Optimal Hedging of Contingent Exposure: The Importance of a Risk Premium

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The focus of this article is how a non-zero risk premium affects an economic agent's optimal hedging decision when exposed to a non-marketed event. The analysis is not confined to the optimal use of one particular hedging instrument, rather, the optimal payoff based on the agent's preferences is derived. We show, for various preferences, how the size of a risk premium affects the degree of nonlinearity in the optimal hedging instrument. This result is in contrast to

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known results for contingent exposure in the case of a zero risk premium. We demonstrate an inefficacy of the approach of confining the analysis to one particular hedging instrument in the case of standard exposure. Jrl Fut Mark 20:823–841, 2000

INTRODUCTION

Faced with a situation of financial risk, an agent may or may not choose to engage in hedging activities. An agent exposed to financial risk with probability one is said to be faced with a *standard exposure*. In this article, we study an exposure contingent on a nonmarketed event, as illustrated by the following standard example.

Consider a contractor who bids 1 unit of foreign currency in an auction for a construction project abroad. The probability of winning the contract is assumed to be p. Hence, with probability p the contractor will receive a fixed amount of foreign currency at a fixed future date, say time 1, and with probability $(1 - p)$ he will receive nothing. Currency and derivatives of currency are by assumption traded in a competitive financial market. However, no assets contingent on the outcome of the auction are available for trade. The auction thus represents a nonmarketed event. We assume that the probability p is independent of the market value of the currency.

Note that by letting p be equal to 1 or 0, we recover a setting of standard exposure or no exposure, respectively. The purpose of this article is to investigate how the contractor should design an optimal hedge. A complete hedge, i.e., a hedge eliminating all uncertainty, is not possible for p strictly positive and less than 1 because then the claim is dependent on a nonmarketed event.

In the hedging literature, a frequently used approach is to confine the analysis to the use of one particular hedging instrument. That is, a decision maker first chooses an instrument like a forward, futures, or option contract; the optimal use of that instrument in order to manage the financial risk is then analyzed.

In this article, we do not take the hedging instrument for given but derive the optimal hedging instrument characterized by its payoff as a function of the value of the underlying asset. We focus on how this optimal payoff is affected by the presence of a nonmarketed exposure and a nonzero risk premium.

The present work is related to that reported by Steil (1993). He works within the contingent exposure setting, and his analysis is not confined to one particular financial derivative. However, he assumes unbiased prices of contingent claims. Several models of optimal hedging (e.g., Lapan, Moschini, & Hanson, 1991; Moschini and Lapan, 1995) are based on the assumption of unbiased prices of contingent claims in the sense that a futures price or option premium equal the agent's expected payoff from the respective contracts. A risk premium is then defined in terms of deviations between the market price and the expected payoffs, defined as biases in the cited papers.

By contrast, we introduce the risk premium directly on the underlying assets as in a standard Black–Scholes economy. Here redundant contingent claims can be priced by arbitrage, thus any risk premium on the contingent claim is a function of the risk premium of the underlying asset. By introducing a nonzero risk premium we are able to study the combined effects of a risk premium and a nonmarketed exposure on the shape of the optimal payoff function.

This article is also related to works on optimal portfolio insurance by Leland (1980) and Brennan and Solanki (1981). Leland (1980) characterizes investors in terms of how their risk aversion and expectations differ from the market average, and focuses on who will benefit from nonlinear payoffs. In particular, he analyzes the demand for portfolio insurance. Brennan and Solanki (1981) provide the explicit form of this optimal payoff, for agents with preferences exhibiting hyperbolic absolute risk aversion (HARA). This problem is further analyzed by Carr and Madan (1997) using a similar framework as in the present article. Relating these papers to our work, their setting resembles the case of no exposure, i.e., our case of $p = 0$.

Other related works within the hedging literature include Briys, Crouhy, and Schlesinger (1993), Moschini and Lapan (1992), Eaker and Grant (1985), Feiger and Jacquillat (1979), Giddy (1985), Lehrbass (1994), Stulz (1984), Kerkvliet and Moffet (1991), and Adler and Detemple (1988).

Our approach yields three important insights. First, we confirm the well known result that a nonzero risk premium induces an expected utility maximizer to hold currency independently of the project considered. Thus, it is natural to consider two components of an optimal contingent claim. The *initial component* is the optimal position before the considered auction. The *optimal hedge component* is the contingent claim required to optimally alter the exposure from the project.

