Pricing American Options with Stochastic Volatility: Evidence from S&P 500 Futures Options

Kian Guan Lim
Xiaoqiang Guo*

This article is the first attempt to test empirically a numerical solution to price American options under stochastic volatility. The model allows for a mean-reverting stochastic-volatility process with non-zero risk premium for the volatility risk and correlation with the underlying process. A general solution of risk-neutral probabilities and price movements is derived, which avoids the common negative-probability problem in numerical-option pricing with stochastic volatility. The empirical test shows clear evidence supporting the occurrence of stochastic volatility. The stochastic-volatility model outperforms the constant-volatility model by producing smaller bias and better goodness of fit in both the in-sample and out-of-sample test. It not only eliminates systematic moneyness bias produced by the constant-volatility model, but also has better prediction power. In addition, both models

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*Correspondence author, Director, Centre for Financial Engineering, Kent Ridge Post Office, P.O. Box 1130, Singapore 911105; e-mail fbalimkg@nus.edu.sg.

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- *Kian Guan Lim is Associate Professor and Director of the Centre for Financial Engineering at the National University of Singapore in Singapore.*
- *Xiaoqiang Guo is a Research associate at the Centre for Financial Engineering at the National University of Singapore in Singapore.*

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perform well in the dynamic intraday hedging test. However, the constant-volatility model seems to have a slightly better hedging effectiveness. The profitability test shows that the stochastic volatility is able to capture statistically significant profits while the constant volatility model produces losses. Jrl Fut Mark 20:625–659, 2000

INTRODUCTION

Option pricing is one of the most exciting areas in modern financial research. Ever since the famous Black and Scholes (1973) study, there are many extensions and modifications to the option-pricing model. One of the most important developments in this area is the assumption of stochastic volatility. Wiggins (1987), Hull and White (1987), Scott (1987), Johnson and Shanno (1987), and Stein and Stein (1991)use different techniques such as finite difference, series solution, Monte Carlo simulations, and Feynman–Kac function, and so on to model and test models with stochastic volatility. Heston (1993) introduces the first closed-form solution for stochastic-volatility models with applications to European-style options.

Stochastic-volatility models address some of the issues such as volatility smiles and skewness, as well as large kurtosis in the underlying distribution. Finucane and Tomas (1996) developed a lattice method to price American-call options. They used interpolations to find out the option prices on the lattice, so recombining nodes on the tree is not an issue in their model. Furthermore, the model is able to calculate several option prices with different underlying asset prices and volatilities from the same lattice as long as the time to maturity and the strike prices are the same. Hence, this method is more efficient than other numerical methods, especially when working with intraday prices or performing scenario analysis.

This article extends Finucane and Tomas's (1996) method in the numerical pricing of American options under stochastic volatility in a nontrivial way involving a three-dimensional lattice. This modified Lattice-Based Approach (LBA) is an efficient and accurate numerical method that is not only theoretically sound, but also easy to implement. As far as we know, this article is the first empirical test of a numerical method in pricing American options with stochastic volatility. We are able to show that the new model fits actual prices better, performs better out-of-sample prediction, and also serves as an effective benchmark to signal under- or overpricing.

Researchers have been looking long for efficient numerical methods to price stochastic-volatility models. The direct application of the CRR binomial method would be computationally explosive due to the nonrecombining nature of the stochastic-volatility model. Moreover, American-style options add additional difficulties as they lack analytical solutions and often require numerical solutions due to the early exercise feature. Cox, Ross, and Rubinstein (1979) introduced the standard binomial option-pricing model. Barone-Adesi and Whaley (1987) and Geske and Roll (1984) have shown various methods to approximate the early exercise premium under the constant-volatility assumption.

There have been many empirical tests of stochastic-volatility models. Chesney and Scott (1989) used the stochastic-volatility model developed by Scott (1987) to study the prices of dollar–Swiss franc exchange rate. Melino and Turnbull (1990) used a similar model as Scott (1987) to study the Canadian dollar to U.S. dollar options traded in Philadelphia Exchange. Nandi (1996) empirically tested the Heston (1993) model with S&P 500 index options. Bates (1996) tested the stochastic-volatility model as part of his stochastic-volatility and jumps model using Deutsche mark American options. However, these empirical test either used European options, or used constant-volatility-model-based approximations for early exercise premium plus a stochastic-volatility-model solution for European options to test American options under stochastic volatility.

Hilliard and Schwartz (1996) developed a transformed recombining binomial tree. Transformations of the variables based on Nelson and Ramaswamy (1990) were performed to make sure that the nodes would recombine. However, the volatility of the process must be in the form of a constant multiple of the instantaneous volatility of the return of the underlying asset. This specification is required in the transformation process to achieve recombination of the tree. The transformation process will increase the magnitude of the underlying asset price and volatility, which could cause the pseudo-probabilities to be negative. In addition, most of the computation time is wasted on those nodes with nearly zero probabilities. Numerical methods for American options with stochastic volatility developed in the past also are less efficient and based on smaller sample size. Our LBA method provides for improvements over these shortcomings.

The rest of this article is divided into four sections. The next section derives the stochastic volatility model. Then we will show the interpolated lattice-based approach to implement the model. The section entitled "Empirical Tests" reports the empirical tests and the findings of the test results, and is followed by the conclusions.

MODEL

Underlying Assumptions

The interpolated-lattice approach in this article uses Heston's (1993) underlying assumption: define the squared volatility as the instantaneous variance of the return of the underlying asset. The underlying assets and squared volatility are assumed to follow the continuous-time-stochastic processes:

$$dS = \mu S dt + S\sqrt{V} dz_S \tag{1}$$

$$dV = \kappa(\theta - V)dt + b\sqrt{V} dz_V, \tag{2}$$

where μ is the expected return of the underlying asset. The squared volatility V is assumed to follow the square-root process, with long-run mean θ and the speed of adjustment κ. Therefore, the half life of the volatility shock is $\ln 2/\kappa$, b is the volatility of the squared volatility process, and ρ is the correlation coefficient between the two standard Brownian motions z_S and z_V.

