a nondividend-paying stock is given by the formula

$$\begin{aligned} P_P = {} & (K - \theta)e^{-rt}N(-\gamma_2) \\ & - \frac{\lambda}{2}e^{(\frac{\sigma^2}{2}-r)t}(\gamma_1 + \sqrt{\gamma_1^2 - 1})N(-\gamma_2 - \sigma\sqrt{t}) \\ & - \frac{\lambda}{2}e^{(\frac{\sigma^2}{2}-r)t}(\gamma_1 + \sqrt{\gamma_1^2 - 1})^{-1}N(-\gamma_2 + \sigma\sqrt{t}) \end{aligned} \tag{12}$$

where

$$\begin{aligned} \gamma_1 &= \frac{Ve^{rt} - \theta}{\lambda \exp(\frac{\sigma^2}{2}t)} \\ \gamma_2 &= \frac{\cosh^{-1}(\gamma_1) - \cosh^{-1}(\frac{K-\theta}{\lambda})}{\sigma\sqrt{t}} \end{aligned} \tag{13}$$

K is the exercise price, and $N(.)$ is the cumulative distribution function of a standard normal random variable.

Proof. Because $g(S) \sim \nabla^Q(\cosh^{-1}[\frac{Ve^{rt} - \theta}{\lambda \exp(\sigma^2 t/2)}], \sigma\sqrt{t}, \theta, \lambda)$, the desired result is obtained.

Corollary 2 (Rubinstein, 1983, and Black & Scholes, 1973). Assume the hypothesis of Corollary 1.

i. Let $\lambda \to 0$. Then, the price of a call (put) option given by the inverse-coshnormal, risk-neutral equation [Equation 11 (12)] converges to the price of a call (put) option given by the displaced diffusion model of Rubinstein (1983).

ii. Let $\lambda \to 0$ and $\theta = 0$. Then, the price of a call (put) option given by the inverse-coshnormal, risk-neutral equation [Equation 11 (12)] converges to the price of a call (put) option given by the Black–Scholes (1973) model.

Proof. See the appendix.

Corollary 3 (call and put option prices on a futures contract). Suppose that the fundamental assumption holds. Then, the prices of European call and put options written on a futures contract are given by Equations 11 and 12, where γ_2 is given by Equation 13 and

$$\gamma_1 = \frac{F - \theta}{\lambda \exp(\frac{\sigma^2}{2}t)}$$

Proof. Because $g(S) \sim \nabla^{Q}(\cosh^{-1}[\frac{F-\theta}{\lambda \exp(\sigma^2 t/2)}], \sigma\sqrt{t}, \theta, \lambda)$, the desired result is obtained.

Corollary 4 (call and put option prices on a foreign currency). Suppose that the fundamental assumption holds. Then, the prices of European call and put options written on a foreign currency are given by Equations 11 and 12, where γ_2 is given by Equation 13 and

$$\gamma_1 = \frac{Ve^{(r-r_f)t} - \theta}{\lambda \exp(\frac{\sigma^2}{2}t)}$$

Proof. Because $g(S) \sim \nabla^{Q}(\cosh^{-1}[\frac{Ve^{(r-r_f)t} - \theta}{\lambda \exp(\sigma^2 t/2)}], \sigma\sqrt{t}, \theta, \lambda)$, the desired result is obtained.

Corollary 5 (Black, 1976, and Garman & Kohlhagen, 1983). Assume the hypothesis of Corollaries 3 and 4.

i. Let $\lambda \to 0$ and $\theta = 0$. Then, the price of a call (put) option on a futures contract given by the inverse-coshnormal, risk-neutral equation converges to the price of a call (put) option given by the Black (1976) model.

ii. Let $\lambda \to 0$ and $\theta = 0$. Then, the price of a call (put) option on a foreign currency given by the inverse-coshnormal, risk-neutral equation converges to the price of a call (put) option given by the Garman and Kohlhagen (1983) model.

Proof. The results follow with arguments similar to those applied in the proof of Corollary 2.