Second, we show that both the presence of a nonmarketed exposure as well as the size of the risk premium has a direct effect on the shape of the optimal payoff. We analyze optimal payoffs for the cases of quadratic, exponential, and logarithmic utilities. For $p = 1$, i.e., the case of standard exposure, the optimal hedge has a linear payoff only for the case of exponential utility. When the exposure is contingent, i.e., p different from 1 or 0, the size of p affects the curvature of the optimal payoff for all preferences considered. The initial component is nonlinear for all analyzed preferences and is, of course, independent of p. When the risk premium approaches 0, the optimal hedge becomes increasingly linear. When the risk premium is equal to zero, a linear hedge is optimal and the initial component is zero.

Third, we illustrate limitations of the standard approach, where the analysis is confined to the optimal use of particular contingent claims, like forward, futures or options. The standard procedure is to find an optimal fraction of the exposure to hedge with one or several favored instruments, and normally these fractions vary with the size of the risk premium. We show that a change in the risk premium demands a different payoff, rather than a different exposure to the same payoff. A replicating strategy for the optimal claim can be implemented either dynamically with the underlying assets or by a static holding of marketed contingent claims (see e.g., Carr and Madan, 1997). Rather than choosing the instruments and their properties somewhat arbitrarily prior to the optimization, knowledge of the optimal payoff guides us to choose the replicating instruments.

This article first describes the economic model and then outlines preferences and specification. The optimal hedging claims are then analyzed under different preferences of the HARA type.

ECONOMIC MODEL

The goal of this study is the optimal design of a contingent claim based on a contractor's preferences and s. Our setup differs from the standard Black–Scholes economy by allowing a nonmarketed outcome. The mentioned contractor receiving a payment in foreign currency with some positive probability is our standard example of this friction.

The existence of a positive risk premium in the financial market encourages the contractor to undertake some investments independently of the project considered. In the following sub-section relevant results for the contractor's *initial problem* by Brennan and Solanki (1981), Carr and Madan (1997), and Cox and Huang (1989) are collected.

The Initial Problem

The contractor makes his investments at the initial date, time 0, and all uncertainty is resolved at the terminal date, time 1. The contractor is not allowed to rebalance his portfolio between time 0 and time 1. The contractor's preferences are represented by a von Neuman-Morgenstern utility function ($U' > 0$ and $U'' < 0$) for time 1 wealth only.

The only source of uncertainty is the foreign exchange rate at time 1, denoted by S (number of domestic units per foreign unit). Formally, S is a random variable with the non-negative real numbers as support. The initial exchange rate is given by the constant S_0.

The contractor seeks the optimal allocation of terminal wealth by investing in a claim contingent on the foreign exchange rate. This claim is denoted by $g(S)$. As described in the next section, this initial claim needs to be supplemented by another claim, termed a hedging claim, because the contractor is faced with an additional currency exposure from the described project.

The contractor's initial wealth is denoted by W_0 and the domestic interest rate r_d is assumed constant. Initially, the contractor wants to invest in the contingent claim solving the problem

$$\max_{g(s)} E[U(W)] = \max_{g(s)} \int_0^\infty U[g(s)]f(s)ds \tag{1}$$

that is, he maximizes expected utility based on his s about the exchange rate at time 1, represented by the probability density $f(\cdot)$. He is constrained by his time zero wealth W_0 and the market prices prevailing at time zero,

$$\int_0^\infty e^{-r_d}\, g(s)q(s)ds = W_0 \tag{2}$$

From standard theory of finance we know that absence of arbitrage opportunities and dynamic completeness is sufficient within our model for the existence of a unique risk-adjusted probability density $q(\cdot)$.

The Lagrangean of the problem is

$$\mathscr{L} = \int_0^\infty U[g(s)]f(s)ds - \lambda\left(\int_0^\infty e^{-r_d}\, g(s)q(s)ds - W_0\right)$$

where λ is the constant Lagrange multiplier. The first-order condition of this problem is

$$\lambda = U'[g(S)]\,\frac{f(S)}{g(S)e^{-r_d}} \tag{3}$$

Consider the Arrow-Debreu security represented by the indicator function $1\ \{\omega \in [s, s + ds)\}$, where ω represents the state of the world. This security pays one unit if and only if a given state ω occurs. Its expected payoff is $f(s)ds$, whereas the time zero market price is $q(s)e^{-r_d}$. From expression (3) we see that the agent chooses the payoff such that the expected gross[1] return of all Arrow-Debreu securities multiplied by the marginal utility of the optimal claim is constant through different states. This is an established result in economics that can be traced back to Borch (1962).

The optimal payoff $g(S)$ in terms of the inverse marginal utility function can be determined from expression (3) as

$$g(S) = [U']^{-1} \left(\lambda \frac{q(S)e^{-r_d}}{f(S)} \right) \tag{4}$$

The claim $g(S)$ represents the payoff of the investments the contractor would like to undertake whether he is exposed to additional foreign exchange risk. In the following section, the described hedging problem is introduced.