Finucane and Tomas (1996) used the assumptions on diffusion processes of Hull and White (1987), in which both the underlying assets and squared volatility are assumed to follow the lognormal process. They then adopted the risk-neutral probability measure and discrete time jumps derived by Boyle (1988), which is an extension of the trinomial tree method for a one-factor model, and applied it to 5-nomial trees for options on two underlying assets. Boyle (1988) derived the probability measure based on the risk-neutral valuation principle, where the expected returns on both underlying assets are assumed to be the risk-free interest rate. While it is appropriate with traded assets like stocks and futures, it is arguable whether it can be used to price the volatility, which is not traded and unobservable. Many researchers believe that the market price or risk premium of the volatility should be taken into account. As a result, it is not appropriate to apply directly Boyle's probability parameters in the stochastic-volatility situation.

While it is difficult to specify the function of the market price of the risk of volatility, a simplified and intuitive way is to assume that it is a linear function of the squared volatility. After adding the market price λ to the above mean-reverting squared-volatility process, it becomes:

$$dV = [\kappa(\theta - V) - \lambda V]dt + \sigma_V\sqrt{V} dz_V. \tag{3}$$

Equation (3) can be reduced to the same initial form as in eq (2) with risk-neutralized parameters:

$$\kappa^* = \kappa + \lambda \qquad \theta^* = \frac{\kappa\theta}{\kappa + \theta}. \tag{4}$$

So when using the square-root process, we allow for mean reversion, correlation, and non-zero risk premium in the squared-volatility process. However, the parameters κ^* and θ^* estimated from the trading prices cannot be interpreted directly as the speed of adjustment and the long-run mean of the squared-volatility process.

Risk-Neutral Probabilities and Jumps

To approximate a continuous-time process using discrete-time jumps and probabilities, the first and second moments of the two processes must be matched. In most options-pricing literature, when the underlying asset is assumed to follow a log-normal process, we have the following result:

$$d \ln S = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dz. \tag{5}$$

We derive in Appendix A the solution for the risk-neutral probabilities and additive jumps (price movements) for our problem setup:

$$\begin{aligned}
u_S &= Sr\Delta t + S\sqrt{V\Delta t} \\
d_S &= Sr\Delta t - S\sqrt{V\Delta t} \\
u_V &= \kappa(\theta - V)\Delta t + b\sqrt{V\Delta t} \\
d_V &= \kappa(\theta - V)\Delta t - b\sqrt{V\Delta t} \\
p_{11} &= p_{22} = 0.25 + 0.25\rho \\
p_{21} &= p_{12} = 0.25 - 0.25\rho,
\end{aligned} \tag{6}$$

where u_S, u_V are upward jumps for S and V, d_S, d_V are downward jumps for S and V, p_{ij} is the joint probability of the jumps where i is 1 for up and 2 for down for the underlying asset process, and j represents the squared-volatility process.

One nice property of the above solution is that the joint probabilities are independent on the jump sizes, so they need not be calculated at every point on the lattice. The jumps are dependent on S, V, and other structural parameters. However, the S and V values are associated with points on the lattice and are the same across time, so the jumps need to be calculated only once. The method is more efficient when compared with other methods attempting to get a recombining tree, in which the jumps and probabilities usually need to be calculated at every node.

In addition, to make sure that the numerical approximation converges to the continuous-time counterpart, it is important to ensure that the four joint probabilities are within (0, 1), and together sum to one. This is not a problem when the volatility is assumed to be constant. However, when the volatility follows a stochastic process, normal binomial methods extended from Cox, Ross, and Rubenstein's (CRR) binomial method often lead to probabilities that are negative or greater than one, when probabilities need to be calculated everywhere in the tree. Not only is it costly in computation, a single node with a negative probability will cause the entire tree to break down. The problem is inherited partly from the fact that the CRR's tree requires a constraint on the magnitude of volatility. The problem is most severe with methods that use transformations to achieve recombining trees (e.g., the bivariate-binomial-tree approach in Hilliard & Schwartz, 1996. Our solution avoids this problem.

INTERPOLATED LATTICE-BASED APPROACH

The option price at time interval t can be calculated by discounting the expected option prices at time $t + 1$. Use C_t to denote the call price at time t, p to denote the respective probabilities of future states, and r_f to denote the risk-free rate. Mathematically, the option price is given by the equation:

$$\begin{aligned} C_t(S_t, V_t) = {} & [p_{11}C_{t+1}(S_{t+1}^+, V_{t+1}^+) + p_{12}C_{t+1}(S_{t+1}^+, V_{t+1}^-) \\ & + p_{21}C_{t+1}(S_{t+1}^-, V_{t+1}^+) + p_{22}C_{t+1}(S_{t+1}^-, V_{t+1}^-)/(1 + r_f\Delta t). \quad (7) \end{aligned}$$

If the option is American-style, early exercise possibility is checked at every node.

However, a direct application of the above solution using the binomial-tree method computationally is explosive. Since the nodes are not recombining, there will be 4^{n+1} nodes after n steps. In other words, it is more than one trillion nodes after 19 steps. Not only that, most of the computations will be wasted in calculating prices of nearly zero probability because the upper part of the tree will have asset prices up to millions and the lower part consists of all nearly zeros. However, it is different when we use the interpolated LBA, because the calculation time only increases linearly with the number of steps chosen, and we can set upper and lower limits of the lattice to improve the efficiency.

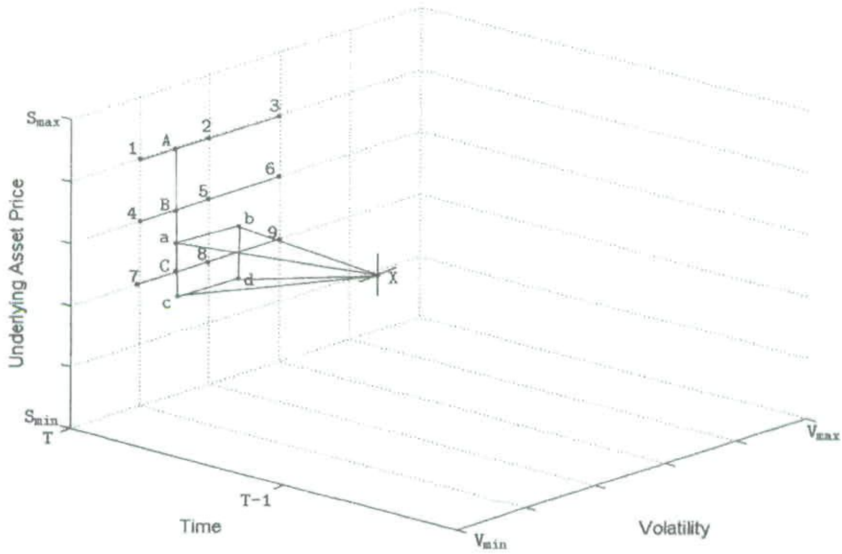

FIGURE 1
Graphical illustration of the lattice-based approach.