Corollary 6 (call and put option prices on a dividend-paying stock). Suppose that the fundamental assumption holds. Then, the prices of European call and put options written on a dividend-paying stock are given by Equations 11 and 12, where γ_2 is given by Equation 13 and

$$\gamma_1 = \frac{(V - D)e^{rt} - \theta}{\lambda \exp(\frac{\sigma^2}{2}t)}$$

Proof. Because $g(S) \sim \nabla^{Q}(\cosh^{-1}[\frac{(V-D)e^{rt} - \theta}{\lambda \exp(\sigma^2 t/2)}], \sigma\sqrt{t}, \theta, \lambda)$, the desired result is obtained.

To compare these equations with the Black–Scholes family of valuation equations, the inverse-coshnormal call option-pricing formula is simulated, holding both the risk-neutral mean and the risk-neutral variance constant, for several combinations of σ, θ, and λ. It is assumed that $E^Q(S_\nabla) = 110$ and $\text{Var}^Q(S_\nabla) = 493.1$. It is also assumed that the underlying is a nondividend-paying stock, that $t = 1$, and that an interest rate of 10% is compounded once a year; that is, one plus the interest rate r is 1.1. Then, the current value of the underlying V is 100. Displaced diffusion and Black–Scholes call prices are calculated with the same formula with $\lambda = 0.001$. Black–Scholes prices have implicit a threshold parameter $\theta = 0$, which implies a volatility parameter σ of 20%. Table I presents the call option prices for several values of the rescale parameter λ. Table II presents the call option prices as a percentage of the Black–Scholes values for the same values of the rescale parameter. The two main conclusions from the analysis of the results of the simulations are the following:

1. If the actual θ of the underlying variable is negative and $\lambda > 0$, the Black–Scholes model overprices out-of-the-money calls and under-

TABLE I
Call Option Prices

	K	*80*	*90*	*100*	*110*	*120*	*130*
Panel A: $\lambda = 0.001$							
θ	σ						
−50	0.138	27.800	19.850	13.138	8.014	4.500	2.332
−30	0.158	27.759	19.795	13.094	8.000	4.519	2.370
−10	0.184	27.707	19.721	13.033	7.980	4.542	2.418
0	0.200	27.674	19.674	12.992	7.965	4.554	2.447
10	0.220	27.637	19.617	12.943	7.946	4.568	2.481
30	0.273	27.541	19.460	12.800	7.885	4.597	2.564
50	0.359	27.411	19.201	12.546	7.760	4.616	2.677
Panel B: $\lambda = 10$							
θ	σ						
−50	0.138	27.799	19.848	13.137	8.014	4.501	2.333
−30	0.158	27.758	19.793	13.092	8.000	4.520	2.372
−10	0.184	27.704	19.718	13.030	7.979	4.543	2.420
0	0.201	27.671	19.669	12.988	7.964	4.556	2.450
10	0.221	27.632	19.611	12.937	7.944	4.570	2.485
30	0.275	27.532	19.447	12.789	7.881	4.600	2.571
50	0.364	27.394	19.168	12.516	7.747	4.620	2.691
Panel C: $\lambda = 30$							
θ	σ						
−50	0.141	27.789	19.836	13.128	8.012	4.506	2.342
−30	0.161	27.743	19.773	13.077	7.996	4.527	2.385
−10	0.190	27.681	19.686	13.004	7.972	4.554	2.441
0	0.208	27.641	19.627	12.954	7.954	4.569	2.476
10	0.230	27.592	19.552	12.890	7.929	4.587	2.519
30	0.294	27.456	19.319	12.681	7.842	4.626	2.634
50	0.416	27.272	18.787	12.167	7.596	4.651	2.824