The Hedging Problem

Now the contractor's total exposure when faced with the prospect of receiving one unit of S with probability p is considered. In addition to the initial claim $g(S)$ from expression (4), the contractor now also invests in a hedging claim, denoted by $h(S)$, to account for the additional currency exposure. The total optimal payoff $y(S)$, excluding the potential currency payoff from the project, is then the sum of the initial claim and the hedging claim, i.e., $y(S) = h(S) + g(S)$.

The contractor now seeks the contingent claim that solves the following problem

$$\max_{y(s)} E[U(W)] = \max_{y(s)} \int_0^{\infty} [pU(s + y(s)) + (1 - p)U(y(s))]\, f(s)ds \tag{5}$$

subject to the budget constraint

$$\int_0^{\infty} e^{-r_d}\, y(s)q(s)ds = W_0 \tag{6}$$

[1]Recall that payoff/price = gross return = 1 + net return.

Note in the formulation of the problem (5) and (6), that if $p = 0$, i.e., the probability of receiving S is zero, $y(S)$ is equal to $g(S)$ given by (1). Note also that if $p = 1$, the problem is reduced to an ordinary hedging problem in which the exposure does not depend on a nonmarketed event.

We form the Lagrangean,

$$\mathcal{L} = \int_0^\infty [pU(s + y(s)) + (1 - p)U(y(s))]f(s)ds - \lambda$$

$$\left(\int_0^\infty e^{-r_d}\, y(s)q(s)ds - W_0\right)$$

where again λ represents the Lagrange multiplier. The first order condition for this problem is

$$pU'(S + y(S)) + (1 - p)U'(y(S)) = \lambda\,\frac{q(S)e^{-r_d}}{f(S)} \tag{7}$$

The structure of this equation is somewhat more complex than the corresponding equation (3) for the case without the additional nonmarketed exposure.

The left-hand side of (7) can be interpreted as the expected marginal utility of the optimal claim, including expectations with respect to nonmarketed outcomes. Thus, it has a close resemblance to the analysis of background risk, see e.g., Franke, Stapleton, and Subrahmanyam (1998).

PREFERENCE AND BELIEF SPECIFICATIONS

In this section preferences are specified. We apply a general HARA utility function with various further restrictions on the parameters to, for example, obtain results for utility functions with constant absolute and constant relative risk aversion.

The Optimal Initial Payoff Under Preference Restrictions

Table 1 displays most important characteristics of the preference specifications considered in this study. To simplify the presentation of the solution of equation (4), the following security payoffs are defined, each corresponding to the four different utility functions of Table I:

$$c_H(S) = \left(\frac{f(S)}{q(S)e^{-r_d}}\right)^{1/\gamma}$$

$$c_Q(S) = \left(\frac{f(S)}{q(S)e^{-r_d}}\right)^{-1}$$

TABLE I

Utility Functions

Name	*General HARA*	*Quadratic*	*Exponential*	*Logarithmic*
		$\tau = 2b$		$\tau = 1$
		$\gamma = -1$	$\gamma \uparrow \infty$	$\gamma \rightarrow 1$
Parameter		$A = -1$	$A = -1$	$A = 1$
$U(W)$	$\frac{\gamma}{1-\gamma}\left(\frac{\tau}{\gamma} W - A\right)^{1-\gamma}$	$W - bW^2$	$-e^{-\tau W}$	$\ln(W)$
$U'(W)$	$\tau\left(\frac{\tau}{\gamma} W - A\right)^{-\gamma}$	$1 - 2bW$	$\tau e^{-\tau W}$	$\frac{1}{W}$
$U''(W)$	$-\tau^2\left(\frac{\tau}{\gamma} W - A\right)^{-\gamma-1}$	$-2b$	$-\tau^2 e^{-\tau W}$	$-\frac{1}{W^2}$
Rest.	$\frac{\tau}{\gamma} W \geq A$	$b > 0,\ W \leq \frac{1}{2b}$	$\tau > 0$	
ARA	$\frac{\tau}{\frac{\tau}{\gamma}} W + A$	$\frac{2b}{1-2bW}$	τ	$\frac{1}{W}$
RRA	$\frac{\tau W}{\frac{\tau}{\gamma}} W + A$	$\frac{2bW}{1-2bW}$	τW	1

$$c_E(S) = \ln\left(\frac{f(S)}{q(S)e^{-r_d}}\right)$$

and

$$c_L(S) = \frac{f(S)}{q(S)e^{-r_d}}$$

The corresponding time zero market prices are $\pi_H = \int_0^\infty c_H(s)\, e^{-r_d}\, q(s)ds$, $\pi_Q = \int_0^\infty c_Q\,(s)e^{-r_d}\, q(s)ds$, $\pi_E = \int_0^\infty c_E(s)\, e^{-r_d}\, q(s)ds$, and $\pi_L = \int_0^\infty c_L(s)\, e^{-r_d}q(s)ds = 1$.