Under the lattice-based approach, a three-dimensional lattice (see Fig. 1) is constructed to represent the behavior of the underlying asset prices, and option prices are calculated from the lattice. S^{max} and S^{min} are used to denote the upper and lower limit for the asset price, and V^{max} and V^{min} to denote the limits for the volatility. The continuous range of S and V then is divided into N_S and N_V discrete intervals of length $\Delta S = (S^{max} - S^{min})/N_S$ and $\Delta V = (V^{max} - V^{min})/N_V$. The time to maturity then is divided into $N_T + 1$ points of time with intervals of $\Delta t = T/N_t$. At any point of time, $(N_S + 1)(N_V + 1)$ option prices are determined jointly by $N_S + 1$ underlying asset prices and $N_V + 1$ squared-volatility values.

At maturity time T, the option price for each S and V is known to be $\max(S - X,0)$ for calls, and $\max(X - S,0)$ for puts. So the $(N_S + 1)(N_V + 1)$ option prices at maturity time T are known. For any one of the $(N_S + 1)(N_V + 1)$ option prices at $t = T - 1$, e.g., for point X in Figure 1, the option price is determined in eq (7) by option prices at the four points a, b, c, and d at time T, which often are not on the lattice. We can obtain the required option prices through interpolation. For example, the option price at point a can be interpolated using the option price of the three points A, B, and C, which is on its vertical intercept with the three nearest S prices. The interpolation is applied again using points 1, 2, and 3 for A, 4, 5, and 6 for B, and 7, 8, and 9 for C. Similar interpolation can be applied to obtain option price at points b, c, and d as well. For S or V with values falling outside the lattice, extrapolation is applied using the top- or bottom-three points on the lattice. The approximating polynomial is a weighted average of the three nearest option prices. The interpolation polynomial is given by the following equation[1]:

$$
\begin{aligned}
C_t(S, V) = {} & \frac{S - S^{min} - (i + 1)\Delta S}{(S^{min} + i\Delta S) - (S^{min} + (i + 1)\Delta S)} \\
& \cdot \frac{S - S^{min} - (\mathrm{i} + 2)\Delta S}{(S^{min} + i\Delta S) - (S^{min} + (i + 2)\Delta S)} \\
& \cdot C_{t+1}(S^{min} + i\Delta S, V) + \frac{S - S^{min} - (i\Delta S)}{(S^{min} + (i + 1)\Delta S) - (S^{min} + i\Delta S)} \\
& \cdot \frac{S - S^{min} - (i - 2)\Delta S}{(S^{min} + (i + 1)\Delta S) - (S^{min} + i + 2)\Delta S)} \\
& \cdot C_{t+1}(S^{min} + (i + 1)\Delta S, V) + \frac{S - S^{min} - i\Delta S}{(S^{min} + (i + 2)\Delta S - (S^{min} + i\Delta S)} \\
& \cdot \frac{S - S^{min} - (i + 1)\Delta S}{(S^{min} + i + 2)\Delta S) - (S^{min} + (i + 1)\Delta S)} \cdot C_{t+1}(S^{min} + (i + 2)\Delta S, V)
\end{aligned} \tag{8}
$$

Graphically, the weight assigned for option price at point A is $(\overrightarrow{aB} \cdot \overrightarrow{aC})/(\overrightarrow{AB} \cdot \overrightarrow{AC})$, where $\overrightarrow{AB}$ is the distance from point A to B. The same polynomial is applied three times using option values of point on the lattice (points 1–9) with the nearest V and S values, respectively, to get the option price required for eq (8). It is applied one more time to get one of the four required option values used in eq (7). The same interpolation technique is applied at time 0 to enable the method to obtain several option prices with different underlying prices and volatilities at the same time. Since the model uses extrapolation for values that fall outside the lattice, the price of the options are not very sensitive to the boundaries of S and V. The discrete time nature of the method also allows it to price American options with discrete dividend payments. Moreover, deltas with respect to the underlying and the squared volatility of the options can be calculated directly from the lattice using the two nearest options price of different underlying prices or volatility.

There are potential problems with the interpolation of call prices. Firstly, at the maturity time, the option price is a kink function against underlying asset price, and thus is not differentiable. Secondly, if the slope of option price changes rapidly, which is observed for near-the-

[1]The equation used here is similar to that used by Finucane and Tomas (1996), except that we add the lower bounds to the lattice to improve computational efficiency.

maturity at-the-money options, interpolation may produce larger errors. Interpolation may pose a problem when the call prices are not very smooth with respect to the underlying asset price and the volatility. Lastly, when the jump size is large with respect to the density of the lattice, the extrapolation at the boundary of the lattice may be inaccurate. In addition, the error will multiply itself over the life of the option due to the nature of the method, and result in unacceptable option prices. However, we solve these problems to a large extent by checking for appropriate option prices where needed, and increasing the number of time intervals to decrease the size of jumps. For example, options that are slightly out of the money often obtain negative values when eq (8) is applied to obtain the interpolated value. We fix this problem by checking for negative values and setting them to zeros. The extrapolation error will be most significant when the term to maturity is long. The options used in this article generally are short-time-to-maturity options, so this is not a problem. However, we find that one possible way to deal with the extrapolation error is to increase the distances among the three points used in the extrapolation. Instead of choosing the top or bottom three points with very small intervals, which often exaggerates the error, we can choose some other points with more appropriate distances between each other.

Validity of LBA

Because interpolation is used, any analytical assessment of the validity of the method will be extremely complicated. A numerical comparison between a known analytical solution is useful to show its numerical convergence when the analytical solution is assumed to be the true price.

