TWO-STATE OPTION PRICING: BINOMIAL MODELS REVISITED

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This article revisits the topic of two-state option pricing. It examines the models developed by Cox, Ross, and Rubinstein (1979), Rendleman and Bartter (1979), and Trigeorgis (1991) and presents two alternative binomial models based on the continuous-time and discrete-time geometric Brownian motion processes, respectively. This work generalizes the standard binomial approach, incorporating the main existing models as particular cases. The proposed models are straightforward and flexible, accommodate any drift condition, and afford additional insights into binomial trees and lattice models in general. Furthermore, the alternative parameterizations are free of the negative aspects associated with the Cox, Ross, and Rubinstein model. © 2001 John Wiley & Sons, Inc. Jrl Fut Mark 21:987–1001, 2001

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INTRODUCTION

After the seminal article by Black and Scholes (BS; Black & Scholes, 1973), several methods for valuing derivative securities were proposed. Merton (1973) extended the BS model to include valuing an option on a stock, or index, that pays continuous dividends. This extension can also be applied to currency options. Barone-Adesi and Whaley (1987) used a quadratic approximation approach to extend the BS framework to the valuation of American options. Cox, Ross, and Rubinstein (CRR; Cox, Ross, & Rubinstein, 1979) and Rendleman and Bartter (RB; Rendleman & Bartter, 1979) pioneered the two-state lattice approach, which is a powerful tool that can be used to value a wide variety of contingent claims. In the binomial setting, valuation by arbitrage arguments is clear. This technique is based on the formation of a risk-free portfolio through the combination of a stock and an option on this stock. Such a portfolio should earn the risk-free interest rate. The idea was not new, and its genesis is the partial differential equation developed by BS. With specific parameters, CRR and RB showed that option values from their binomial models converge to the "celebrated BS model" values.

The CRR and RB methodology was later used, slightly modified, or extended by, among others, Hull and White (1988) and Boyle (1986). Tian (1993) proposed an alternative binomial model and compared its performance to that of the CRR model. However, the results were later criticized in Easton (1996). Trigeorgis (1991) delineated a log-transformed variation of the binomial model, which supposedly overcomes the problems of consistency, stability, and efficiency associated with the CRR specification and various other numerical techniques. According to Trigeorgis, this methodology relative to that of CRR's "compares favorably in terms of computational efficiency due to the log-transformation" (p. 319).

This article revisits two-state option pricing and presents two alternative models based on continuous-time and discrete-time geometric Brownian motion (GBM) processes. The proposed methodology proves extremely flexible because it accommodates any centering condition. This flexibility is achieved by two parameters being solved as a function of the third. Additionally, the RB model is extended, and the log-transformed parameterization suggested by Trigeorgis is shown to be mathematically identical to a particular case of the extended RB model. Thus, the enhanced computational efficiency attributed to the log transformation proves mistaken because it is the result of an exact solution and the specified centering condition.

The main contribution of this work is pedagogical in nature. The proposed parameterizations are simple and flexible alternatives to the popular existing specifications and afford additional insights into binomial trees and lattice models in general.

The format of this article is as follows: The first section swiftly reviews the continuous-time GBM process; the second presents the RB and CRR models; the third discusses the continuous-time GBM process and presents the alternative binomial model (ABMC); the fourth provides the discrete-time GBM process and presents the alternative binomial model (ABMD); the fifth reviews the log-transformed binomial model and proves this to be a particular case of an extended RB parameterization; the sixth revisits binomial models, extends the RB model to accommodate numerous centering conditions, and compares it to the ABMC and ABMD models; and the seventh provides the conclusions.

CONTINUOUS-TIME GBM

In a risk-neutral world, if one assumes that the stock price S follows a continuous-time GBM process,

$$dS = rS\,dt + \sigma S\,dz \tag{1}$$

where r is the instantaneous risk-free interest rate, σ is the instantaneous volatility of the stock price, dt is an infinitely small increment of time, and dz is a Wiener process. By using Ito's lemma, one can show that