TABLE II
Call Option Prices as a Percentage of the Black–Scholes Values

K		80	90	100	110	120	130
Panel A: $\lambda = 0.001$							
θ	σ						
−50	0.138	100.46%	100.89%	101.12%	100.62%	98.81%	95.30%
−30	0.158	100.31%	100.62%	100.79%	100.44%	99.23%	96.85%
−10	0.184	100.12%	100.24%	100.32%	100.19%	99.74%	98.81%
0	0.200	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
10	0.220	99.87%	99.71%	99.62%	99.76%	100.31%	101.39%
30	0.273	99.52%	98.91%	98.52%	99.00%	100.94%	104.78%
50	0.359	99.05%	97.60%	96.57%	97.43%	101.36%	109.40%
Panel B: $\lambda = 10$							
θ	σ						
−50	0.138	100.45%	100.88%	101.12%	100.62%	98.84%	95.34%
−30	0.158	100.30%	100.60%	100.77%	100.44%	99.25%	96.94%
−10	0.184	100.11%	100.22%	100.29%	100.18%	99.76%	98.90%
0	0.201	99.99%	99.97%	99.97%	99.99%	100.04%	100.12%
10	0.221	99.85%	99.68%	99.58%	99.74%	100.35%	101.55%
30	0.275	99.49%	98.85%	98.44%	98.95%	101.01%	105.07%
50	0.364	98.99%	97.43%	96.34%	97.26%	101.45%	109.97%
Panel C: $\lambda = 30$							
θ	σ						
−50	0.141	100.42%	100.82%	101.05%	100.59%	98.95%	95.71%
−30	0.161	100.25%	100.50%	100.65%	100.39%	99.41%	97.47%
−10	0.190	100.03%	100.06%	100.09%	100.09%	100.00%	99.75%
0	0.208	99.88%	99.76%	99.71%	99.86%	100.33%	101.19%
10	0.230	99.70%	99.38%	99.21%	99.55%	100.72%	102.92%
30	0.294	99.21%	98.20%	97.61%	98.46%	101.58%	107.64%
50	0.416	98.55%	95.49%	93.65%	95.37%	102.13%	115.41%

prices in-the-money calls. This is consistent with an underlying having a distribution with a thinner right tail and a fatter left tail than the distribution implied by the Black–Scholes equation (Hull, 1997, pp. 492–494.), that is, with Figure 3. Stutzer (1996), assuming efficient markets, documented that the Black–Scholes model overprices out-of-the-money calls and underprices in-the-money calls in the S&P 500 index options market.

2. If either (a) $\theta > 0$ or (b) $\theta = 0$ and the actual λ of the underlying is positive, the Black–Scholes model underprices out-of-the-money calls and overprices in-the-money calls. This is consistent with an underlying having a distribution with a fatter right tail and a thinner left tail than the distribution implied by the Black–Scholes equation (Hull, 1997, pp. 492–494.), that is, with Figure 4. Assuming efficient markets, Black (1975) and Gultekin et al. (1982) showed that

> the Black–Scholes equation underprices out-of-the-money calls and overprices in-the-money calls in the stock options market. However, results opposite to the ones just mentioned were obtained by Macbeth and Merville (1979) for a sample of six stocks during 1976. Chang et al. (1998) documented that the Black–Scholes model overprices in-the-money calls and underprices out-of-the-money calls on foreign currencies. Zhou (1998), on the basis of 1995 prices, reported implied volatilities by futures options consistent with a market where the Black–Scholes equation underprices out-of-the-money calls and overprices in-the-money calls.

The inverse-coshnormal distribution gives rise to implied volatility sneers (skews or smiles). To illustrate this effect, implied volatilities are calculated for the option prices of Table I with the Black–Scholes formula. Typical volatility sneers are plotted in Figures 5 and 6 (which are consistent with the figures of the implied densities). Figure 5 shows a typical volatility sneer generated by an inverse coshnormal with $\theta < 0$ and $\lambda > 0$. For S&P 500 options, this is totally consistent with "the sneer pattern displayed in Figure 1, . . . with call option implied volatilities decreasing monotonically as the call goes deeper out of the money" (Dumas et al., 1998). Figure 6 shows a typical volatility sneer generated by a positively skewed inverse coshnormal with $\theta > 0$ and $\lambda > 0$. As remarked by Zhou (1998, p. 1709), "for agricultural options, implied volatility is an increasing function of option strike price." Sneers with similar patterns were documented by Chang et al. (1998) for currency options.