Here $c_H(S)$, $c_Q(S)$, and $c_L(S)$ are non-negative in all states ($f(\cdot)$ is a probability density) and represent the expected gross return of an Arrow-Debreu security raised to a power depending on the agent's risk aversion. In contrast, the payoff $c_E(S)$ may take negative values.

The solution for $g_H(S)$ of expression (4) for general HARA utility is

$$g_H(S) = \frac{\gamma}{\tau} A + \frac{W_0 - e^{-r_d}\frac{\gamma}{\tau} A}{\pi_H}\, c_H(S)$$

(see, e.g., Carr and Madan, 1997). The contractor wants to invest the amount $\gamma/\tau\ Ae^{-r_d}$ to the interest rate r_d and the remaining in the risky security $c_H(S)$ (if it exists). This property is usually referred to as two-fund separation. The quantity invested to the rate r_d is independent of initial wealth and the corresponding payoff $\gamma/\tau\ A$ may in some cases be interpreted as the subsistence wealth level. Since investments in the risky securities $c_H(S)$, $c_Q(S)$, and $c_L(S)$ cannot yield negative payoffs, the subsistence level also represents the minimum payoff of the total investments for agents characterized by these three preferences.

If quadratic utility is assumed, the optimal payoff is

$$g_Q(S) = \frac{1}{2b} + \frac{W_0 - \frac{1}{2b}\, e^{-r_d}}{\pi_Q}\, c_Q(S) \tag{8}$$

When exponential utility is assumed we obtain

$$g_E(S) = \left(W_0 - \frac{1}{\tau}\, \pi_E\right) e^{r_d} + \frac{1}{\tau}\, c_E(S) \tag{9}$$

The investment in the risky security is decreasing with increasing absolute risk aversion τ and is independent of initial wealth W_0.

Finally, logarithmic utility is assumed. The optimal payoff is now

$$g_L(S) = W_0 c_L(S) \tag{10}$$

Observe that the market price at time zero of $c_L(S)$ is 1. In this case, one-fund separation applies; i.e., the contractor wants to invest his complete initial wealth in the security $c_L(S)$.

Based on this analysis we may conclude that each of the four securities introduced earlier represents the only risky securities an investor wants to invest in (if they exist) under the four different preference scenarios.

Belief Specification

Both the risk-adjusted probability density $q(\cdot)$ and the contractor's belief density $f(\cdot)$ are assumed to be log-normal. A log-normal probability density with parameters m and v for the terminal stock price S is

$$\ell(m, v) = \frac{1}{\sqrt{2\pi} v S} \exp\left(-\frac{1}{2v^2}\left(\ln\left(\frac{S}{S_0}\right) - \left(m - \frac{1}{2}\, v^2\right)\right)^2\right)$$

We assume that $f(S) = \ell(\mu, \sigma)$ and $q(S) = \ell(r_d - r_f, \sigma)$. This choice is consistent with the setup in continuous time by Garman and Kohlhagen (1983) which is based on the standard model[2] by Black and Scholes (1973) and Merton (1973). Note that both densities, i.e., both the contractor and the market, agree on the volatility rate σ. Here μ can be interpreted as the contractor's belief about the expected instantaneous proportional change of the currency, whereas the constant r_f is interpreted as the risk-free rate in the foreign country.

We define

$$\beta = \frac{\mu - (r_d - r_f)}{\sigma^2}$$

a constant, commonly termed Sharpe's ratio. We may interpret $\mu - (r_d - r_f)$ as the risk premium connected to investments in foreign currency, and β may be viewed as the market price of risk. We see that the expected proportional change in the exchange rate, μ, is equal to $r_d - r_f + \beta\sigma^2$, which corresponds with theories suggesting that this change is related to an interest rate differential and a risk premium, see Hodrick (1987) and Lewis (1995). Note that the assumption of a risk premium equal to zero implies that $\mu + r_f = r_d$, i.e., $q(\cdot) = f(\cdot)$.

With this complete belief specification we are able to explicitly characterize the following payoff, related to the Arrow-Debreu security and discussed earlier under equation (3),

$$\frac{f(S)}{q(S)e^{-r_d}} = CS^{\beta} \tag{11}$$

where the constant $C = S_0^{-\mu-(r_d-r_f)/\sigma^2} \exp(1/2\, \beta(r_f - r_d + \sigma^2 - \mu) + r_d)$. Based on this belief specification and the expressions for the optimal claims described earlier, we arrive at the following fully specified payoff functions:

$$c_Q(S) = \frac{1}{C} S^{-\beta}$$

$$c_E(S) = \ln(C) + \beta \ln(S)$$

$$c_L(S) = CS^{\beta}$$

[2]where the price process is specified as $dS_t = mS_t dt + \sigma S_t dW_t$, where dW_t represents the increment of a standard Wiener process.