$$d\ln(S) = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\,dz \tag{2}$$

or

$$dX = \alpha\,dt + \sigma\,dz \tag{3}$$

where $X = \ln(S)$ and $\alpha = (r - \frac{\sigma^2}{2})$. As a result, $\ln(S)$ follows a generalized Wiener process for the time period $(0, t)$, where t is a point in time; the variable $\hat{X} = X_t - X_0 = \ln(\frac{S_t}{S_0})$ is normally distributed with a mean $\alpha \cdot t$ and a variance of $\sigma^2 t$, and S_0 and S_t represent the stock prices at time 0 and t, respectively. More succinctly,

$$\hat{X} \sim N(\alpha \cdot t, \sigma\sqrt{t}) \tag{4}$$

The continuously compounded rate of return (R) realized during the period $(0, t)$ can be defined by the following equation:

$$S_t = S_0 e^{Rt} \tag{5}$$

Accordingly,

$$R = \frac{1}{t}\ln\left(\frac{S_t}{S_0}\right) = \frac{1}{t}\cdot\hat{X} \tag{6}$$

Or

$$t \cdot R = \hat{X} \tag{7}$$

Thus, R is normally distributed with a mean of $\alpha = r - \frac{\sigma^2}{2}$ and a variance of $\frac{\sigma^2}{t}$. Taking the expected value of both sides of Equation 7, one can obtain the following:

$$E(t \cdot R) = t \cdot \alpha = E(\hat{X}) = E\left[\ln\left(\frac{S_t}{S_0}\right)\right] \tag{8}$$

In a binomial model, the stock price can move either up from S_0 to $u \cdot S_0$ or down to $d \cdot S_0$, where u and d are two parameters such that u is greater than 1 and d is less than 1. Because the stock price follows a binomial model, the variable $\frac{S_t}{S_0}$ has the following distribution:

$$\begin{cases} u & \text{with risk-neutral probability } p \\ d & \text{with risk-neutral probability } 1 - p \end{cases} \tag{9}$$

For the lattice approach, we obtain

$$E\left(\ln\frac{S_t}{S_0}\right) = p\ln(u) + (1 - p)\ln(d) \tag{10}$$

or

$$E(\hat{X}) = pU + (1 - p)D \tag{11}$$

where

$$U = \ln(u), \quad D = \ln(d) \tag{12}$$

Therefore, the variable $\hat{X}$ follows the following distribution:

$$\begin{cases} U & \text{with risk-neutral probability } p \\ D & \text{with risk-neutral probability } 1 - p \end{cases} \tag{13}$$

It can be shown that the variance, for the lattice approach, of the variable $\ln\frac{S_t}{S_0}$ is given by the following:

$$Var\left(\ln\frac{S_t}{S_0}\right) = p(1 - p)\left[\ln\frac{u}{d}\right]^2 \tag{14}$$

or

$$Var(\hat{X}) = p(1 - p) \cdot (U - D)^2 \tag{15}$$

RB AND CRR MODELS

RB (1979) and CRR (1979) proposed the following system:

$$E\left[\ln\frac{S_{\Delta t}}{S_0}\right] = p\ln(u) + (1-p)\ln(d) = \mu\cdot\Delta t \tag{16}$$

$$Var\left[\ln\frac{S_{\Delta t}}{S_0}\right] = p(1-p)[\ln(u/d)]^2 = \sigma^2\Delta t \tag{17}$$

where Δt is equal to T/n, T is the time to maturity, and n is the number of time steps. RB suggested the following value for μ:

$$\mu = \frac{r - \frac{1}{2}\sigma\sqrt{T}\left[\sqrt{\frac{1-p}{p}} - \sqrt{\frac{p}{1-p}}\right] - \frac{1}{4}\sigma^2\left[\frac{1-p}{p} + \frac{p}{1-p}\right]}{\left[1 + \left(\sqrt{\frac{1-p}{p}} - \sqrt{\frac{p}{1-p}}\right)\frac{1}{\sqrt{T}}\right]} \tag{18}$$

and indicated that "the best approximation would occur if $\mu = r - \frac{\sigma^2}{2}$" (p. 1101), which implies that $p = \frac{1}{2}$. This should not be an approximation because by substituting Δt for t in Equation 8, one can obtain

$$E\left(\ln\frac{S_{\Delta t}}{S_0}\right) = \Delta t\cdot\alpha = \Delta t\cdot\left(r - \frac{\sigma^2}{2}\right) \tag{19}$$

Therefore, in a risk-neutral world, $\mu = r - \frac{\sigma^2}{2} = \alpha$ is not just the best approximation; it is the only correct value for Equation 16.

