
EXPORTS AND HEDGING

EXCHANGE RATE RISKS:

THE MULTI-COUNTRY

CASE

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This article examines the optimal production, export allocation, and hedging decisions of a risk-averse international firm that exports to several foreign markets with different currencies. The firm faces multiple exchange rate risks. Optimal decisions are analyzed under two scenarios. In the first, there is a forward market for one currency only. Then, the export allocation to different markets is separable from the firm's preferences and the joint distribution of the exchange rates. In contrast, total production is not separable except for a special case. In the second scenario, there is a forward market for each currency. Then, both production and export allocation are separable. Hedging with forward contracts depends on risk premia and on the joint distribution of the exchange rates. If tradable exchange rate risk is a linear function of untradable exchange rate risk plus noise, there is a conflict between cross hedging and taking a basis risk. If, alternatively, the untradable exchange rate risk is a linear function of the tradable exchange rate risk and noise, there is no such conflict. A speculative position in a biased forward market for one currency can be cross hedged using an unbiased forward market for another currency.

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INTRODUCTION

In recent years, foreign exchange rates have continued to fluctuate extensively. In many cases, exchange rate risk has a negative impact on the operations of firms engaged in international activities. Consequently, these firms have been using various ways and instruments to protect themselves against the adverse effects of exchange rate risk on production, exports, and, hence, profits. Since international firms are active on markets in several countries, they are jointly exposed to fluctuations in several exchange rates. The nature of this joint exposure together with the characteristics of the product markets determine the optimal production, export allocation, and risk management decisions. Financial instruments such as forward contracts play a particularly important role in the management of exchange rate risk.

This article addresses two questions: What are the optimal decisions on domestic production and export allocation of a risk-averse exporting firm in the presence of a joint exposure to several exchange rates? What is the optimal exchange rate risk management using forward contracts? These questions will be answered for two scenarios that differ with respect to the existence of currency forward markets.

There is a close link between the existence of currency forward markets and more general economic characteristics of the foreign country. Industrialized countries have quite sophisticated foreign exchange derivative markets with active trading in forward and futures contracts, swaps, options, and various other financial instruments. If a firm exports to such countries, it is natural to assume that forward contracts are available. If the firm exports to one of the newly industrialized countries in Latin America and Asia, where currency derivative markets began to develop, assuming the existence of forward contracts seems reasonable as well, since forward contracts are comparatively simple derivatives. However, in a number of less developed countries, the availability of currency forward contracts cannot be taken as given. Typically, the foreign exchange markets of these countries are strictly regulated. Then, forward contracts often do not exist or, if they do exist, forward rates and liquidity are not the outcome of a market process, but rather are subjected to direct political influence. In addition, the capital markets of these countries are often heavily regulated as well, so that it is hardly possible to create the cash flows of forward contracts by borrowing or lending in these countries' currencies indirectly. Hence, if a firm exports to such a country, it is likely that a currency forward market does not exist. For this reason, the article analyzes two scenarios with different availability of forward contracts.

In the first scenario, forward contracts exist only for part of the export markets' currencies. In this case, the export allocation of the firm's total production to the export markets is separable from the firm's preferences and the joint distribution of the exchange rates. In contrast, total production itself is not separable except for the special case of competitive export markets. Existing currency forward markets are used for directly hedging the associated exchange rate risk and for indirectly hedging the otherwise untradable risks of correlated exchange rates (cross hedging). Optimal forward positions depend on the risk premia in the forward markets and on the joint distribution of the exchange rates.

In the second scenario, there is a forward market for each currency. Both export allocation and total production are then separable from the firm's preferences and expectations. The firm can completely eliminate exchange rate risk from its profits by fully hedging its foreign currency revenues. However, full hedging is not necessarily optimal. If there is a non-zero risk premium in one forward market, the firm will speculate on it and simultaneously use correlated forward markets to cross hedge.

In recent years, various studies have examined the problem of the exporting firm under exchange rate uncertainty. Holthausen (1979) was among the first to show that, given the existence of a competitive forward market, the risk-averse firm's production decision is independent of its preferences and expectations. This separation result has been derived for the case of a single export market. Another important result is that full hedging is optimal if the forward market is unbiased. The basic model of Holthausen (1979), in which the exchange rate is the only source of randomness in the firm's profits, has been extended in a number of subsequent studies. Briys and Schlesinger (1993) analyze the problem under state-dependent utility. Wolf (1995) examines the related problem of an importing firm; risky foreign revenue is incorporated by Adam-Müller (1997). Briys, Crouhy, and Schlesinger (1993) analyze the effects of an independent background risk arising from other activities of the firm. Adam-Müller (2000) deals with the effects of untradable inflation risk. Other extensions focus on cross hedging with forward contracts for correlated currencies in which no revenue is generated. Broll, Wahl, and Zilcha (1995) and Lence (1995) have shown that the correlation between the exchange rate and the cross hedging instrument's payoff determines whether full hedging is optimal. These authors have also shown that Holthausen's (1979) separation result cannot be derived in this case. Broll (1997) analyzes a triangular cross hedging problem.

In all these models, the firm is restricted to export to one single foreign market such that production and exports are identical. By allowing

the firm to export to several foreign markets, this article provides a generalization whereby generating revenue in several currencies exposes the firm to multiple exchange rate risks. In addition, the firm has to decide on the allocation of its total production to the export markets. This problem of export allocation does not exist in the case of a single export market.

In a recent paper, Broll, Wong, and Zilcha (1999) consider a related problem of a firm that exports to two markets. In contrast to the assumptions on the availability of forward markets made here, they assume that there is a forward market for exchange of the export markets' currencies. For this case, they derive the optimal export allocation, production, and hedging decisions. Alternatively, they consider the existence of a single forward market between one export market's currency and the firm's domestic currency. For this case, they characterize optimal export allocation and production decisions, but not the hedging decision.

This article is organized as follows. First, the model is described. The firm's optimal decisions are then analyzed under the assumption that there is a forward market for some currencies, but no forward market for other currencies. The discussion continues with an analysis of how to deal with the problem if there is a forward market for each currency. Finally, a numerical example is given. All proofs are summarized in the Appendix.

THE MODEL

Consider a two-date model of a risk-averse exporting firm whose preferences can be summarized in a von Neumann-Morgenstern utility function U defined over profits in domestic currency $\tilde{\Pi}$. $U(\tilde{\Pi})$ is at least twice continuously differentiable and strictly concave. The firm makes decisions at date 0, with uncertainty resolved at date 1. The firm is a single-product firm that produces in its home country and exports its output completely to several export markets. For simplicity, restrict the number of export markets to two. In each market, the firm faces an increasing and concave revenue function $R_i(q_i)$, $i = 1, 2$. q_i denotes export volume allocated to market i . The firm's total production ($q_1 + q_2$) gives rise to costs $C(q_1 + q_2)$ in domestic currency. The cost function is increasing and convex. In each market, the firm invoices in local currency such that revenues in country i are denominated in currency i . The exchange rate between currency i and the domestic currency, denoted by $\tilde{s}_i$, is stochastic with $s_i > 0$.¹ The firm is exposed simultaneously to fluctuations in two exchange rates. The joint distribution of $\tilde{s}_1$ and $\tilde{s}_2$ is known to the firm.