Firstly, the LBA model is compared with the Black–Scholes Model in valuing a European call option. We shall recognize that the LBA is actually a general form of the constant-volatility model. When the parameters κ, θ, b and ρ are all set to zero, the LBA is effectively valuing a constant-volatility option. In the comparison, N_S, N_V, and N_T are set to 100, 50, and 200, respectively, S^{max} and S^{min} are set to $1.5 \times X$ and $0.5 \times X$, respectively, and V^{max} and V^{min} are set to $5 \times V$ and 0, respectively.[2] The CRR model with 200 steps also is used to price the options. As we can see from Table I, the two numerical methods' prices are very close to the true price. The average absolute-percentage pricing error is around 0.1%.

[2]Finucane and Tomas (1996) used $2 \times X$ for S^{max} and zero for S^{min}, which is inefficient in valuing options with large underlying prices. We followed their boundary set for V, i.e., $5V$ for V^{max} and 0 for V^{min}.

TABLE 1

Comparison Between BS, CRR,and LBA Models

	$T = 1/12$			$T = 3/12$			$T = 6/12$		
	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
LBA									
S = 90	0.0886	0.4896	1.1451	0.8982	2.3063	3.9322	2.3474	4.7143	7.2063
S = 95	0.6552	1.5593	2.5568	2.2753	4.1069	5.9838	4.2559	6.9293	9.6218
S = 100	2.5101	3.6666	4.8082	4.6195	6.5820	8.5547	6.8856	9.6366	12.4094
S = 105	5.9894	6.8851	7.9056	7.9250	9.7130	11.6154	10.2024	12.8006	15.5466
S = 110	10.5206	10.9790	11.7168	11.9877	13.4022	15.1122	14.0729	16.3684	19.0055
CRR									
S = 90	0.0869	0.4910	1.1476	0.8990	2.3032	3.9358	2.3508	4.7219	7.2091
S = 95	0.6563	1.5537	2.5541	2.2745	4.1114	5.9840	4.2602	6.9342	9.6150
S = 100	2.5092	3.6543	4.7996	4.6100	6.5756	8.5427	6.8817	9.6244	12.3711
S = 105	5.9881	6.8775	7.8985	7.9268	9.7168	11.6187	10.2071	12.8072	15.5105
S = 110	10.5202	10.9800	11.7159	11.9893	13.4079	15.1168	14.0772	16.3663	18.9471
Black-Scholes									
S = 90	0.0878	0.4902	1.1451	0.8975	2.3087	3.9316	2.3494	4.7140	7.1993
S = 95	0.6557	1.5534	2.5547	2.2712	4.1053	5.9823	4.2545	6.9282	9.6072
S = 100	2.5121	3.6586	4.8053	4.6150	6.5831	8.5526	6.8887	9.6349	12.3850
S = 105	5.9901	6.8810	7.9036	7.9229	9.7123	11.6130	10.2013	12.7986	15.5057
S = 110	10.5202	10.9799	11.7163	11.9883	13.4025	15.1097	14.0754	16.3655	18.9359
ErrorLBA									
S = 90	0.90%	−0.12%	−0.00%	0.07%	−0.10%	0.01%	−0.08%	0.01%	0.10%
S = 95	−0.08%	0.38%	0.08%	0.18%	0.04%	0.03%	0.03%	0.02%	0.15%
S = 100	−0.08%	0.22%	0.06%	0.10%	−0.02%	0.02%	−0.05%	0.02%	0.20%
S = 105	−0.01%	0.06%	0.03%	0.03%	0.01%	0.02%	0.01%	0.02%	0.26%
S = 110	0.00%	0.01%	0.00%	−0.01%	−0.00%	0.02%	−0.02%	0.02%	0.37%
Mean	0.06%	MAE	0.09%	RMSE	0.18%				
Error CRR									
S = 90	−0.98%	0.17%	0.22%	0.17%	−0.24%	0.10%	0.06%	0.17%	0.14%
S = 95	0.09%	0.01%	−0.02%	0.14%	0.15%	0.03%	0.13%	0.09%	0.08%
S = 100	−0.11%	−0.12%	−0.12%	−0.11%	−0.11%	−0.12%	−0.10%	−0.11%	−0.11%
S = 105	−0.03%	−0.05%	−0.06%	0.05%	0.05%	0.05%	0.06%	0.07%	0.03%
S = 110	−0.00%	0.00%	−0.00%	0.01%	0.04%	0.05%	0.01%	0.01%	0.06%
Mean	0.00%	MAE	0.10%	RMSE	0.18%				

This table shows and compares European-call-option prices under constant-volatility assumption calculated using different models, namely the Black–Scholes model (BS) the Cox, Ross, and Rubinstein model (CRR), and our Lattice-Based Approach (LBA). To illustrate the accuracy of the model, we calculated prices for call options under different moneyness by choosing the underlying stock price from 90 to 110, under different volatilities by choosing σ of 0.2, 0.3, and 0.4, and under different time to maturities of 1 month, 3 months, and 6 months. Values chosen for common parameters are $X = 100$ and $r = 0.05$. The prices are simulated using these parameters. Two-hundred steps are used for both the CRR model and the LBA. For LBA, N_s and N_v are set to 100 and 50, respectively. The Pricing Error is measured as percentage of (model price − true price)/true price, taking the BS price as the true price. Mean Error (ME), Mean Absolute Error (MAE), and Root Mean Squared Error (RMSE) of the percentages are reported for the entire sample.

Secondly, to assess the LBA model's ability to price European options with stochastic volatility, it is compared with Heston's (1993) solution for valuing European options with stochastic volatility. The standard binomial-tree method is not used here because it is not computationally feasible to price options with stochastic volatility. As we can see from Table II, the LBA prices are very close to the true prices calculated using the analytical solution. Except for three cases of the out-of-the-money option, the absolute-percentage pricing errors are below 0.2%. Since there is no analytical solution or other known efficient and yet accurate numerical methods available, we cannot assess directly the validity of the LBA in valuing American options under stochastic volatility. However, it should be consistent with its accuracy in valuing European options.