RB (1979) suggested the following exact solution to the system (Equations 16 and 17):

$$u = e^{\mu\cdot\Delta t + a\cdot\sigma\sqrt{\Delta t}},\quad d = e^{\mu\cdot\Delta t - b\cdot\sigma\sqrt{\Delta t}} \tag{20}$$

where $a = \frac{(1-p)}{\sqrt{p(1-p)}}$ and $b = \frac{p}{\sqrt{p(1-p)}}$. If one sets $p = \frac{1}{2}$, the values u and d are given by the following:[1]

$$u = e^{(r-\frac{\sigma^2}{2})\cdot\Delta t + \sigma\sqrt{\Delta t}},\quad d = e^{(r-\frac{\sigma^2}{2})\cdot\Delta t - \sigma\sqrt{\Delta t}} \tag{21}$$

CRR (1979) suggested different parameters as a solution to the system (Equations 16 and 17):

$$u = e^{\sigma\sqrt{\Delta t}},\quad d = e^{-\sigma\sqrt{\Delta t}} \tag{22}$$

$$p = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\Delta t} \tag{23}$$

[1]The property that the risk-neutral probability is equal to one-half is generally attributed to Jarrow and Rudd (1983).

These values for u, d, and p from Equations 22 and 23 satisfy Equation 16 exactly and Equation 17 approximately. Substituting the values for u, d, and p from Equations 22 and 23 into the left-hand side of Equation 17, one obtains the following:

$$\sigma^2 \Delta t \cdot \left(1 - \frac{\mu^2 \Delta t}{\sigma^2}\right) \tag{24}$$

When terms of order higher than Δt are ignored, one obtains

$$\sigma^2 \Delta t \cdot \left(1 - \frac{\mu^2 \Delta t}{\sigma^2}\right) = \sigma^2 \Delta t - \mu^2 (\Delta t)^2 \approx \sigma^2 \Delta t$$

For sufficiently small Δt, Equation 17 can be approximately satisfied. However, when $(1 - \frac{\mu^2 \Delta t}{\sigma^2}) < 0$, that is, $\Delta t > \frac{\sigma^2}{\mu^2}$, the left-hand side of Equation 17, which is the variance of the lattice distribution, is negative, and Equation 17 cannot be satisfied. Although CRR suggested a value for p given by Equation 23, the following value is actually applied:

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{25}$$

If $\Delta t > \frac{\sigma^2}{r^2}$, the CRR model gives negative probabilities because

$$p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} > 1$$

Therefore,

$$1 - p < 0$$

As a direct result of the approximate solution to Equation 17, for the CRR model the actual volatility at any node of the lattice is downward biased unless Δt is sufficiently small. This is not the case for the RB model because if $p = \frac{1}{2}$, $\mu = r - \frac{\sigma^2}{2}$ and Equation 21 are an exact solution to the system (Equations 16 and 17). Thus, the RB parameters have the same mean and variance of the underlying lognormal diffusion process for any step size, and the CRR parameters result in the same mean for any step size but the same variance only in the limit.

ABMC

In a risk-neutral world, the expected value and volatility (δ) of the stock price at time $t + \Delta t$ are given by (see the Appendix)

$$E(S_{t+\Delta t}) = S_t e^{r\Delta t} \tag{26}$$

$$\delta(S_{t+\Delta t}) = S_t \cdot e^{r\Delta t} \cdot \sqrt{(e^{\sigma^2 \cdot \Delta t} - 1} \tag{27}$$

where t is the current time and Δt is a relatively short period of time. Therefore,[2]

$$r\Delta t = \ln \frac{E(S_{t+\Delta t})}{S_t} = \ln E\left(\frac{S_{t+\Delta t}}{S_t}\right) \tag{28}$$