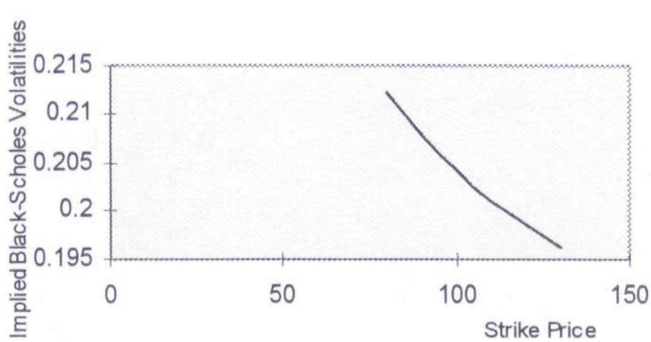

FIGURE 5

This figure displays a typical pattern of implied Black–Scholes volatilities generated by the inverse-coshnormal option-pricing model when $\theta < 0$ and $\lambda > 0$. In this context, Dumas et al. (1998, p. 2062) highlight that such "sneer . . is indicative of the pattern [seen in the S&P 500 option market] since the crash, with call option implied volatility decreasing monotonically as the call goes deeper out of the money."

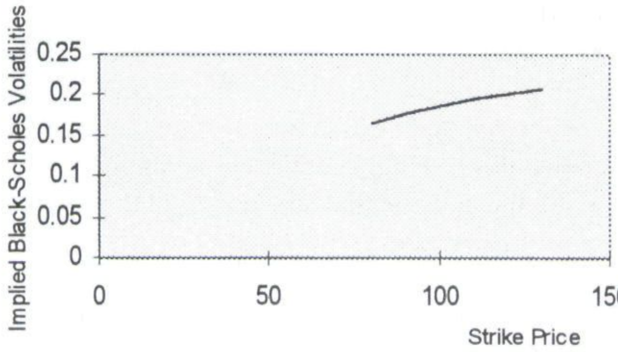

FIGURE 6
This figure displays a typical pattern of implied Black–Scholes volatilities generated by the inverse-coshnormal option-pricing model when $\theta > 0$ and $\lambda > 0$. Such a pattern is consistent with recent findings in some commodities markets. As Zhou (1988, p. 1709) reported, "For agricultural options, implied volatility is an increasing function of option strike price."

CONCLUSIONS

This article obtained new option-pricing results based on the following fundamental assumption: The representative agent has an extended power utility function of consumption, and aggregate consumption and the underlying asset are bivariate-displaced-lognormal-inverse coshnormal distributed.

The same pricing formulae might be obtained under another set of hypotheses, namely, that the representative agent has a negative exponential utility function of consumption and aggregate consumption and the underlying asset are bivariate-normal-inverse-coshnormal-distributed. In particular, the following two definitions replace Definitions 1 and 2:

Definition. The negative exponential marginal utility function of consumption is given by

$$U'(C) = \exp(\varphi C) \tag{14}$$

with $\varphi < 0$. This utility function represents constant absolute risk aversion.

Definition. The bivariate-normal-inverse-coshnormal distribution of aggregate consumption, C, and underlying, S, is defined by

$$\left[C, \cosh^{-1}\left(\frac{S-\theta}{\lambda}\right)\right] =^{d} N^{P}\left([\mu_c, \mu], \begin{bmatrix} \sigma_c^2 & \rho\sigma\sigma_c \\ \rho\sigma\sigma_c & \sigma^2 \end{bmatrix}\right)$$

The joint distributions imply that the marginal distribution of the underlying asset is an inverse coshnormal. This is a four-parameter distribution that depends on a scale parameter μ, a shape parameter σ, a threshold parameter θ, and a rescale parameter λ. This article shows that the lognormal, risk-neutral density implied by the Black–Scholes model is the limit of the inverse-coshnormal, risk-neutral density when the threshold parameter is zero and the rescale parameter approaches zero.