However, one should note that, due to the interpolation, the relationship between the density and the boundary of the lattice and the accuracy of the method is not so straightforward. For example, since there are three dimensions of the lattice, just increasing one dimension will not necessarily increase the accuracy. Our simulated validity test result shows that our choice of the boundaries of the lattice, as well as N_S, N_T, and N_V is reasonably good in terms of both accuracy and computational speed.[3]

Lastly, we show graphically the effect of stochastic volatility on (American) options in Figure 2. The upper-left graph shows us the surface of European-call-option prices over time and across different moneyness. The European call options are calculated using the Black–Scholes formula using $X = 100$, $r = 0.05$, $\sigma = 0.2$, a dividend yield of 0.05, with S ranges from 90 to 110, and time to maturity from 0.5 to 0. Using a dividend yield that is equal to the risk-free interest rate is the same as valuing a futures option. As we can see, at the maturity ($t = T$), the option value is actually a kink function of S. As the time to maturity increases, the kinky line smoothes out and the time value increases. If we look carefully enough, we also can see that the time value of at-the-money option ($S = X = 100$) is the largest compared to in-the-money or out-of-the money options.

When taking stochastic volatility into consideration, the surface of the option value is basically the same as the constant-volatility value, since the differences are small with respect to the option values, so a separate graph is not shown. However, when we just look at the differences, there are some interesting patterns. The upper-right graph shows the differences of the European-call-options value with stochastic vola-

[3]The algorithms are implemented partly in Matlab and partly in C. The LBA model costs around 1 second on a Pentium II 333Mhz PC for one 3D lattice simulation.

TABLE II

Comparison Between Heston and LBA Models

	$T = 1/12$			$T = 3/12$			$T = 6/12$		
	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
LBA									
S = 90	0.0865	0.4331	1.0123	0.8859	1.9028	3.1373	2.3263	3.6462	5.1797
S = 95	0.6504	1.4623	2.3730	2.2631	3.6178	5.0855	4.2374	5.7545	7.4444
S = 100	2.5088	3.5552	4.6061	4.6151	6.0719	7.6212	6.8810	8.4396	10.1737
S = 105	5.9938	6.7944	7.7231	7.9310	9.2412	10.7125	10.2071	11.6580	13.3332
S = 110	10.5244	10.9245	11.5778	11.9998	13.0099	14.2906	14.0896	15.3362	16.8742
Heston									
S = 90	0.0857	0.4335	1.0115	0.8852	1.9023	3.1355	2.3272	3.6447	5.1748
S = 95	0.6509	1.4551	2.3696	2.2588	3.6147	5.0826	4.2367	5.7520	7.4382
S = 100	2.5107	3.5455	4.6017	4.6105	6.0703	7.6177	6.8817	8.4366	10.1664
S = 105	5.9944	6.7897	7.7198	7.9290	9.2385	10.7087	10.2068	11.6550	13.3245
S = 110	10.5241	10.9253	11.5766	12.0006	13.0087	14.2873	14.0910	15.3337	16.8632
Percentage Error									
S = 90	0.96%	−0.09%	0.08%	0.08%	0.02%	0.06%	−0.04%	0.04%	0.09%
S = 95	−0.07%	0.49%	0.15%	0.19%	0.09%	0.06%	0.02%	0.04%	0.08%
S = 100	−0.08%	0.27%	0.10%	0.10%	0.03%	0.05%	−0.01%	0.04%	0.07%
S = 105	−0.01%	0.07%	0.04%	0.03%	0.03%	0.03%	0.00%	0.03%	0.06%
S = 110	0.00%	−0.01%	0.01%	−0.01%	0.01%	0.02%	−0.01%	0.02%	0.07%
Mean	0.07%	MAE	0.09%	RMSE	0.18%				

This table shows and compares European-call-option prices under stochastic volatility using the Heston model and the Lattice-Based Approach (LBA). Standard models like BS and CRR cannot handle stochastic volatility. Values chosen for common parameters are $X = 100$, $r = 0.05$, $t = (1/12, 3/12, 6/12)$, $\sqrt{V} = (0.2, 0.3, 0.4)$, $\kappa = 3$, $\theta = 0.04$, $b = 0.1$, $\rho = -0.1$. For the LBA, $N_S = 100$, $N_r = 200$, and $N_V = 50$. The prices are simulated using these parameters. The Pricing Error is measured as percentage of (model price − true price)/true price, taking the Heston price as the true price. Mean Error (ME), Mean Absolute Error (MAE), and Root Mean Square Error (RMSE) of the percentages are reported for the entire sample.

tility minus the option values with constant volatility. For the additional parameters of the LBA, we use $\kappa = 1$, $\theta = 0.04$, $b = 0.3$, $\rho = -0.3$, $N_T = N_V = 50$, and $N_S = 100$. We can see that the constant-volatility model systematically underprices in-the-money calls and overprices out-of-the-money calls with respect to stochastic-volatility models. The degree of mispricing increases proportionately with the time to maturity and the absolute moneyness. The findings are just the opposite when the correlation is positive, as shown in the lower-left graph. The constant-volatility model overprices in-the-money calls and underprices out-of-the-money options.

The lower-right graph of Figure 2 shows us the early exercise premium of the American option over its European counterpart. As we can

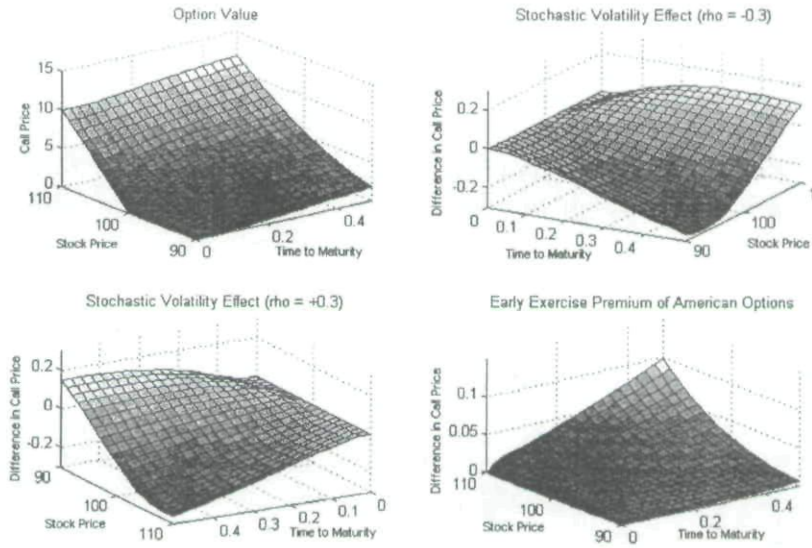

FIGURE 2
Effects of stochastic volatility in pricing (American) options.

see, the premium decreases as the option approaches the maturity, and is largest for long time-to-maturity and in-the-money options.