As a direct result of the binomial distribution (Equation 9), one obtains the following:

$$E\left(\frac{S_{t+\Delta t}}{S_t}\right) = pu + (1 - p)d \tag{29}$$

By using Equations 26 and 29, one can find

$$pu + (1 - p)d = e^{r\Delta t} \tag{30}$$

The central moment of order n consistent with distribution (Equation 9) is given by

$$M_n = p(u - m)^n + (1 - p)(d - m)^n \tag{31}$$

where m denotes the expected value of $(\frac{S_{t+\Delta t}}{S_t})$

$$m = pu + (1 - p)d$$

By substitution,

$$M_n = p(1 - p)[(1 - p)^{n-1} + p^{n-1}(-1)^n](u - d)^n$$

Therefore,

$$M_1 = 0,$$
$$M_2 = p(1 - p)(u - d)^2,$$
$$M_3 = p(1 - p)(1 - 2p)(u - d)^3$$

To match the variance V (or volatility δ) of the stock price with the lattice parameters, using the second central moment, one has to satisfy the following equation:

$$M_2 = p(1 - p)(u - d)^2 = V\left(\frac{S_{t+\Delta t}}{S_t}\right)$$

[2]According to Equation 28, $r\Delta t = \ln\left(E\frac{S_{t+\Delta t}}{S_t}\right)$. It is tempting to assume that $r\Delta t = E\left(\ln\frac{S_{t+\Delta t}}{S_t}\right)$. However, $\ln\left(E\frac{S_{t+\Delta t}}{S_t}\right) \neq E\left(\ln\frac{S_{t+\Delta t}}{S_t}\right) = \Delta t \cdot \left(r - \frac{\sigma^2}{2}\right)$ as a result of Equation 8.

or

$$\sqrt{p(1-p)}(u-d) = \delta\left(\frac{S_{t+\Delta t}}{S_t}\right) = e^{r\cdot\Delta t}\cdot\sqrt{(e^{\sigma^2\cdot\Delta t}-1)} = e^{r\cdot\Delta t}\cdot\tilde{\sigma}\sqrt{\Delta t} \tag{32}$$

where $\tilde{\sigma}\sqrt{\Delta t} = \sqrt{(e^{\sigma^2\cdot\Delta t}-1)}$.[3]

By solving the system (Equations 30 and 32) with respect to u and d, one can obtain the following parameters for the ABMC model:

$$u = e^{r\Delta t} + a\cdot e^{r\Delta t}\cdot\tilde{\sigma}\sqrt{\Delta t}, \quad d = e^{r\Delta t} - b\cdot e^{r\Delta t}\cdot\tilde{\sigma}\sqrt{\Delta t}$$

or

$$u = e^{r\Delta t}\cdot(1 + a\cdot\tilde{\sigma}\sqrt{\Delta t}), \quad d = e^{r\Delta t}\cdot(1 - b\cdot\tilde{\sigma}\sqrt{\Delta t}) \tag{33}$$

where $a = \frac{(1-p)}{\sqrt{p(1-p)}}$ and $b = \frac{p}{\sqrt{p(1-p)}}$.

ABMD

The discrete-time version of the GBM process for a sufficiently short period of time Δt is given by

$$\frac{S_{t+\Delta t} - S_t}{S_t} = \frac{\Delta S}{S_t} = r\Delta t + \sigma\vartheta\cdot\sqrt{\Delta t}$$

where ϑ is a random drawing from a standard normal distribution $N(0, 1)$. It assumes that the proportional change in the stock price $\frac{\Delta S}{S_t}$ is normally distributed with a mean of $r\Delta t$ and a standard deviation of $\sigma\sqrt{\Delta t}$; that is, $\frac{S_{t+\Delta t}}{S_t}$ is normally distributed with a mean of $1 + r\Delta t$ and the same standard deviation[4] of $\sigma\sqrt{\Delta t}$. As for the case of the ABMC, to match the expected value and volatility of the stock price to the lattice parameters for the ABMD, one has to solve the following system:

$$pu + (1-p)d = 1 + r\cdot\Delta t \tag{34}$$

$$\sqrt{p(1-p)}(u-d) = \sigma\sqrt{\Delta t} \tag{35}$$

[3]If terms of order $(\Delta t)^2$ are ignored, $\tilde{\sigma}\sqrt{\Delta t} = \sqrt{(e^{\sigma^2\cdot\Delta t}-1)} \approx \sigma\sqrt{\Delta t}$.

[4]The discrete-time process is lognormal in the limit because $\ln(1+x) \to x$ as $x \to 0$. Therefore, as n, the number of steps, increases and $\Delta t \to 0$, the discrete-time process approximates the lognormal diffusion process.