This article establishes a risk-neutral valuation relation when either set of assumptions holds, which means that option prices have an inverse-coshnormal, risk-neutral density implicit. Then, closed-form expressions for calls and puts written on nondividend-paying stocks, futures, foreign currencies, and dividend-paying stocks are derived.

There are five related reasons the valuation equations derived in this article might be useful to some market participants:

1. The inverse coshnormal accurately fits four moments of the actual data, whereas the lognormal underlying Black–Scholes probably only fits accurately two moments (see Figs. 1 and 2).
2. The tails of the densities implied by option market prices and the tails of the densities implied by inverse-coshnormal option prices have similar behaviors compared with the risk-neutral lognormal implicit in the Black–Scholes model (see Figs. 3 and 4).
3. The direction of the deviates of the Black–Scholes prices compared with option market prices is the same as the direction of the deviates of the Black–Scholes prices compared with inverse-coshnormal option prices (see Tables I and II). Inverse-coshnormal option prices when the threshold parameter is negative and the rescale parameter is positive are consistent with the empirical results provided by Stutzer (1996) for the S&P 500 options market. Inverse-coshnormal option prices when the threshold parameter and the rescale parameters are positive are in line with the empirical results provided by Chang et al. (1998) for currency options.
4. The inverse-coshnormal option model is able to generate sneers similar to those observed in the market (see Figs. 5 and 6).
5. The inverse-coshnormal valuation equation preserves the risk-neutral property underlying Black–Scholes and has Black–Scholes as a special case (see Corollaries 1 and 2).

In-sample tests and out-of-sample prediction tests of the valuation equations derived in this article (vs the Black–Scholes family of equations) should be done in future research. Estimates for the model $(\hat{\sigma}, \hat{\theta}, \hat{\lambda})$ can be obtained with Whaley's (1982) method, which minimizes the sum of squares of price deviates.[27] Then, the following null hypothesis should be tested:

1. The estimated parameters are statistically equal to zero.
2. There is no difference between risk-neutral parameters (implicit in option market prices) and actual parameters (estimated from the observed realizations of the underlying).
3. The estimated parameters depend systematically on some option market-related variables (e.g., underlying value, strike price, time to maturity, volume of trade, and open interest).
4. The market price might be expressed as a linear function of the model price (with intercept 0 and slope 1).
5. Price deviates depend systematically on some option-pricing inputs (e.g., estimated parameters, strike price, time to maturity, interest rate, and underlying value).
6. The root-mean-squared valuation error (RMSVE) and the Akaike information criterion give better results for the inverse-coshnormal model than for the Black–Scholes model.

In addition, further research might develop and test new error-based statistics (of the RMSVE type) to measure the ability of the model to generate sneers or smiles and to reproduce Bates' skewness premiums. A study of risk management applying the hedge portfolio error analysis of Dumas et al. (1998) can also be performed.

APPENDIX

Proof of Lemma 2

Let $x \sim \nabla(\mu, \sigma, \theta = 0, \lambda = 1)$ be a standardized inverse coshnormal. The mean, variance, skewness, and kurtosis of x are given by

$$E^P(x_\nabla) = \omega^{\frac{1}{2}}\Omega$$

$$\mathrm{Var}^P(x_\nabla) = \omega\Omega^2(\omega - 1) - \frac{1}{2}(\omega^2 - 1)$$

$$\mathrm{Ske}^P(x_\nabla) = \frac{(\omega + 2)(\omega - 1)^{1/2}}{(1 - 1/2\frac{\omega+1}{\omega\Omega^2})^{3/2}} - 3/4\frac{(\omega - 1)^{1/2}(\omega + 1)^2}{\omega\Omega^2(1 - 1/2\frac{\omega+1}{\omega\Omega^2})^{3/2}}$$

[27] A deviate or an error is, by definition, the difference between a market value and a theoretical or model value.