EMPIRICAL TEST

We empirically test the LBA model. As a comparison, the CRR model is used to perform an empirical test of the constant-volatility model.

Empirical Data

The empirical test of the model is performed with intraday trading prices of S&P 500 futures options at the Chicago Mercantile Exchange (CME) from January 4, 1995 to March 16, 1995. We chose the data based on two considerations. Firstly, the options were actively traded American options. Secondly, S&P 500 index and S&P 500 futures options are used widely in existing literature. Both the underlying futures and the options on futures expired on March 16, 1995. There are very few trades before the sample period.

Option prices were matched with the nearest corresponding futures price preceding the option transaction. Both the futures and the options prices are quoted in index points. The futures prices in the sample period ranged from 460.35 to 492.1, and the most actively traded options over the life of the futures were at-the-money options, which had strike prices ranging from 475 to 495. There were 3742 call option prices in the sample. Several common filtering rules were applied to the raw sample before the data were used in the study.

Firstly, call prices that were less than six days to maturity were taken out of the sample to avoid possible liquidity-related bias. In fact, we observed many cases in the last few days when the call prices moved in the opposite direction of the underlying asset price movements. This can only be explained as relatively large intraday fluctuations of the volatility, which may have potentially distorted the parameter estimation. This is one of the filtering rules used by Bakshi, Cao, and Chen (1997).

Secondly, call prices that were less than 0.5 were not used to mitigate the impact of price discreteness (the tick size for S&P 500 futures option is 0.05). This is because most option-pricing models assume continuous price movements, while in real world the prices move in ticks. Nandi (1996) used this rule with his S&P index options data.

Lastly, the options with a time lag of more than 30 seconds with the corresponding futures were eliminated to ensure the concurrence between the option and its underlying asset. A similar rule was applied in Chang, Chang, and Lim (1998). Due to the more frequent trading of the futures, this rule did not exclude many options from the sample. The filtered sample consists of 2957 call prices. A more-detailed breakdown of the raw data and filtered data statistics according to moneyness and time-to-maturity is shown in Table III.

The risk-free rate was calculated from quotes of U.S. Treasury Bill prices best matching the options maturity. The U.S. Treasury Bill used in this study matured on March 23, 1995. The quotes were obtained through Reuters Fixed Income 3000. The average of bid and ask discount quotes was used to calculate the yield.[4]

Parameter Estimation

To implement option-pricing models, the spot volatility and other unobservable structural parameters need to be backed out implicitly using the observed-traded-option prices on each day. Since the models here assume that the structural parameters are constant, there is some internal inconsistency between the models and this daily estimation. However, as daily estimations greatly increased the fit, we followed many existing empirical studies to use this method.[5]

[4]There are four missing data in the 46 sample days, where the averages of the two nearest yields are used.

[5]See Bakshi et al. (1997) and Chang et al. (1998).

TABLE III

Sample Statistics

	$T \geq 15$		$35 > T > 15$		$15 \geq T$		*All Maturity*	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
<0.98								
Number	401	388	67	65	61	6	529	459
Mean	1.804	1.836	1.170	1.174	0.385	0.608	1.560	1.726
Standard Deviation	0.767	0.752	0.197	0.198	0.101	0.038	0.822	0.743
0.98–0.99								
Number	241	239	295	282	250	212	786	733
Mean	3.031	3.026	1.909	1.903	0.945	0.997	1.947	2.007
Standard Deviation	0.762	0.749	0.516	0.515	0.293	0.271	0.993	0.972
0.99–1.00								
Number	194	189	391	384	610	376	1,195	949
Mean	4.782	4.788	3.539	3.537	1.813	2.322	2.860	3.305
Standard Deviation	0.654	0.657	0.789	0.786	0.906	0.706	1.415	1.173
1.00–1.01								
Number	71	70	257	260	474	215	802	535
Mean	7.005	7.012	5.958	5.965	3.646	4.679	4.684	5.585
Standard Deviation	0.754	0.757	0.973	0.977	1.465	1.059	1.804	1.278
1.01–1.02								
Number	42	41	92	89	153	55	287	185
Mean	10.255	10.271	9.150	9.140	7.323	7.963	8.338	9.041
Standard Deviation	0.864	0.869	1.121	1.124	1.459	1.422	1.715	1.432
≥1.02								
Number	—	—	74	74	69	22	143	96
Mean	NaN	NaN	13.049	13.049	12.812	11.691	12.935	12.738
Standard Deviation	NaN	NaN	1.130	1.130	2.688	0.907	2.032	1.221
All Moneyness								
Number	949	927	1,176	1,144	1.617	886	3,742	2,957
Mean	3.488	3.509	4.561	4.582	3.153	3.148	3.680	3.816
Standard Deviation	2.260	2.250	3.168	3.184	2.980	2.431	2.942	2.769

The sample data are intraday trading prices of S&P 500 futures options at the Chicago Mercantile Exchange (CME) from January 4, 1995 to March 16, 1995. They are divided into 3 time-to-maturity and 6 moneyness groups so we have 3 + 6 = 9 subgroups, 3 × 6 = 18 subsubgroups, 1 grand total, 28 groups altogether. Number of observations, mean, and standard deviation are given for each group.

For the LBA model, five parameters of the process V, κ, θ, and ρ needed to be implied from the traded-option prices. For the CRR model, only the volatility needed to be implied.[6]

Several possible objective-function candidates needed to be minimized in order to imply out the volatility and other necessary parameters.

[6]The algorithms of the LBA are implemented in C Programming Language for computational efficiency. The optimization procedure uses Sequential Quadratic Programming methods in Matlab.

TABLE IV

Parameter Estimation Results

	CRR	*LBA*				
	σ	$\sqrt{V}$	b	κ	θ	ρ
Mean	0.10	0.09	0.61	3.04	0.03	−0.52
Std Deviation	0.01	0.01	0.19	0.90	0.01	0.13

For example, the parameters could be estimated by minimizing the sum of squared-percentage pricing error. This method would put more weight on matching the cheaper options since the same dollar pricing error would lead to larger percentage pricing errors on cheaper options. Alternatively, we could minimize the sum of squared-index-point pricing error. However, this would lead to more weight on matching the more expensive options. We followed the convention in most research to minimize the sum of squared-index-point error.