By inserting these expressions into the expressions (8), (9), and (10), respectively, we derive the optimal payoff functions for our assumed beliefs. It is apparent that the numerical value of β affects the shape of the optimal payoff functions. The sign of β determines the sign of the slope, e.g., a positive β leads to optimal payoffs increasing in S. For $-1 < \beta < 0$, the optimal payoff functions are convex (and decreasing); for $0 < \beta < 1$, the optimal payoff functions are concave (and increasing). In the numerical example we have chosen a positive β less than one. Under the hypothesis of uncovered interest parity, one should not expect values of β far from zero.

Figure 1 depicts the optimal initial contingent claim for the three preference scenarios. The three utility functions are calibrated so that the coefficient of relative risk aversion (RRA) equals one[3] for wealth level $W_0 e^{r_d}$, i.e., as all wealth was invested in the risk-free domestic asset. The three plotted g functions are similar, i.e., increasing and concave functions of roughly same shape for our choice of parameters. In addition, the 45-degrees line is plotted as well as the probability density function for S under the original probability measure to indicate the likelihood of the outcomes. The numerical values of the financial parameters are assumed to be $r_d = 0.04$, $r_f = 0.06$, $\mu = 0$, and $\sigma = 0.3$, then $\beta = 2/9$ and the constant $C = 1.0536$.

THE OPTIMAL HEDGING CLAIM

In this section the optimal payoffs for the quadratic, exponential, and logarithmic utility functions are analyzed. Before we proceed we recall the connection between the forward price and the spot price of currency in this model given by $F_0 = S_0 e^{r_d - r_f}$.

Quadratic Utility

Using the quadratic utility from Table I and equation (7) we find by first solving for $y_Q(S)$, then determining λ from equation (6), and substituting back in the expression for $y(S)$ that

$$y_Q(S) = \frac{1}{2b} - pS + \frac{W_0 + \left(pF_0 - \frac{1}{2b}\right) e^{-r_d}}{\pi_Q} c_Q(S)$$

[3]by increasing the RRA coefficient the resulting curves for the quadratic and exponential utility functions are still similar, but flatter.

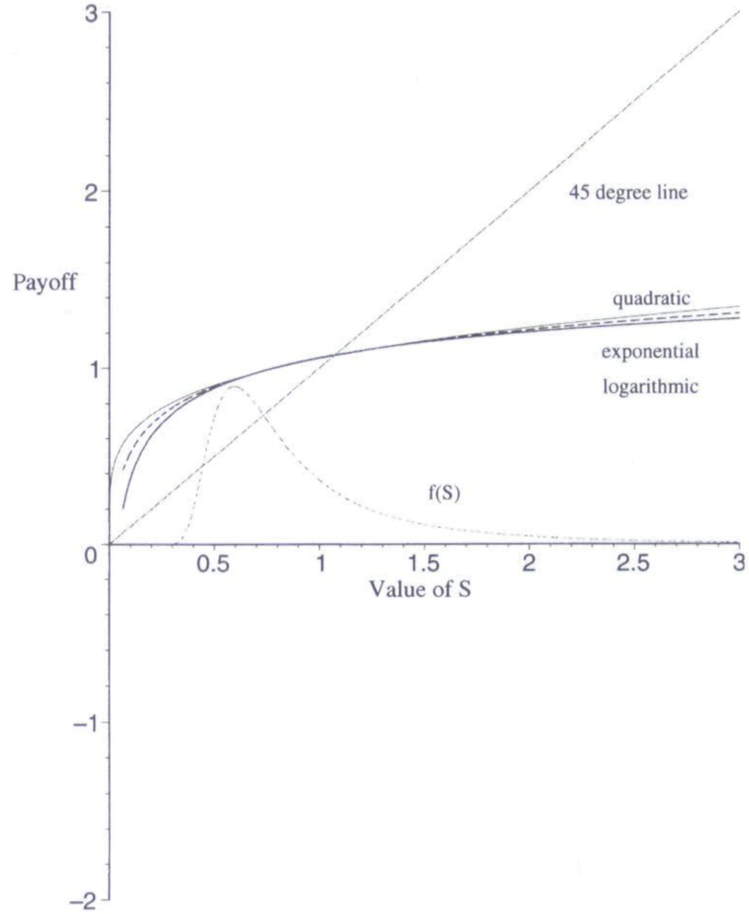

FIGURE 1
Plot of $g_Q(S)$, $g_E(S)$, and $g_L(S)$ for the parameter values $\beta = \frac{2}{9}$, $C = 1.0536$, $W_0 = 1$, $b = 0.2402$, $\tau = 0.9608$.