¹Random variables have a tilde; their realizations do not. Optimal values have an asterisk.

If a competitive forward market for currency i exists, exchange rate risk arising from $\tilde{s}_i$ can be traded. f_i is the exogenously determined forward price at date 0 for delivery of one unit of currency i at date 1. F_i is the amount of currency i the firm sells forward. Given the existence of forward markets for each currency, the firm's profits in domestic currency at date 1 are given by

$$\begin{aligned}\tilde{\Pi} = & \tilde{s}_1 R_1(q_1) + \tilde{s}_2 R_2(q_2) + F_1(f_1 - \tilde{s}_1) \\ & + F_2(f_2 - \tilde{s}_2) - C(q_1 + q_2).\end{aligned}\quad (1)$$

The firm's optimization problem at date 0 is

$$\max_{q_1, q_2, F_1, F_2} E[U(\tilde{\Pi})] \quad (2)$$

where $\tilde{\Pi}$ is defined in eq. (1). The firm has to make three decisions simultaneously: a production decision on the sum $(q_1 + q_2)$, an export allocation decision on how much to export to country 1 and how much to export to country 2, and a hedging decision on the optimal forward position(s). To keep the problem interesting, assume that the optimal export allocation always implies a positive export volume to each market, $q_1^*, q_2^* > 0$. Since the problem is concave, the following first order conditions are both necessary and sufficient for the unique optimum.

$$E[U'(\tilde{\Pi}^*) (\tilde{s}_1 R_1'(q_1^*) - C'(q_1^* + q_2^*))] = 0 \quad (3)$$

$$E[U'(\tilde{\Pi}^*) (\tilde{s}_2 R_2'(q_2^*) - C'(q_1^* + q_2^*))] = 0 \quad (4)$$

$$E[U'(\tilde{\Pi}^*) (f_1 - \tilde{s}_1)] = 0 \quad (5)$$

$$E[U'(\tilde{\Pi}^*) (f_2 - \tilde{s}_2)] = 0. \quad (6)$$

Before analyzing the firm's optimal decisions, some commonly used definitions concerning forward markets and forward positions are repeated. The risk premium in the forward market for currency i is $E[f_i - \tilde{s}_i]$. The forward market is said to be unbiased [biased] if it is zero [not zero]. If $E[f_i - \tilde{s}_i] < [>] 0$, the forward market exhibits backwardation [contango]. The situation in which the firm sells less [more] in the forward market than its revenues in currency i is called underhedging [overhedging], $F_i < [>] R_i(q_i)$. $F_i = R_i(q_i)$ is called full hedging.

OPTIMAL DECISIONS WITH ONE FORWARD MARKET

This section analyzes the firm's optimal production, export allocation, and hedging decisions, assuming that there is a forward market for currency

1 but no forward market for currency 2. As there is no forward market for currency 2, it is not possible to trade the $\tilde{s}_2$ risk and eq. (6) is irrelevant. In this case, the firm's profits are denoted by $\tilde{\Pi}_\diamond$.

The remainder of this section is organized as follows: The firm's optimal export allocation to the two export markets is analyzed first. Then, the optimal production decision is characterized for a special case. Finally, the optimal hedging decision will be derived for different exchange rate distributions. The optimal export allocation decision is as follows.

Proposition 1 Suppose there is a forward market for currency 1 but no forward market for currency 2. Then, the optimal export allocation decision is independent of the firm's preferences and the joint distribution of the exchange rates. It is given by

$$f_1 R'_1(q_1^*) = C'(q_1^* + q_2^*). \quad (7)$$

The export allocation rule in eq. (7) requires that the export volume to country 1 is determined such that marginal revenue, converted at the deterministic forward rate f_1 , equals marginal costs of total production. For any given production decision, if the export volume to country 1 is below the level implied by (7), reallocating export volume from country 2 to country 1 generates a riskless profit. Equation (7) implies a direct relation between export volume q_1 and export volume q_2 . Rewriting eq. (7), this relation is given by $q_2^* = C'^{-1}(f_1 R'_1(q_1^*)) - q_1^*$, where $C'^{-1}(\cdot)$ denotes the inverse of the function of marginal costs.² Neither the firm's utility function nor the joint distribution of the exchange rates appear in this relation. Thus, Proposition 1 is a separation result; the optimal export allocation decision is separable from the firm's preferences and its assessment of the joint distribution of the exchange rates despite the fact that there is untradable exchange rate risk arising from $\tilde{s}_2$.

In contrast to the export allocation decision, the optimal production decision is, in general, not separable, but it depends on preferences and on the joint distribution of the two exchange rates. However, there is a special case in which the production decision is separable as well. This case is characterized by a competitive export market in country 1, since eq. (7) can then be solved for optimal production $(q_1^* + q_2^*)$. The following result, which is a direct consequence of Proposition 1, deals with this special case.

Corollary 1 Suppose there is a forward market for currency 1 but no forward market for currency 2. If the export market in country 1 is

²Equation (7) also shows that assuming $q_1^* > 0$ is equivalent to assuming $f_1 R'_1(0) > C'(0)$.

competitive with $R'_1(\cdot) = p_1$, the optimal production decision is independent of the firm's preferences and the joint distribution of the exchange rates. It is given by

$$q_1^* + q_2^* = C'^{-1}(f_1 p_1). \quad (8)$$

Under the assumptions of Corollary 1, separation is a direct implication of the following argument: As long as marginal costs are lower than constant marginal revenue from exporting to country 1 converted at the deterministic forward rate, $f_1 p_1$, the firm can earn a riskless profit by increasing total production and exporting the additional output to country 1. Nonsatiation guarantees that this riskless profit opportunity will be used until it vanishes at the level of production implied by Corollary 1.

Holthausen (1979) and others prove a related separation result for the case of a single export market. Their result holds for decreasing as well as for constant marginal revenue. Corollary 1 shows that, in general, the optimal production decision is no longer separable if a second export market with untradable exchange rate risk is introduced. It is not surprising that the decision on the exposure toward the untradable $\tilde{s}_2$ risk is affected by the firm's preferences and the joint distribution of the exchange rates. This exposure is created by exporting q_2 to country 2. Since q_2 is directly linked to q_1 by Proposition 1, the firm's total production ($q_1 + q_2$) is not separable as well. According to Corollary 1, Holthausen's (1979) separation result remains valid for constant marginal revenue in the export market whose currency is tradable with forward contracts.

For the case of two export markets and the existence of one forward market, Broll, Wong, and Zilcha (1999) derive a closely related result. These authors show that the optimal production decision is separable. This coincides with Corollary 1. Since they assume constant marginal revenue for both export markets, their firm only exports to the country whose currency is tradable with forward contracts; exports to the other country are zero. Under the assumptions used here, the firm still exports to country 2, where marginal revenue is decreasing.