$$\mathrm{Kur}^{P}(x_{\nabla}) = \frac{\omega^4 + 2\omega^3 + 3\omega^2 - 3}{(1 - 1/2\frac{\omega+1}{\omega\Omega^2})^2} + \frac{\omega - 3\omega^2 + 3\omega^4 - \omega^7}{[\omega^{1/2}\Omega(\omega - 1) - 1/2\frac{\omega^2 - 1}{\omega^{1/2}\Omega}]^2}$$
$$+ \frac{1/8\omega^8 - 1/2\omega^2 + 3/8}{[\omega\Omega^2(\omega - 1) - 1/2(\omega^2 - 1)]^2}$$

where[28] $\mathrm{Ske}(.) = \mu_3/\mu_2^{3/2}$, $\mathrm{Kur}(.) = \mu_4/\mu_2^2$, $\Omega = \cosh(\mu)$, and $\omega = \exp(\sigma^2)$. For an arbitrary, positively skewed inverse coshnormal $f(S) \sim \nabla(\mu, \sigma, \theta, \lambda)$ the mean, variance, skewness, and kurtosis are given by

$$E^P(S_{\nabla}) = \theta + \lambda E^P(x_{\nabla})$$
$$\mathrm{Var}^P(S_{\nabla}) = \lambda^2 \mathrm{Var}^P(x_{\nabla})$$
$$\mathrm{Ske}^P(S_{\nabla}) = \mathrm{Ske}^P(x_{\nabla})$$
$$\mathrm{Kur}^P(S_{\nabla}) = \mathrm{Kur}^P(x_{\nabla})$$

For an arbitrary, negatively skewed inverse coshnormal $f(S) \sim \nabla(\mu, \sigma, \theta, \lambda)$, although the variance and kurtosis are given by previous expressions, the mean and skewness are given by

$$E^P(S_{\nabla}) = \theta - \lambda E^P(x_{\nabla})$$
$$\mathrm{Ske}^P(S_{\nabla}) = -\mathrm{Ske}^P(x_{\nabla})$$

Proof of Lemma 3

Suppose that the fundamental assumption holds. The marginal distribution of consumption is by definition $f(C) = \int_{-\infty}^{\infty} f(C, S)\, dS$. Hence, the marginal distribution of consumption is a univariate-displaced-lognormal distribution $\Lambda^P(\mu_c, \sigma_c, \tau)$. Using the definition of the φ moment about τ of a univariate-displaced-lognormal random variable yields

$$E[U'(C)] = \exp\left[\varphi\mu_c + \frac{1}{2}\varphi^2\sigma_c^2\right]$$

The conditional distribution of consumption given the value of the underlying variable is by definition $f(C|S) = \frac{f(C,S)}{f(S)}$. Then, the conditional distribution of consumption is a univariate-displaced-lognormal distribution; that is, $C|S$ is

$$\Lambda^P\left[\mu_c + \rho\frac{\sigma_c}{\sigma}\left(\cosh^{-1}\left(\frac{S - \theta}{\lambda}\right) - \mu\right), (\sigma_c^2(1 - \rho^2))^{\frac{1}{2}}, \tau\right]$$

[28]The ith central moment of the random variable about its first moment is μ_i.

Using the definition of the φ moment about τ of a univariate-displaced-lognormal random variable yields

$$E[U'(C)\,|\,S] = \exp\left[\varphi\mu_c + \rho\varphi\frac{\sigma_c}{\sigma}\left[\cosh^{-1}\left(\frac{S-\theta}{\lambda}\right) - \mu\right] + \frac{1}{2}\varphi^2\sigma_c^2(1-\rho^2)\right]$$

Then, the pricing kernel (Equation 4) can be put in the desired form.

Proof of Corollary 2

Simplifying, taking the limits, and simplifying again yield

$$\lim_{\lambda\to 0} P_c = (V - \theta e^{-rt})N(d_1) - e^{-rt}(K-\theta)N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{Ve^{rt}-\theta}{K-\theta}\right) + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}$$

which is the displaced diffusion call price equation.[29] When $\theta = 0$, the Black–Scholes call price is obtained. The proof for the put is similar.

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[29]This can be verified in two steps. First, define $\theta = bV$, and substitute this in Equation 18. Second, define $1 - be^{-rt} = a$, and substitute this in the previous expression.

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