Two-fund separation does not longer hold since a total of three different securities are used: Riskfree investment, the security $c_Q(S)$, and the forward contract. Instead three fund separation is obtained.

The optimal hedging claim is

$$h_Q(S) = p\left(e^{-r_d} F_0 \frac{c_Q(S)}{\pi_Q} - S\right)$$

The optimal hedging strategy for quadratic utility is to sell p forward contracts on the foreign currency, borrow the present value of the forward contracts at time zero ($pF_0e^{-r_d}$) to invest in the security $c_Q(S)$.

The corresponding strategy by Steil (1993) is to sell p forward contracts. The positive risk premium introduces the asset $c_Q(S)$, which again leads to a somewhat more complicated strategy. The contractor's total investments in the security $c_Q(S)$ has increased, which again influences the optimal payoff $h(S)$. In particular, $h(S)$ is a nonlinear function of S, which is not the case in Steil's model. Furthermore, Steil's statement that the hedging strategy does not depend on initial wealth is proved.

Exponential Utility

Performing the same exercise for the exponential utility function, we obtain

$$y_E(S) = e^{r_d}\left(W_0 - \frac{\pi_p}{\tau} - \frac{\pi_E}{\tau}\right) + \frac{c_p(S)}{\tau} + \frac{c_E(S)}{\tau} \tag{12}$$

where $c_p(S) = \ln(pe^{-\tau S} + 1 - p)$ and $\pi_p = \int_0^\infty c_p(s)\, e^{-r_d}q(s)ds$, represents the corresponding time zero market price.

Also in this case, three-fund separation applies, the first term represents the terminal value of the investment to the interest rate r_d, the last term represents the investment in the security $c_E(S)$, and the term in the middle represents the investment in the security $c_p(S)$. In contrast to the quadratic case, the third fund is not the foreign currency, but a particular hedging claim with payoff $c_p(S)$.

Solving for the optimal hedging claim we obtain

$$h_E(S) = \frac{-\pi_p e^{r_d}}{\tau} + \frac{c_p(S)}{\tau} \tag{13}$$

Observe that the higher absolute risk aversion, the lower investment in the risky securities. Also note that $h_E(S)$ only involves the hedging claim $c_p(S)$ and, in particular, is independent of the risk premium.

A Note on the Classic Hedging Analysis with Exponential Utility

In order to compare our results with the classic approach of confining the analysis to the optimal use of forward contracts only, we now focus on the case of a standard, noncontingent exposure, i.e., $p = 1$.

From equation (13) with $p = 1$, it follows that

$$h_E(S) = F_0 - S$$

which is the payoff from a short forward contract. Receiving one unit of S from the project with probability one, is basically the same as increasing the initial wealth by one unit of S. Exponential utility implies that the agent has constant absolute risk aversion, so that the optimal amount invested in the risky asset should not vary with different levels of wealth. Hence, when $p = 1$, the optimal hedge, $h(S)$, should exactly offset the payoff from the project. This is obviously achieved by a forward contract.

We now denote the risk premium by $\rho = \mu - (r_d - r_f)$ so that $\beta = \rho/\sigma^2$, and compute for $p = 1$ from expressions (11) and (12)

$$y_E(S) = e^{r_d}W_0 + F_0 - S - \frac{\rho}{\tau\sigma^2}\,[e^{r_d}\pi_{\ln(S)} - \ln(S)] \tag{14}$$

where $\pi_{\ln(S)} = \int_0^\infty \ln(S)e^{-r_d}q(s)ds$.

The classic analysis differs from ours by solely focusing on the use of forward contracts. Only a brief review is given here; for more details, see, e.g., Newbery (1989).

Within the classical approach the total payoff from a similar exposure at time 1 can be written as

$$\kappa = S + z(f - S)$$

where z is the number of forward contracts used to hedge the cash-flow S.

The analysis proceeds by maximizing the certainty equivalent CE

$$CE = E[\kappa] - \frac{1}{2}\,\tau\sigma_\kappa^2$$

where σ_κ^2 is the variance of κ.

The expression for CE holds exactly for exponential utility and a normally distributed exchange rate. For a lognormally distributed exchange rate and exponential utility the expression for CE can only be considered as a second-order Taylor approximation.