Now, turn to the firm's optimal hedging decision. Since the firm exports to both export markets, the joint stochastic behavior of the two exchange rates has to be taken into account when making the hedging decision. Omitting the arguments of $R_1(q_1)$, $R_2(q_2)$, and $C(q_1 + q_2)$ for notational simplicity, profits can be written as $\tilde{\Pi}_\diamond = \tilde{s}_1(R_1 - F_1) + f_1 F_1 + \tilde{s}_2 R_2 - C$. Thus, it is straightforward that the firm can directly hedge against fluctuations in $\tilde{s}_1$. Whether it can manage the randomness of $\tilde{s}_2$ through cross hedging depends on the joint distribution of the exchange rates.

The case of stochastically independent exchange rates is analyzed first. Under independence, there is no systematic relationship between $\tilde{s}_1$ and $\tilde{s}_2$ that would allow for cross hedging. Hence, the optimal forward position relative to the revenues in currency 1, $(F_1^* - R_1^*)$, depends only on the sign of the risk premium in the forward market, $E[f_1 - \tilde{s}_1]$.

Proposition 2 Suppose there is a forward market for currency 1 but no forward market for currency 2. Suppose further that $\tilde{s}_1$ and $\tilde{s}_2$ are stochastically independent. The firm fully hedges [overhedges] [underhedges] its revenues in currency 1 if, and only if, the forward market exhibits unbiasedness [contango] [backwardation].

Stochastic independence implies that the exposure to $\tilde{s}_2$ created by R_2^* is an additive independent background risk in the firm's profits. Since the $\tilde{s}_2$ risk is neither directly nor indirectly tradable, the best the firm can do in the forward market for currency 1 is to manage the $\tilde{s}_1$ risk. Thus, it is only the risk premium in the forward market for currency 1 that matters. If this risk premium is zero, the firm can eliminate the $\tilde{s}_1$ risk from its profits without altering expected profits. Risk aversion ensures that this behavior is optimal. If the risk premium is negative (contango), taking a long position in the $\tilde{s}_1$ risk yields an expected profit. Hence, the firm speculates on a negative risk premium by optimally assuming an overhedging position, $F_1^* > R_1^*$. Similarly, if the risk premium is positive (backwardation), a speculative short position in the $\tilde{s}_1$ risk increases expected profits. Thus, an underhedging position is optimal, $F_1^* < R_1^*$.

How much the firm speculates depends on its willingness to bear risks created by speculation as well as on the attractiveness of speculation as measured by the size of the risk premium. But Proposition 2 states that, given a non-zero risk premium, at least some $\tilde{s}_1$ risk will be taken. Hence, this result is an extension of the well-known result of Arrow (1965, 39) that any risk averter takes a risky position when there is a non-zero risk premium, derived under the existence of one single, tradable risk. Proposition 2 shows that this result still holds for additively combined independent risks where the initial exposure to both risks is endogeneously determined by export allocation and one risk is tradable. For multiplicatively combined risks, a related result has been derived by Adam-Müller (2000).

As Proposition 2 shows, the question of whether the forward market is unbiased or not is crucial for the optimal forward position. According to the unbiased forward rate hypothesis, currency forward markets should be unbiased. Nevertheless, testing this hypothesis is difficult. In spite of numerous tests and various methodologies employed, empirical evidence

is far from conclusive. For example, Naka and Whitney (1995) find some weak support for the unbiased forward rate hypothesis whereas Luintel and Paudyal (1998) reject the hypothesis. A detailed overview is provided by Engel (1996).

Proposition 2 is based on stochastic independence of the exchange rates. However, independence is a very strong assumption which does not necessarily represent a realistic description of exchange rate behavior. Typically, if a currency devalues (or revalues) relative to another currency, it tends to devalue (or revalue) in relation to other currencies as well. In this case, there is a positive correlation between exchange rates. Consequently, the optimal hedging decision should be analyzed for stochastically dependent exchange rates as well. This will be done in the following analysis, which concentrates on cases in which one exchange rate is a linear function of the other plus noise or vice versa. More precisely, it is alternatively assumed that either

(A.1) $\tilde{s}_1 = \alpha + \beta\tilde{s}_2 + \tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\beta \neq 0$ and stochastic independence of $\tilde{s}_2$ and $\tilde{\varepsilon}$

or

(A.2) $\tilde{s}_2 = \alpha' + \beta'\tilde{s}_1 + \tilde{\varepsilon}'$ with $E\tilde{\varepsilon}' = 0$, $\beta' \neq 0$ and stochastic independence of $\tilde{s}_1$ and $\tilde{\varepsilon}'$ holds.

Under (A.1), the exchange rates are positively correlated for $\beta > 0$. For negative β , the exchange rates are expected to move in opposite direction.³ For (A.2), the corresponding statements have to be based on β' .

Briys, Crouhy, and Schlesinger (1993) applied (A.1) and (A.2) to describe the behavior of spot and futures prices. The following proposition generalizes the result reported by Lence (1995), who used an assumption slightly weaker than (A.2) to analyze cross hedging in the presence of a single export market. Under (A.1) and (A.2), the optimal hedging decision can be characterized as follows.

Proposition 3 Suppose there is an unbiased forward market for currency 1 but no forward market for currency 2.

- a. Suppose (A.1) holds. The optimal forward position F_1^* is characterized by $F_1^* \in (R_1^*, R_1^* + R_2^*/\beta)$.
- b. Suppose (A.2) holds. The optimal forward position is given by $F_1^* = R_1^* + \beta' R_2^*$.

For $\beta, \beta' > 0$, the firm overhedges, for $\beta, \beta' < 0$ it underhedges. Under (A.2), the optimal forward position can be unambiguously determined whereas under (A.1), only a weaker statement can be derived. To

³ $\beta = 0$ is precluded, since it implies stochastic independence between $\tilde{s}_1$ and $\tilde{s}_2$.

see why, it is helpful to understand that the optimal position in the forward market for currency 1 serves two functions: First, it is used to directly manage the risk arising from the revenues in currency 1. Second, the forward market for currency 1 is also used to cross hedge the risk arising from revenues in currency 2. Cross hedging is possible, as the two exchange rates are stochastically dependent.

To fulfill the first function, the firm hedges its direct exposure toward $\tilde{s}_1$ as if the exchange rates were independent. Therefore, the direct hedging part of the optimal forward position is the same for (A.1) and (A.2). Since the forward market is assumed to be unbiased, there is no incentive for speculation. Hence, the direct hedging part is a full hedge with respect to the revenues in currency 1 and is thus equal to R_1^* . This full hedge eliminates the firm's direct exposure toward $\tilde{s}_1$. This is optimal since it does not affect expected profits but reduces risk. This direct hedging part does not appear in the cross hedging models of Broll, Wahl, and Zilcha (1995) and Lence (1995), since there is no second export market in these models.