By solving the system (Equations 34 and 35) with respect to u and d, we obtain the following parameters for the ABMD model:

$$u = 1 + r \cdot \Delta t + a \cdot \sigma\sqrt{\Delta t}, \quad d = 1 + r \cdot \Delta t - b \cdot \sigma\sqrt{\Delta t} \tag{36}$$

where $a = \frac{(1-p)}{\sqrt{p(1-p)}}$ and $b = \frac{p}{\sqrt{p(1-p)}}$.

LOG-TRANSFORMED BINOMIAL METHOD

Trigeorgis's (1991) log-transformed binomial model is based on Equations 11 and 15. For the period of time $\Delta t = \frac{T}{n}$, where T is the time to maturity and n is the number of time steps, the system can be given as follows:

$$E[\Delta X] = pU + (1-p)D = \alpha \cdot \Delta t \tag{37}$$

$$Var[\Delta X] = p(1-p)[U - D]^2 = \sigma^2 \Delta t \tag{38}$$

where $\Delta X = \ln(\frac{S_{\Delta t}}{S_0})$. By solving the system (Equations 37 and 38) with respect to U and D, one can obtain the following parameters for the log-transformed binomial model:

$$U = \alpha \cdot \Delta t + a \cdot \sigma\sqrt{\Delta t}, \quad D = \alpha \cdot \Delta t - b \cdot \sigma\sqrt{\Delta t} \tag{39}$$

or, according to Equation 12,

$$u = e^{\alpha \cdot \Delta t + a \cdot \sigma\sqrt{\Delta t}}, \quad d = e^{\alpha \cdot \Delta t - b \cdot \sigma\sqrt{\Delta t}} \tag{40}$$

where $a = \frac{(1-p)}{\sqrt{p(1-p)}}$ and $b = \frac{p}{\sqrt{p(1-p)}}$. Comparing Equations 20 and 40, one can conclude that the log-transformed binomial model is equivalent to the RB model adjusted for the parameter μ ($\mu \equiv \alpha$) and subject to a drift-free constraint. Given the additive nature afforded via the log transformation, the condition $u \cdot d = 1$ is equivalent to the condition $U + D = 0$ suggested by Trigeorgis. Thus, in short, the stated "increased numerical efficiency" provided by the log-transformed model is not the result of the alternative specification but the exact solution to the system and the specified centering condition.

BINOMIAL MODELS REVISITED

The values u and d proposed by RB (1979; Equation 21) and CRR (1979; Equation 22) are different. The reason for this is that the former is an exact solution of the system (Equations 16 and 17), and the latter is an approximation. For CRR, negative probabilities and variance of the lattice distribution under specific conditions are the direct result of this approximation.

However, there is no need for approximations to solve the system (Equations 16 and 17). The drift-free condition

$$u \cdot d = 1 \tag{41}$$

specified by the CRR model and implied in Equation 22 can be easily replicated by the extended RB, ABMC, and ABMD models. This can be achieved by a determination of the appropriate value of the probability p. To find this value for p, one must substitute the values u and d from Equations 20, 33, and 36 into Equation 41 for the extended RB, ABMC, and ABMD models, respectively, and solve these equations for p. After basic mathematical transformations, one can obtain the following:

$$p = \frac{1}{2}\left[1 - \frac{q}{\sqrt{4 + q^2}}\right] \tag{42}$$

where for the RB model

$$q_{\mathrm{RB}} = \frac{(\sigma^2 - 2r)\Delta t}{\sigma\sqrt{\Delta t}}$$

for the ABMC model

$$q_{\mathrm{ABMC}} = \frac{(1 + \tilde{\sigma}^2\Delta t \cdot e^{2r\Delta t})e^{-r\Delta t} - e^{r\Delta t}}{\tilde{\sigma}\sqrt{\Delta t} \cdot e^{r\Delta t}}$$

and for the ABMD model

$$q_{\mathrm{ABMD}} = \frac{(1 + \sigma^2\Delta t) - (1 + r \cdot \Delta t)^2}{(1 + r \cdot \Delta t) \cdot \sigma\sqrt{\Delta t}}$$

Note that when terms of order higher than Δt are ignored, $q_{\mathrm{ABMC}} \approx q_{\mathrm{ABMD}} \approx q_{\mathrm{RB}}$ Subsequently, all three models have the same probability. Equation 42 gives a unique value for p, which cannot be negative or greater than 1, and satisfies the drift-free condition (Equation 41) for all models considered.[5]

Another example of the flexibility of the RB, ABMC, and ABMD models is that one may also grow the tree along the forward. One may set

$$u \cdot d = e^{2r\Delta t} \tag{43}$$

[5]According to Equation 42, $p \in (0,1)$. Moreover, the smaller Δt is, the closer p is to 0.5.