It is straightforward to show that the optimal number of forward contracts is equal to

$$z^* = 1 - \frac{E[S] - F_0}{\tau\sigma^2}$$

Multiplying this with the payoff from a forward contract, yields the payoff of the optimal hedge from the classic analysis, denoted $\tilde{y}(S)$:

$$\tilde{y}(S) = (F_0 - S) - \frac{E[S] - F_0}{\tau\sigma^2}\,(F_0 - S) \tag{15}$$

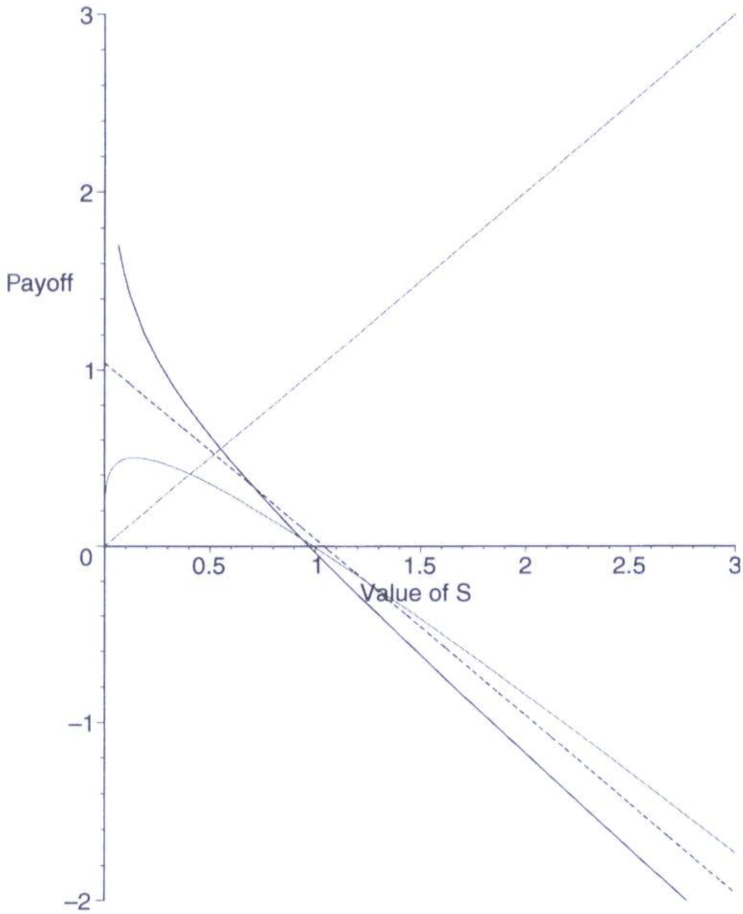

FIGURE 2
A plot of $h_Q(S)$, $h_E(S)$, and $h_L(S)$ for the parameter values $p = 1$, $\beta = \frac{2}{9}$, $C = 1.0536$, $W_0 = 1$, $b = 0.2402$, $\tau = 0.9608$.

Comparing equations (14) and (15), we first note the relationship between ρ and $(E[S] - F_0)$. Recall that $F_0 = S_0 e^{r_d - r_f}$ in our model. Furthermore, $E[S] = S_0 e^{\mu}$. A risk premium ρ equal to zero, implies that $\mu = r_d - r_f$. Hence, $E[S] = F_0$ and $z^* = 1$. We see that in the case of zero risk premium, $y_E(S) = \tilde{y}(S)$.

However, $y_E(S)$ and $\tilde{y}(S)$ are different when the risk premium is positive. A positive risk premium implies that $E[S] > F_0$, and z^* becomes less than one. Hence, the optimal forward position is reduced in the classic case, but still maintaining a linear exposure to the risky asset S. Our analysis, on the other hand, shows that the optimal exposure in the hedging claim is independent of the risk premium. Rather, a change in the risk premium should be absorbed by a change in the payoff of the initial