The second function is cross hedging. In general, the optimal cross hedging position depends on the type of co-movement between the exchange rates. If $\tilde{s}_1$ and $\tilde{s}_2$ tend to move in parallel, cross hedging requires selling currency 1 forward in order to hedge against changes in exchange rate 2. However, the exact determination of the cross hedging part is sensitive to the specification of the joint distribution of the exchange rates as can be seen from Proposition 3. Therefore, the cross hedging part has to be explained separately for (A.1) and (A.2).

Under (A.1), a forward market for currency 1 can be interpreted as an opportunity to trade a 'package' that adds up to $\tilde{s}_1$. This package contains $\tilde{s}_2$ risk as well as $\tilde{\varepsilon}$ risk, where the proportion depends on β . The higher β , the more $\tilde{s}_2$ risk per unit of $\tilde{\varepsilon}$ risk is in the package. The firm's profits are given by $\tilde{\Pi}_\diamond = \tilde{s}_2[\beta(R_1 - F_1) + R_2] + (\alpha + \tilde{\varepsilon})(R_1 - F_1) + f_1 F_1 - C$ under (A.1). This indicates a definite relation between taking $\tilde{s}_2$ risk and taking $\tilde{\varepsilon}$ risk. Any deviation from full hedging, $F_1 = R_1$, and, thus, any cross hedging of the $\tilde{s}_2$ risk with forward contracts on $\tilde{s}_1$ adds undesirable basis risk $\tilde{\varepsilon}$ to the firm's profits. Hence, there is a trade-off between cross hedging the $\tilde{s}_2$ risk and increasing the $\tilde{\varepsilon}$ risk: The $\tilde{s}_2$ -risk could be eliminated by setting $F_1 = R_1 + R_2/\beta$, the $\tilde{\varepsilon}$ risk by setting $F_1 = R_1$. Proposition 3 states that the firm takes both risks into account and chooses an optimal forward position between these extremes. Thus, the firm optimally takes a position in the $\tilde{s}_2$ risk and in the $\tilde{\varepsilon}$ risk as well. How much of these risks is taken depends on the firm's preferences and the relative riskiness of $\tilde{s}_2$ and $\tilde{\varepsilon}$.

Under (A.2), $\tilde{s}_2$ risk is a package of $\tilde{s}_1$ risk and $\tilde{\varepsilon}'$ risk. The forward market for currency 1 allows the firm to trade the $\tilde{s}_1$ part of the $\tilde{s}_2$ risk separately. The firm's profits are given by $\tilde{\Pi}_\diamond = \tilde{s}_1[(R_1 - F_1) + \beta'R_2] + (\alpha' + \tilde{\varepsilon}')R_2 + f_1F_1 - C$ under (A.2). The basis risk $\tilde{\varepsilon}'$ is independent of F_1 , since the exposure toward the $\tilde{\varepsilon}'$ risk is exclusively determined by the optimal export volume to country 2, R_2^* . Thus, the optimal forward position is aimed at managing the $\tilde{s}_1$ risk only. A conflict between hedging the $\tilde{s}_1$ risk and running a basis risk from $\tilde{\varepsilon}'$ does not exist. Since the forward market for currency 1 is assumed to be unbiased, the $\tilde{s}_1$ -risk is eliminated by setting $F_1^* = R_1^* + \beta'R_2^*$, where $\beta'R_2^*$ is the cross hedging part. In contrast, the cross hedging part cannot be quantified under (A.1).

Proposition 3 characterizes the optimal forward position for two alternative assumptions on the joint distribution of the exchange rates. Whether (A.1) or (A.2) holds is a econometric question. A direct test of (A.1) against (A.2) is not possible, since the two equations in (A.1) and (A.2) form an underidentified system. Thus, there is an identification problem. However, using some forecasting model in which the exchange rates are explained by a vector of observable variables allows one to generate a testable identified system of equations. The only prerequisite is that there is at least one explanatory variable for each exchange rate that does not affect the other exchange rate. Details on this solution of the identification problem can be found in, for example, Manski (1995, 112–115). Of course, the econometric methods employed to estimate the identified system depend on the type of forecasting model used.

OPTIMAL DECISIONS WITH TWO FORWARD MARKETS

In this section, it is assumed that there are competitive forward markets for both currencies. As in the previous section, the optimal production decision and the optimal export allocation decision are analyzed first. Then, the hedging decision will be derived using the same assumptions on the joint distribution of the exchange rates as in the previous section. The optimal production decision and the optimal export allocation decision are as follows:

Proposition 4 Suppose there is a forward market for each currency. Then, the optimal export allocation decision as well as the optimal production decision are independent of the firm's preferences and the joint distribution of the exchange rates, since

$$f_1 R'_1(q_1^*) = f_2 R'_2(q_2^*) = C'(q_1^* + q_2^*). \quad (9)$$

Proposition 4 characterizes both the optimal export allocation decision

and the optimal production decision. The first equation of (9) is the optimal export allocation rule. If the firm allocated its total production differently, it could earn a riskless profit by reallocating export volume from one export market to the other. The profit were riskless, since revenues from both countries were converted at the deterministic forward rates.

Both equations of (9) taken together describe the optimal production decision. If the firm produced less than indicated by eq. (9), it could earn a riskless profit by increasing total production, exporting additional output according to the export allocation rule and converting the revenues at the known forward rates.

As is obvious from eq. (9), Proposition 4 is again a separation result. Thus, Proposition 4 implies that Holthausen's (1979) separation result is robust to the introduction of a second export market if additional exchange rate risk is tradable.

Comparison of Propositions 1 and 4 shows that the export allocation decision is separable from preferences and expectations disregard of whether there are one or two forward markets. When Proposition 4 is compared with Corollary 1, it becomes evident that a competitive export market in country 1 is not necessary for the production decision to be separable if there are two forward markets. However, if this export market is competitive, total production is the same with one and with two forward markets as follows from eqs. (8) and (9).

Now, consider optimal hedging. Since optimal exports are determined by Proposition 4, the exposure toward both exchange rates is given as well. Then, the optimal hedging decision is equivalent to a portfolio problem with two risky assets if full hedging in each forward market is used as a starting point. As in the previous section, optimal forward positions depend on the joint distribution of $\tilde{s}_1$ and $\tilde{s}_2$. Independently distributed exchange rates are analyzed first.

Proposition 5 Suppose there is a forward market for each currency. Suppose further that $\tilde{s}_1$ and $\tilde{s}_2$ are stochastically independent. In each currency, the firm fully hedges [overhedges] [underhedges] its revenues if, and only if, the forward market exhibits unbiasedness [contango] [backwardation].

In each currency, the firm assumes a speculative position if there is a non-zero risk premium. In the absence of a risk premium, risk aversion ensures that the associated exchange rate risk will be fully hedged. Cross hedging is impossible, since the exchange rates are independent. As in the previous section, the extent of speculation is determined by the firm's risk tolerance and the size of the risk premium but at least some risk will be taken whenever there is a non-zero risk premium.