This can be achieved by the determination of the appropriate value of the probability p. To find this value, one must substitute the values of u and d from Equations 20, 33, and 36 into Equation 43 for the extended RB, ABMC, and ABMD models, respectively, and solve for p. After algebraic manipulation, one obtains the following for the RB model:

$$q_{\mathrm{RB}} = \sigma\sqrt{\Delta t}$$

for the ABMC model:

$$q_{\mathrm{ABMC}} = \tilde{\sigma}\sqrt{\Delta t}$$

and for the ABMD model:

$$q_{\mathrm{ABMD}} = \frac{\sigma\sqrt{\Delta t}}{1 + r \cdot \Delta t}$$

Finally, rather than imposing conditions (Equation 41 or 43), one may select a value for p that corresponds to the drift parameter of the GBM process, which is consistent with the BS model. This is achieved by the third central moment of the lattice distribution being set equal to 0. This implies $p = \frac{1}{2}$. Subsequently, the RB model has the following two parameters, which are also given by Equation 21:

$$u = e^{\alpha \cdot \Delta t + \sigma\sqrt{\Delta t}}, \quad d = e^{\alpha \cdot \Delta t - \sigma\sqrt{\Delta t}} \tag{44}$$

and

$$ud = e^{2 \cdot \alpha \cdot \Delta t} \tag{45}$$

The values u and d proposed for the ABMC model for the case $p = \frac{1}{2}$ are given by

$$u = e^{r\Delta t} \cdot (1 + \tilde{\sigma}\sqrt{\Delta t}), \quad d = e^{r\Delta t} \cdot (1 - \tilde{\sigma}\sqrt{\Delta t}) \tag{46}$$

and

$$ud = e^{2r\Delta t} \cdot (1 - \tilde{\sigma}^2 \Delta t) \tag{47}$$

Finally, the values u and d proposed for the ABMD model for the case $p = \frac{1}{2}$ are given as follows:

$$u = 1 + r \cdot \Delta t + \sigma\sqrt{\Delta t}, \quad d = 1 + r \cdot \Delta t - \sigma\sqrt{\Delta t} \tag{48}$$

and

$$ud = (1 + r \cdot \Delta t)^2 - \sigma^2 \Delta t \tag{49}$$

Thus, when terms of order higher than Δt are ignored for all models,

$$ud = 1 + 2\left(r - \frac{\sigma^2}{2}\right) \cdot \Delta t = 1 + 2 \cdot \alpha \cdot \Delta t \tag{50}$$

where $2 \cdot \alpha \cdot \Delta t$ represents the drift for the time period $2\Delta t$. Note that in all of the cases presented, the ABMC, ABMD, RB, and extended RB models will not under any circumstance give negative probabilities. Furthermore, the variance of the lattice distribution and the local volatility at each node are precise. This is a direct result of the exact solutions proposed and the underlying methodology. Given the system proposed by CRR and RB, one has three unknowns and only two equations. CRR and RB both initially imposed a constraint on the solution by choosing a value for one of the unknowns a priori. Contrary to this, it is obvious that one can first solve the system for two unknowns as a function of the third one and then select a desired value for this unknown.

CONCLUSIONS

Two alternative binomial lattice models are developed in this article. Although for option-pricing purposes, the binomial lattice has been supplanted by more sophisticated techniques, it is still the most widely used device for pedagogical purposes.