The nonlinear optimization result achieved using the Sequential Quadratic Programming method did not ensure a global minimum of the objective function, and to some extent was dependent on the initial guesses. Since our objective surface was very irregular, and we were estimating five variables at the same time, the estimated parameters were very sensitive to the initial values. To ensure stability of the estimates, we added a constraint on the long-run mean and the volatility: $\theta - \sigma^2 < 0.05$. This is not a restrictive constraint since the estimated θ is about 0.09. After we imposed the constraint, the sample implied parameters appeared to be in order. Due to the computational time constraint, we performed a two-step estimation. Firstly, we estimated the parameters using the results of previous studies with similar data as the initial values. After that, we re-estimated the parameters using the average of the previous estimates as the initial values.

The same method was used to obtain the daily implied volatility of the standard-binomial-tree model for comparison purpose.

The estimates of parameters of the LBA and CRR model are reported in Table IV.

As was mentioned before, the risk-neutralized theta cannot be interpreted directly as the long-run mean of volatility. However, using eq (4), it can be inferred that the true long-run mean will be larger than our estimates for any positively defined, unobservable market price of the risk of volatility. We found that the long-run mean of the squared volatility, θ,

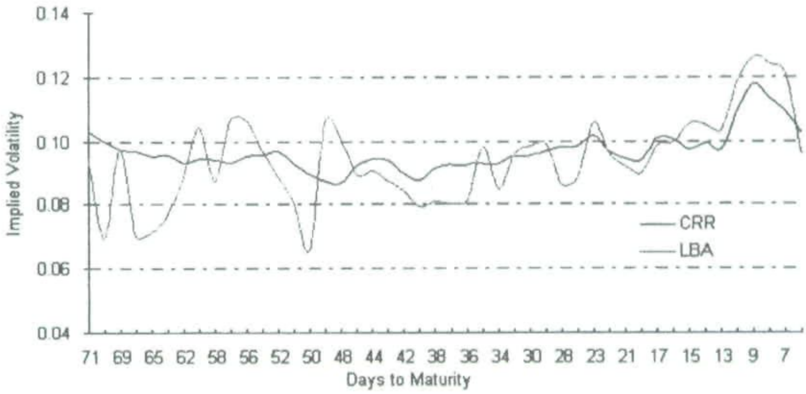

FIGURE 3
Implied volatility estimates.

is always larger than the squared volatility. This implies that the volatility has a tendency to increase during the life of the futures. This also is reflected by the lower mean of the implied volatilities of the LBA in relation to the CRR because the implied volatility for the LBA is just the initial volatility, and that of the CRR is the average volatility over the life of the option. We can see from Figure 3 the increasing trend in the implied-volatility time series of the LBA. To verify this point, we ran a simple OLS regression of the implied volatility over the days to maturity. The coefficient was negative (indicating an increasing trend of the volatility), and statistically different from zero at 1% significance level for both the LBA and CRR. The half life of the volatility shock, as measured in days, is $365 \cdot \ln 2/\kappa$, or 83.1 days.

Tests of Model Performance

The model's performance was examined using four tests: in-sample test, out-of-sample test, dynamic intraday-hedging-performance test, and profitability test. We divided the moneyness into six subcategories, and the time-to-maturity into three. Including both the subtotals and grand totals, there are 28 categories. For the first three tests, we reported the Mean error, Mean of Absolute Error (MAE), and Root Mean Square Error (RMSE) statistics, in terms of both index and percentage points to assess different aspects of the error. For the profitability test, the mean profit/loss for each category is reported. The index point error is defined as the difference between actual price and model price, and the percentage error is the index point error divided by the model price. Some studies base the percentage error on the actual price. Mathematically,

$$\varepsilon_i \equiv C_a - C_m \tag{9}$$

$$\varepsilon_{\%} \equiv \frac{C_a - C_m}{C_m} \times 100\% \tag{10}$$

where C_a is the actual option price, C_m is the model option price, ε_i is the index point error, and $\varepsilon_{\%}$ is the percentage error.

In-Sample Test

The in-sample test is to assess the goodness of fit of the model price compared to actual prices using the parameter estimated employing all the data in the same day. We expected that the stochastic volatility model would fit the prices better simply because it uses more parameters and therefore allows for more degree of freedom. Moneyness and Time-To-Maturity biases were examined to assess the extent of model misspecification.

The in-sample test results are shown in Table V. The LBA model shows better goodness-of-fit with no evidence of bias related to moneyness or maturity. In contrast, the CRR model displays evidence of moneyness bias.

Bias. The overall in-sample performances showed no significant bias for both models. The mean errors in index-point terms and percentage terms of all call options in the sample for both models were very small. There was no significant maturity bias for both models. This is partly due to the relatively overall short time to maturity of the options on futures. However, we noticed that the CRR model displayed systematic bias in terms of moneyness. In every time-to-maturity category, the CRR model overpriced (shown by a negative number) out-of-money calls and underpriced in-the-money calls, and the degree of mispricing was proportional to the moneyness. The bias was evident according to both the mean errors for index points and percentage. This finding is consistent with Macbeth and Merville (1979).

As we expected, the moneyness bias disappeared when we took into account the stochastic volatility. There was no obvious pattern of moneyness related bias for the LBA model. In addition, the LBA produced a much smaller bias. The CRR model's degree of bias was generally 10 times more than that of the LBA model.

We also found that the relative magnitude of the mean-absolute-index-points pricing error with respect to its mean-index-points pricing error was larger in each subcategory for the LBA than the CRR model.

For example, in the first category in Table V (Panel A), where $T \geq 35$ and $S/X < 0.98$, CRR produces an MAE of 0.18 and a Mean of -0.14, while LBA has 0.06 and 0.003, respectively. If we take MAE/Mean, we obtain 1.3 for CRR and 20 for LBA. This also can be interpreted as the evidence of CRR model's bias. When most of the errors are of the same sign within each moneyness category, the mean-index-points error will be close to the mean-absolute index-points error, and vice versa. The LBA model's pricing error is distributed more evenly around the true price, so the mean-index-points error is much smaller than the mean-absolute-index-points error.