Now, the optimal forward positions are analyzed for the case of dependent exchange rates using (A.1) and (A.2).

Proposition 6 Suppose there is a forward market for each currency and the forward market for currency 1 is unbiased. Suppose further that either (A.1) or (A.2) holds:

- a. If the forward market for currency 2 exhibits backwardation, an underhedging position in the forward market for currency 2 is optimal. The optimal position in the forward market for currency 1 is given by $F_1^* \in (R_1^*, R_1^* + (R_2^* - F_2^*)/\beta)$ for (A.1) and by $F_1^* = R_1^* + \beta'(R_2^* - F_2^*)$ for (A.2).
- b. If the forward market for currency 2 exhibits contango, an overhedging position in the forward market for currency 2 is optimal. The optimal position in the forward market for currency 1 can be characterized as in part a.
- c. If both forward markets are unbiased, full hedging is optimal.

Given linearly dependent exchange rates, Proposition 6 shows how the unbiased forward market for currency 1 will be used to cross hedge open positions taken using the biased forward market for currency 2.

First, consider the optimal position in the forward market for currency 2. Proposition 6 states that its sign is exclusively determined by the risk premium in this forward market. In particular, if the forward market for currency 2 is biased, the firm assumes a speculative position.

Second, consider the optimal position in the forward market for currency 1. Since the two exchange rates are stochastically dependent, this position again serves two functions. The first is direct hedging exchange rate risk from revenues in currency 1, the second is cross hedging an open position in currency 2.

The direct hedging part of the forward position in currency 1 is the same as in the previous section. It leads to a full hedge of the revenues in currency 1, since this forward market is unbiased. It is the same for (A.1) and (A.2) and equals R_1^* .

In addition, the optimal position in the forward market for currency 1 is affected by an open position in currency 2. As in the previous section where there was no forward market for currency 2 (Proposition 3) the firm can make use of the forward market for currency 1 to cross hedge an exposure toward $\tilde{s}_2$. Qualitatively, the optimal positions in the forward market for currency 1 in Propositions 3 and 6 are closely related. The only difference is that in the previous section (Proposition 3), the exposure toward $\tilde{s}_2$ was given by $R_2^* > 0$, whereas in the present section, the

exposure toward $\tilde{s}_2$ is given by $(R_2^* - F_2^*)$, which can be either positive, negative, or zero. The mechanics of cross hedging an exposure toward $\tilde{s}_2$ using the forward market for currency 1 are the same as in the previous section.

There is, however, an important economic difference. In the previous section, the exposure toward $\tilde{s}_2$ was created by exporting to country 2, the $\tilde{s}_2$ risk was untradable, and cross hedging with forward contracts for currency 1 was the only alternative for managing the $\tilde{s}_2$ risk. Thus, it was impossible to eliminate both the $\tilde{s}_1$ risk and the $\tilde{s}_2$ risk. In the present case, there is a forward market for both currencies. Hence, it is possible to eliminate exchange rate risk altogether by fully hedging in both forward markets. But if hedging is costly, it is not optimal to do so. In the present context, the exposure toward $\tilde{s}_2$ is the result of both exporting to country 2 and directly managing the $\tilde{s}_2$ risk with forward contracts on currency 2. In contrast to the previous section, there are two opportunities for hedging the $\tilde{s}_2$ risk: A direct hedging opportunity in the form of the forward market for currency 2 and an indirect hedging opportunity in the form of the forward market for currency 1. Parts a and b of Proposition 6 deal with the case in which the direct hedging opportunity is costly and the indirect hedging opportunity is costless, both in terms of expected profits. In part c, hedging is costless in both forward markets such that full hedging in both forward markets is optimal and cross hedging is not needed.

As an example, consider part a of Proposition 6 in more detail. Here, directly hedging the $\tilde{s}_2$ risk is costly, as there is a positive risk premium in the forward market for currency 2. Therefore, the firm assumes an underhedging position in the forward market for currency 2, $R_2^* > F_2^*$. Cross hedging the position in the $\tilde{s}_2$ risk using the forward market for currency 1 can be interpreted similar to Proposition 3: For positively [negatively] correlated exchange rates, an overhedging [underhedging] position in the forward market for currency 1 is optimal. Part b can be interpreted analogously.

Proposition 6 shows that it is optimal to partly substitute the biased forward market for currency 2 by the unbiased forward market for currency 1. The firm saves hedging costs in the forward market for currency 2. However, this can only be done at the cost of running a basis risk under (A.1) and (A.2). This can be seen as follows: Under (A.1), the firm's profits are given by $\tilde{\Pi} = \tilde{s}_2[\beta(R_1 - F_1) + (R_2 - F_2)] + (\alpha + \tilde{\epsilon})(R_1 - F_1) + K$, where $K = f_1R_1 + f_2R_2 - C$. For $F_1 = R_1 + (R_2 - F_2)/\beta$, there is no $\tilde{s}_2$ risk in the firm's profits. For $F_1 = R_1$, there is no basis risk from $\tilde{\epsilon}$. Hence, basis risk is created by cross hedging in the forward market for

currency 1. According to Proposition 6, the firm chooses a compromise by taking a position between these extremes. A numerical example in which an increase in basis risk leads to a reduction of cross hedging is given in the next section. Under (A.2), profits are given by $\tilde{\Pi} = \tilde{s}_1[(R_1 - F_1) + \beta'(R_2 - F_2)] + (\alpha' + \tilde{\varepsilon}')(R_2 - F_2) + K$. Exposure towards basis risk ε' is created by deviating from the (costly) full hedge in the forward market for currency 2. Since this exposure cannot be managed in the forward market for currency 1, this forward market is used for a costless elimination of the $\tilde{s}_1$ risk. This requires $F_1^* = R_1^* + \beta'(R_2^* - F_2^*)$.

For both (A.1) and (A.2), the optimal position in the forward market for currency 1 is qualitatively the same in Proposition 3. Of course, the quantities sold in the forward markets will be different, owing to wealth effects arising from different optimal production and export allocation.

As is clear from Proposition 6, full hedging in an unbiased forward market is not optimal if the other forward market is biased and the risks are dependent. If the unbiased market is also used for cross hedging other risks, over- or underhedging is optimal. In a single export market model, this effect cannot be captured.