It is widely recognized that convergence of a discrete-time lattice model to the continuous-time BS model is ensured as long as in the limit the first two moments of the discrete-time process are consistent with the lognormal diffusion process of the continuous-time GBM. Although the CRR result is asymptotic, this study demonstrates that the specification of the parameters is the result of an unnecessary approximation. For the CRR model, negative probability and downward biased variance are a direct result of the approximate solution to the specified system. It is true that the consequences are of minimal concern, yet an exact solution, which is free of the negative aspects of approximations, is always preferable. It is often argued that the drift-free condition is preferred because it is easier to calculate the hedge parameters from this specification. This article demonstrates that the drift-free condition is easily obtainable without approximations. By solving for two parameters, u and d, as a function of the third, p, the methodology employed in this article affords one the flexibility to specify any centering condition.

This study also shows that although RB (1979) believed that the "best approximation would occur" if the drift parameter was set equal to the drift of the GBM process, this is the only correct value for their system. The RB model is extended to accommodate numerous centering conditions. Finally, the log-transformed binomial model delineated by Trigeorgis (1991) is shown to be mathematically identical to the case of the extended RB model with the drift parameter set equal to 0. That is, the log transformation is not the source of increased numerical efficiency, as suggested by Trigeorgis, but the result of an exact solution and specified centering condition.

In conclusion, it is obvious that the binomial-pricing method has three independent parameters subject to only two constraints, which are the mean and variance that converge in the limit to the appropriate diffusion process. This work demonstrates that the proposed alternatives, which are exact solutions based on continuous-time and discrete-time GBM and are implemented on a lattice, not only converge to the appropriate diffusion process but are simple, flexible alternatives, free of the negative aspects associated with the CRR parameterization and theoretically tractable.

APPENDIX

Consider a variable w that is normally distributed with a mean of 0 and a variance of 1, $w \sim N(0.1)$, and an arbitrary constant a. The expected value of the variable $A = e^{a \cdot w}$ is given by

$$E(e^{a \cdot w}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{a \cdot w} \cdot e^{-\frac{w^2}{2}} \, dw = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{a^2}{2}} \cdot e^{-\frac{(w-a)^2}{2}} \, dw$$

$$= e^{\frac{a^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(w-a)^2}{2}} dw \tag{A1}$$

Substituting $x = w - a$, one can obtain

$$E(e^{a \cdot w}) = e^{\frac{a^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \, dx = e^{\frac{a^2}{2}} \tag{A2}$$

According to Equation 2,

$$d \ln(S) = \alpha \, dt + \sigma \, dz \tag{A3}$$

where $\alpha = (r - \frac{\sigma^2}{2})$.

Integrating the previous equation, we obtain

$$\int_0^T d \ln(S) = \int_0^T \alpha \, dt + \int_0^T \sigma \, dz \tag{A4a}$$

which is equivalent to

$$\ln\left(\frac{S_T}{S_0}\right) = \alpha \cdot T + \sigma \cdot z_T \tag{A4b}$$

Therefore,

$$S_T = S_0 e^{\alpha \cdot T + \sigma \cdot z_T} \tag{A5}$$

Given this, the expected value and variance of the stock price at time T are

$$E(S_T) = S_0 \cdot e^{\alpha \cdot T} \cdot E(e^{\sigma \cdot z_T}) = S_0 \cdot e^{\alpha \cdot T} \cdot E(e^{b \cdot w}) \tag{A6}$$

$$V(S_T) = E(S_T^2) - [E(S_T)]^2 = S_0^2 \cdot e^{2 \cdot \alpha \cdot T} \cdot E(e^{2 \cdot b \cdot w}) - [E(S_T)]^2 \tag{A7}$$

where $w = \frac{z_T}{\sqrt{T}}$ and $b = \sqrt{T} \cdot \sigma$.
According to Equation A2,

$$E(S_T) = S_0 \cdot e^{\alpha \cdot T + \frac{b^2}{2}} = S_0 \cdot e^{r \cdot T} \tag{A8}$$

$$V(S_T) = S_0^2 \cdot (e^{2 \cdot \alpha \cdot T + 2 \cdot b^2} - e^{2 \cdot r \cdot T}) = S_0^2 \cdot e^{2 \cdot r \cdot T} \cdot (e^{\sigma^2 \cdot T} - 1) \tag{A9}$$

Thus, the standard deviation of the stock price at time T is

$$\delta(S_T) = S_0 \cdot e^{r \cdot T} \cdot \sqrt{(e^{\sigma 2 \cdot T} - 1)} \tag{A10}$$

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