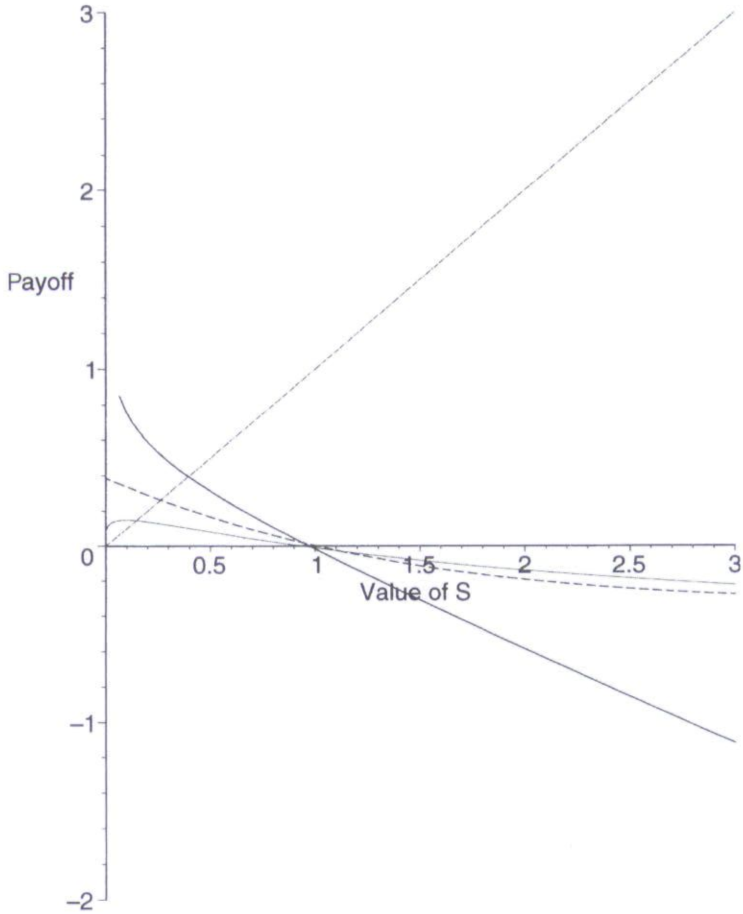

FIGURE 3
A plot of $h_Q(S)$, $h_E(S)$, and $h_L(S)$ for the parameter values $p = \frac{1}{2}$, $\beta = \frac{2}{9}$, $C = 1.0536$, $W_0 = 1$, $b = 0.2402$, $\tau = 0.9608$.

claim. This interesting and important result is overlooked by the restrained approach of confining the analysis to one particular hedging instrument.

Logarithmic Utility

Also in this case we proceed by substituting the marginal utility from Table I into equation (7). This leads to a quadratic expression for $y(S)$ with solution

$$y_L(S) = \frac{1}{2}\left(\frac{1}{\lambda} c_L(S) - S\right) \pm \sqrt{\frac{1}{4}\left(\frac{1}{\lambda} c_L(S) - S\right)^2 + \frac{1}{\lambda} S(1 - p)c_L(S)}$$

This can be rewritten as

$$y_L(S) = \frac{1}{2}\left(\frac{1}{\lambda}c_L(S) - S \pm \sqrt{\left(\frac{1}{\lambda}c_L(S) - (2p - 1)S\right)^2 + 4p(1 - p)S^2}\right)$$

From the last expression, we see that it may be hard to find an analytical expression for the square root on the right-hand side for values of p that are different from 0 or 1.

We proceed by analyzing the case $p = 1$ and find that

$$y_L(S) = (F_0e^{-r_d} + W_0)c_L(S) - S$$

With the help of equation (10), the optimal hedging claim is determined as

$$h_L(S) = F_0e^{-r_d}\, c_L(S) - S$$

Comparing this payoff with the corresponding one based on quadratic utility for $p = 1$, we see that both utility functions imply the same optimal strategy: Sell one forward contract, borrow the present value of the forward price at time zero, invest the borrowed amount in the risky asset, and $c_L(S)$ in the logarithmic case and $c_Q(S)$ in the quadratic case.

In Figure 2 the optimal hedging claims are plotted for $p = 1$. Whereas the corresponding initial claims were similar for the three utility functions, the optimal hedging claims show fundamental differences: The curve for the quadratic utility function is convex, the curve for exponential utility is linear, and the curve for logarithmic utility is concave four our choice of parameters.

The similar plot for $p = 1/2$ is shown in Figure 3. The most important difference for this case is that the curve for the exponential utility function is now convex, i.e., nonlinear.

CONCLUDING REMARKS

The prime focus of this article is how a non-zero risk premium affects an economic agent's optimal hedging decision. Our approach differs from that of Steil (1993), Brennan and Solanki (1981), Cox and Huang (1989), and Carr and Madan (1997) by introducing a risk premium, and by focusing on optimal contingent claims for hedging a nonmarketed exposure, rather than on optimal consumption and portfolio policies.

Our main result is to describe how the size of a risk premium influence the shape of the optimal payoff, both in the case of standard exposure as well as with a contingent exposure. The payoff is characterized explicitly for three different sets of preferences. We split the optimal contingent claim into a sum of an initial component and the hedge component. In the case of exponential utility, we demonstrate that the hedge component is independent of the risk premium. In case of quadratic and logarithmic utility, the hedge component exhibits similar nonlinearities as the initial component. We may interpret the pure hedge under both quadratic and logarithmic utility, as forward contracts plus the present value of the forward price invested in the same manner as the initial component.

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