AN EXAMPLE

The purpose of this example is to illustrate Propositions 4 and 6, derived under the assumption of forward markets for both currencies. The firm's preferences are represented by a utility function that belongs to the hyperbolic absolute risk aversion (HARA) class of utility functions. This class is given by

$$U^{\text{HARA}}(\Pi) = \frac{1 - \gamma}{\gamma} \left(\frac{a\Pi}{1 - \gamma} + b \right)^{\gamma} \quad (10)$$

with constants a , b , and γ .⁴ Let $a = 1$, $b = 10$, and $\gamma = -1$ such that absolute risk aversion and absolute prudence are decreasing in profits. The revenue functions are given by $R_i(q_i) = q_i(500 - q_i)$, $i = 1, 2$. The cost function is $C(q_1 + q_2) = 5(q_1 + q_2)^2$. The distributions of the exchange rates are based on (A.1) and (A.2) where each exchange rate is a linear function of the other and noise. For simplicity, $\alpha = \alpha' = 0$ and $\beta = \beta' = 1$. Then, (A.1) reduces to $\tilde{s}_1 = \tilde{s}_2 + \tilde{\varepsilon}$. $\tilde{s}_1$ is riskier than $\tilde{s}_2$ in the sense of Rothschild and Stiglitz (1970) since $\tilde{s}_1$ comprises $\tilde{\varepsilon}$ as an

⁴By choosing particular values of b and γ , widely used utility functions such as the negative exponential ($\gamma \rightarrow -\infty$, $b = 1$), the power ($\gamma < 1$, $b = 0$), the logarithmic ($\gamma = b = 0$) and the quadratic ($\gamma = 2$) class of utility functions can be obtained from the HARA class.

TABLE I
Distributional Assumptions

For	Realization of	State 1	State 2	State 3	State 4	Expectation
(A.1)	$\tilde{s}_2$	0.9	0.9	1.1	1.1	1.0
	$\tilde{\varepsilon}$	$+\varepsilon$	$-\varepsilon$	$+\varepsilon$	$-\varepsilon$	0.0
(A.2)	$\tilde{s}_1$	0.9	0.9	1.1	1.1	1.0
	$\tilde{\varepsilon}'$	$+\varepsilon'$	$-\varepsilon'$	$+\varepsilon'$	$-\varepsilon'$	0.0

TABLE II
Optimal Decisions for Different Levels of f_2

f_2	Export Alloc.		open pos. under (A.1)		open pos. under (A.2)	
	q_1^*	q_2^*	$(R_1^* - F_1^*)$	$(R_2^* - F_2^*)$	$(R_1^* - F_1^*)$	$(R_2^* - F_2^*)$
1.00	22.73	22.73	0	0	0	0
0.99	23.77	21.48	-1,406	7,074	-1,410	1,410
0.98	24.82	20.22	-2,784	14,261	-2,811	2,811
0.97	25.88	18.95	-4,133	21,856	-4,220	4,220
0.96	26.95	17.66	-5,461	30,271	-5,658	5,658
0.95	28.04	16.36	-6,781	40,163	-7,145	7,145
0.94	29.14	15.04	-8,116	52,706	-8,705	8,705
0.93	30.25	13.71	-9,495	70,340	-10,367	10,367
0.92	31.37	12.36	-10,974	99,455	-12,165	12,165
0.91	32.50	10.99	-12,692	165,929	-14,144	14,144

additional noise term. Similarly, (A.2) simplifies to $\tilde{s}_2 = \tilde{s}_1 + \tilde{\varepsilon}'$ such that $\tilde{s}_2$ is riskier than $\tilde{s}_1$. At date 1, there are four states with equal probability. The specifications of $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{\varepsilon}$, and $\tilde{\varepsilon}'$ are summarized in Table I. In the following discussion, basis risks $\tilde{\varepsilon}$ and $\tilde{\varepsilon}'$ will be varied.

It follows from Table I that $E\tilde{s}_1 = E\tilde{s}_2 = 1$. Throughout, the forward market for currency 1 is assumed to be unbiased, $f_1 = 1$. The forward market for currency 2 exhibits varying levels of backwardation. Hence, the example refers to part a of Proposition 6.

Table II exhibits the optimal export allocation and hedging decisions for different levels of backwardation in the forward market for currency 2. Basis risk is fixed at $\varepsilon = \varepsilon' = 0.20$. The first column shows the forward rate for currency 2. At $f_2 = 0.99$, the forward market is only slightly biased, at $f_2 = 0.91$, there is strong backwardation. Columns 2 and 3 show the optimal export allocation to both markets. This illustrates Proposition 4. The optimal hedging decisions, presented in columns 4 and 5 for (A.1) and in columns 6 and 7 for (A.2), illustrate Proposition 6.

Table II can be interpreted as follows: The lower the forward rate for currency 2, the less attractive exporting to country 2 is. This is true in absolute terms as well as in relation to exports to country 1. Hence, exports to country 2 decrease and exports to country 1 increase as f_2 declines. However, the net effect on total production is negative. According to Proposition 4, it is independent of the exchange rate distribution and, hence, the same for (A.1) and (A.2).

Owing to backwardation, the optimal open position in currency 2 is an underhedging position, $(R_2^* - F_2^*) > 0$. The higher the risk premium in the forward market for currency 2, the higher the speculative position. Positive correlation between $\tilde{s}_1$ and $\tilde{s}_2$ allows the firm to cross hedge the open position in currency 2 by selling forward additional units of currency 1. Hence, overhedging in currency 1 is optimal, $(R_1^* - F_1^*) < 0$. As the open position in currency 2 grows, the open position in currency 1 increases as well (in absolute terms).

Under (A.1), the open position in currency 1 cross hedges only part of the $\tilde{s}_2$ risk from the open position in currency 2. This is because any cross hedging creates basis risk $\tilde{\varepsilon}$ that were absent otherwise. Therefore, the proportion of $(R_2^* - F_2^*)$ cross hedged by $(R_1^* - F_1^*)$ decreases from nearly 20% at $f_2 = 0.99$ to less than 8% at $f_2 = 0.91$. Under (A.2), the basis risk $\tilde{\varepsilon}'$ is independent from cross hedging. Hence, the firm cross hedges the entire exposure toward $\tilde{s}_1$ created by the speculative position in the forward market for currency 2. Thus, $-(R_1^* - F_1^*) = (R_2^* - F_2^*)$.

Comparing the optimal forward position under (A.1) and (A.2), it is obvious that the firm speculates much more aggressively under (A.1). This is because the $\tilde{s}_2$ risk is much less volatile under (A.1) than under (A.2). Hence, a given exposure toward $\tilde{s}_2$ is much more risky under (A.2) than under (A.1).

Table III shows the effect of a mean preserving spread in basis risk. A mean preserving spread is equivalent to an increase in ε for (A.1) and in ε' for (A.2). This is shown in the first column. f_2 is fixed at 0.99. In each scenario, $q_1^* = 23.77$ and $q_2^* = 21.48$. q_1^* and q_2^* are independent of basis risk, in line with Proposition 4.

For (A.1), an increase in basis risk implies that cross hedging creates higher basis risks. This reduces the attractiveness of cross hedging and, indirectly, the attractiveness of taking speculative positions in the forward market for currency 2 as well. For (A.2), the exposure toward basis risk $\tilde{\varepsilon}'$ is created by underhedging in currency 2. A mean preserving spread of $\tilde{\varepsilon}'$ reduces the attractiveness of speculation directly. Accordingly, the open

TABLE III
Optimal Decisions for Different Levels of ε and ε'

ε for (A.1) ε' for (A.2)	open pos. under (A.1)		open pos. under (A.2)	
	$(R_1^* - F_1^*)$	$(R_2^* - F_2^*)$	$(R_1^* - F_1^*)$	$(R_2^* - F_2^*)$
0.10	-5,668	11,336	-5,672	5,672
0.15	-2,508	8,184	-2,510	2,510
0.20	-1,406	7,074	-1,410	1,410
0.25	-901	6,580	-902	902
0.30	-626	6,306	-626	626
0.35	-459	6,140	-460	460
0.40	-352	6,033	-352	352
0.45	-278	5,959	-278	278
0.50	-225	5,906	-225	225

position $(R_2^* - F_2^*)$ and the associated cross hedging position $(R_1^* - F_1^*)$ decrease in ε' (in absolute terms).⁵

CONCLUSION

This article examines the optimal decisions of a risk-averse international firm that exports to two segmented foreign markets and manages exchange rate risk with forward contracts.

If there is a forward market for one currency but no forward market for the other, the results are as follows. The allocation of the firm's total production to the export markets is independent of its preferences and the joint distribution of the exchange rates. However, optimal production itself is not independent, except for the case of constant marginal revenue in the country for which currency there is a forward market. For stochastically independent exchange rates, cross hedging is impossible. Hence, the optimal forward position only depends on the risk premium in the forward market. For dependent exchange rates, results differ significantly. If tradable exchange rate risk is a linear function of the untradable exchange rate risk plus independent basis risk, as in (A.1), cross hedging and running a basis risk are conflicting objectives. The optimal forward position is characterized by a compromise between these objectives. Alternatively, if the untradable exchange rate is a function of the tradable exchange rate and basis risk, as in (A.2), basis risk does not affect cross hedging.

⁵The open positions $(R_1^* - F_1^*)$ for (A.1) and (A.2) converge as basis risk increases. This is attributable to the four-state nature of the example.

If there is a forward market for each currency, the following decisions are optimal: The firm's total production and its allocation between the two export markets is independent of its preferences and the joint distribution of the exchange rates. For stochastically independent exchange rates, only the risk premium in the forward market under consideration matters. For linearly dependent exchange rates as in (A.1) and (A.2), it is optimal to speculate on a non-zero risk premium in one forward market and to cross hedge the risk from speculation using the other, unbiased forward market. Again, the effect of basis risk depends on whether (A.1) or (A.2) holds.

These results indicate that the existence of currency forward markets and the joint distribution of the exchange rates play an important role for decisions made by exporting firms on production, export allocation, and risk management. In particular, optimal currency risk management with forward contracts is very sensitive to the specification of the joint distribution of the exchange rates.

APPENDIX

Proof of Proposition 1 and Corollary 1

From (5), replace $E[\tilde{s}_1 U'(\tilde{\Pi}_\diamond^*)]$ by $f_1 E[U'(\tilde{\Pi}_\diamond^*)]$ in (3). (7) immediately follows. Using $R'_1(\cdot) = p_1$, (8) follows from rearranging (7).

Proof of Proposition 2

(5) is equivalent to $E[U'(\tilde{\Pi}_\diamond^*)] E[f_1 - \tilde{s}_1] = \text{cov}(\tilde{s}_1, U'(\tilde{\Pi}_\diamond^*)) = \text{cov}(\tilde{s}_1, E[U'(\tilde{\Pi}_\diamond^*)|\tilde{s}_1])$. Hence, $U''(\cdot) < 0$ and $dE[U'(\tilde{\Pi}_\diamond^*)|\tilde{s}_1]/d\tilde{s}_1 = E[U''(\tilde{\Pi}_\diamond^*)|\tilde{s}_1] (R_1 - F_1)$ imply $\text{sgn } E[f_1 - \tilde{s}_1] = \text{sgn } (F_1^* - R_1^*)$.

Proof of Proposition 3

Using $E[f_1 - \tilde{s}_1] = 0$ and (A.1), (5) can be written as

$$-\text{cov}(\tilde{s}_1, U'(\tilde{\Pi}_\diamond^*)) = -\beta \text{cov}(\tilde{s}_2, U'(\tilde{\Pi}_\diamond^*)) - \text{cov}(\tilde{\varepsilon}, U'(\tilde{\Pi}_\diamond^*)) = 0. \quad (11)$$

(A.1) implies $\tilde{\Pi}_\diamond = \tilde{s}_2[\beta(R_1 - F_1) + R_2] + (\alpha + \tilde{\varepsilon})(R_1 - F_1) + f_1 F_1 - C$. Then, $U''(\cdot) < 0$ and $dE[U'(\tilde{\Pi}_\diamond^*)|\tilde{s}_2]/d\tilde{s}_2 = E[U''(\tilde{\Pi}_\diamond^*)|\tilde{s}_2](\beta(R_1 - F_1) + R_2)$ imply

$$\begin{aligned} \operatorname{sgn} \operatorname{cov}(\tilde{s}_2, E[U'(\tilde{\Pi}_\diamond)|s_2]) &= \operatorname{sgn} \operatorname{cov}(\tilde{s}_2, U'(\tilde{\Pi}_\diamond)) \\ &= -\operatorname{sgn}(\beta(R_1 - F_1) + R_2). \end{aligned} \tag{12}$$

Similarly, $U''(\cdot) < 0$ and $dE[U'(\tilde{\Pi}_\diamond)|\varepsilon]/d\varepsilon = E[U''(\tilde{\Pi}_\diamond)|\varepsilon](R_1 - F_1)$ imply

$$\operatorname{sgn} \operatorname{cov}(\tilde{\varepsilon}, E[U'(\tilde{\Pi}_\diamond)|\varepsilon]) = \operatorname{sgn} \operatorname{cov}(\tilde{\varepsilon}, U'(\tilde{\Pi}_\diamond)) = \operatorname{sgn}(F_1 - R_1). \tag{13}$$

Now, one can show that $F_1^* < [>] R_1^* + R_2^*/\beta$ if $\beta > [<] 0$. Let $\hat{F}_1 = R_1 + R_2/\beta$ such that $(\beta(R_1 - \hat{F}_1) + R_2) = 0$ and evaluate the LHS of (11) at $\hat{F}_1$. (12) implies $\operatorname{cov}(\tilde{s}_2, U'(\tilde{\Pi}_\diamond)) = 0$, (13) implies $\operatorname{cov}(\tilde{\varepsilon}, U'(\tilde{\Pi}_\diamond)) > [<] 0$ for $\beta > [<] 0$. Thus, $-\operatorname{cov}(\tilde{s}_1, U'(\tilde{\Pi}_\diamond)) < [>] 0$ such that the first order condition evaluated at $\hat{F}_1$ is negative [positive] for $\beta > [<] 0$. It follows from the concavity of the problem that $F_1^* < [>] R_1^* + R_2^*/\beta$.

Next, it will be shown that $F_1^* > [<] R_1^*$ if $\beta > [<] 0$. Evaluate the LHS of (11) at $\check{F}_1 = R_1$. (13) implies $\operatorname{cov}(\tilde{\varepsilon}, U'(\tilde{\Pi}_\diamond)) = 0$; (12) implies $\operatorname{cov}(\tilde{s}_2, U'(\tilde{\Pi}_\diamond)) < 0$. Therefore, the first order condition at $\check{F}_1$ is positive [negative] if $\beta > [<] 0$. The concavity of the problem implies $F_1^* > [<] R_1^*$. This proves part a.

To prove part b, use (A.2) to rewrite profits as $\tilde{\Pi}_\diamond = \tilde{s}_1 [(R_1 - F_1) + \beta'R_2] + (\alpha' + \tilde{\varepsilon}')R_2 + f_1F_1 - C$. Since $\tilde{s}_1$ and $\tilde{\varepsilon}'$ are independent, $(\alpha' + \tilde{\varepsilon}')R_2$ is an additive independent background risk. Since the $\tilde{s}_1$ risk can be costlessly hedged, it is optimal to eliminate it completely. Hence $F_1^* = R_1^* + \beta'R_2^*$.

Proof of Propositions 4 and 5

These proofs are omitted since the proof of Proposition 4 [5] is based on exactly the same arguments as the proof of Proposition 1 [2].

Proof of Propositions 6

As a result of Proposition 4, R_1^* and R_2^* can be taken as given. Let $X_i^* = R_i^* - F_i^*$ denote the optimal open position in currency i . Then, profits are given by $\tilde{\Pi} = (\tilde{s}_1 - f_1)X_1 + (\tilde{s}_2 - f_2)X_2 + K$, with $K = f_1R_1 + f_2R_2 - C$. For $E[f_1 - \tilde{s}_1] = 0$, (5) is equivalent to $\operatorname{cov}(\tilde{s}_1, U'(\tilde{\Pi}^*)) = 0$.

First, the optimal forward positions are derived for (A.1). To derive X_1^* under (A.1), rewrite (5) as

$$\operatorname{cov}(\tilde{s}_1, U'(\tilde{\Pi}^*)) = \beta \operatorname{cov}(\tilde{s}_2, U'(\tilde{\Pi}^*)) + \operatorname{cov}(\tilde{\varepsilon}, U'(\tilde{\Pi}^*)) = 0. \tag{14}$$

(A.1) implies $\tilde{\Pi} = \tilde{s}_2(\beta X_1 + X_2) + (\alpha + \tilde{\varepsilon})X_1 - f_1X_1 - f_2X_2 + K$. Then, $U''(\cdot) < 0$ and $dE[U'(\tilde{\Pi})|s_2]/ds_2 = E[U''(\tilde{\Pi})|s_2](\beta X_1 + X_2)$ imply

$$\begin{aligned}\text{sgn cov}(\tilde{s}_2, E[U'(\tilde{\Pi})|s_2]) &= \text{sgn cov}(\tilde{s}_2, U'(\tilde{\Pi})) \\ &= -\text{sgn}(\beta X_1 + X_2).\end{aligned}\quad (15)$$

Similarly, $U''(\cdot) < 0$ and $dE[U'(\tilde{\Pi})|\varepsilon]/d\varepsilon = E[U''(\tilde{\Pi})|s_2]X_1$ imply

$$\text{sgn cov}(\tilde{\varepsilon}, E[U'(\tilde{\Pi})|\varepsilon]) = \text{sgn cov}(\tilde{\varepsilon}, U'(\tilde{\Pi})) = -\text{sgn } X_1. \quad (16)$$

Using $E[f_2 - \tilde{s}_2] < 0$ for part a, eq. (6) requires $\text{cov}(\tilde{s}_2, U'(\tilde{\Pi}^*)) < 0$, which, by eq. (15), is equivalent to $(\beta X_1^* + X_2^*) > 0$. Then, by eqs. (14) and (16), $\text{cov}(\tilde{\varepsilon}, U'(\tilde{\Pi}^*)) > [<] 0$ for $\beta > [<] 0$ is equivalent to $X_1^* < [>] 0$.

Now, derive X_2^* under (A.1). Since eq. (6) requires $\text{cov}(\tilde{s}_2, U'(\tilde{\Pi}^*)) < 0$ and $\beta X_1^* < 0$ as just shown, $X_2^* > 0$ directly follows from eq. (15). Combining and rearranging yields $F_1^* \in (R_1^*, R_1^* + (R_2^* - F_2^*)/\beta)$. This completes the proof of part a for (A.1).

Second, derive the optimal forward positions for (A.2). X_2^* will be derived first. Rewrite eq. (6) using (A.2) and $\text{cov}(\tilde{s}_1, U'(\tilde{\Pi}^*)) = 0$ as

$$E[U'(\tilde{\Pi}^*)] E[f_2 - \tilde{s}_2] = \text{cov}[\tilde{\varepsilon}', U'(\tilde{\Pi}^*)]. \quad (17)$$

Under (A.2), profits are given by $\tilde{\Pi} = \tilde{s}_1(X_1 + \beta'X_2) + (\alpha' + \tilde{\varepsilon}')X_2 - f_1X_1 - f_2X_2 + K$. Hence, $U''(\cdot) < 0$ and $dE[U'(\tilde{\Pi})|\varepsilon']/d\varepsilon' = E[U''(\tilde{\Pi})|\varepsilon']X_2$ imply

$$\text{sgn cov}(\tilde{\varepsilon}', E[U'(\tilde{\Pi})|\varepsilon']) = \text{sgn cov}(\tilde{\varepsilon}', U'(\tilde{\Pi})) = -\text{sgn } X_2. \quad (18)$$

Since $E[f_2 - \tilde{s}_2] < 0$ in part a, eqs. (17) and (18) imply $X_2^* > 0$.

Now, derive X_1^* under (A.2). Using $U''(\cdot) < 0$ and $dE[U'(\tilde{\Pi})|s_1]/ds_1 = E[U''(\tilde{\Pi})|s_1](X_1 + \beta'X_2)$ yields

$$\begin{aligned}\text{sgn cov}(\tilde{s}_1, E[U'(\tilde{\Pi})|s_1]) &= \text{sgn cov}(\tilde{s}_1, U'(\tilde{\Pi})) \\ &= -\text{sgn}(X_1 + \beta'X_2).\end{aligned}\quad (19)$$

By eq. (19), $\text{cov}[\tilde{s}_1, U'(\tilde{\Pi}^*)] = 0$ requires $X_1^* + \beta'X_2^* = 0$. Thus, $X_1^* < [>] 0$ for $\beta' > [<] 0$, since $X_2^* > 0$. Rearranging yields $F_1^* = R_1^* + \beta'(R_2^* - F_2^*)$. This completes the proof of part a for (A.2).

Since the proofs of parts b and c use exactly the same arguments as the proof of part a they are omitted.

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