Goodness of Fit. The overall mean-absolute-index-points pricing error of CRR is 0.18, while it is 0.08 for the LBA. In percentage terms, the mean absolute error is 2.9% for LBA and 6.1% for the CRR model. The findings in the RMSE measure are consistent with this. Furthermore, the LBA dominates the CRR in almost all the categories.

We reported the MAE as well as RMSE to measure the model's goodness of fit. Mean of absolute error gives us the sense of the expected average error. On the other hand, RMSE punishes larger errors because larger errors receive more weights when squared, so it is more appropriate for use with risk-averse investors. In both measures, the LBA shows better goodness of fit, as was expected. The findings in terms of the index points are consistent with the percentage error.

Implications. The smaller bias and the better goodness of fit in the in-sample test for the LBA over the CRR show that there is some evidence of stochastic volatility in the S&P 500 futures options. The stochastic-volatility model is a better specification than the constant-volatility model. However, we still are not sure whether the better performance of the LBA is only an illusion due to more degrees of freedom from more parameters. The question is answered by the result from the out-of-sample test.

Out-of-Sample Test

We test the out-of-sample goodness of fit using lagged-parameter estimation. In this test, we used the previous trading day's implied volatility and parameter estimates to price the current day's options. This is an important test of model misspecification because, if there is no stochastic term in the volatility, the over-fitting in the in-sample test of the stochastic volatility model will be penalized. Since the previous day's data was used to estimate the parameters, this test can also be interpreted as the prediction test of the models and of parameter stability.

TABLE V

In-Sample Test Result

	$T \geq 15$		$35 > T > 15$		$15 \geq T$		*All Maturity*	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
Panel A: Index-Points Error								
<0.98								
Mean	−0.142	0.003	−0.176	0.009	−0.026	0.062	−0.146	0.004
MAE	0.185	0.064	0.176	0.050	0.049	0.064	0.182	0.062
RMSE	0.210	0.086	0.194	0.063	0.055	0.074	0.206	0.083
0.98–0.99								
Mean	−0.081	−0.011	−0.179	−0.005	−0.104	0.005	−0.125	−0.004
MAE	0.187	0.073	0.179	0.062	0.113	0.052	0.163	0.063
RMSE	0.214	0.095	0.196	0.076	0.132	0.064	0.186	0.079
0.99–1.00								
Mean	0.091	0.004	−0.088	0.007	−0.062	−0.001	−0.042	0.003
MAE	0.170	0.075	0.123	0.070	0.106	0.083	0.125	0.076
RMSE	0.208	0.096	0.143	0.089	0.131	0.106	0.154	0.097
1.00–1.01								
Mean	0.245	0.008	0.128	−0.008	0.117	−0.008	0.139	−0.006
MAE	0.248	0.087	0.146	0.095	0.189	0.118	0.177	0.103
RMSE	0.281	0.117	0.191	0.120	0.232	0.146	0.222	0.131
1.01–1.02								
Mean	0.550	0.003	0.371	−0.022	0.299	0.021	0.389	−0.004
MAE	0.550	0.099	0.374	0.106	0.324	0.133	0.398	0.112
RMSE	0.568	0.146	0.414	0.134	0.394	0.164	0.447	0.146
≥1.02								
Mean	NaN	NaN	0.544	0.046	0.383	0.024	0.507	0.041
MAE	NaN	NaN	0.544	0.130	0.389	0.088	0.509	0.120
RMSE	NaN	NaN	0.570	0.169	0.440	0.119	0.543	0.159
All Moneyness								
Mean	−0.019	0.000	0.009	0.001	0.005	0.001	−0.001	0.001
MAE	0.203	0.072	0.191	0.079	0.148	0.087	0.182	0.079
RMSE	0.243	0.096	0.247	0.103	0.197	0.114	0.232	0.105
Panel B: Percentage Error								
<0.98								
Mean	−9.31%	0.10%	−12.62%	1.21%	−3.31%	12.21%	−9.70%	0.42%
MAE	10.77%	3.84%	12.62%	4.47%	7.65%	12.59%	10.99%	4.04%
RMSE	12.90%	5.26%	13.53%	5.69%	8.37%	14.97%	12.94%	5.56%
0.98–0.99								
Mean	−3.44%	−0.37%	−9.01%	−0.45%	−9.06%	0.82%	−7.21%	−0.06%
MAE	6.22%	2.45%	9.02%	3.32%	10.32%	5.59%	8.48%	3.69%
RMSE	7.18%	3.15%	10.01%	4.09%	11.90%	7.04%	9.81%	4.90%
0.99–1.00								
Mean	1.75%	0.16%	−2.80%	0.17%	−2.69%	−0.11%	−1.85%	0.06%
MAE	3.58%	1.61%	3.61%	2.01%	4.97%	4.03%	4.14%	2.73%
RMSE	4.35%	2.14%	4.31%	2.52%	6.45%	5.44%	5.27%	3.90%

TABLE V (Continued)

In-Sample Test Result

	$T \geq 15$		$35 > T > 15$		$15 \geq T$		*All Maturity*	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
			Panel B: Percentage Error (Continued)					
1.00–1.01								
Mean	3.56%	0.15%	2.09%	−0.02%	2.12%	−0.20%	2.30%	−0.07%
MAE	3.61%	1.29%	2.42%	1.61%	4.01%	2.66%	3.22%	1.99%
RMSE	4.09%	1.75%	3.07%	2.06%	4.78%	3.36%	3.98%	2.63%
1.01–1.02								
Mean	5.68%	0.05%	4.14%	−0.26%	3.70%	0.21%	4.35%	−0.05%
MAE	5.68%	0.98%	4.18%	1.16%	4.05%	1.71%	4.48%	1.28%
RMSE	5.88%	1.47%	4.53%	1.47%	4.82%	2.12%	4.94%	1.69%
≥1.02								
Mean	NaN	NaN	4.37%	0.36%	3.41%	0.21%	4.15%	0.32%
MAE	NaN	NaN	4.38%	0.99%	3.46%	0.75%	4.17%	0.93%
RMSE	NaN	NaN	4.58%	1.27%	3.90%	1.01%	4.44%	1.22%
All Moneyness								
Mean	−3.91%	−0.01%	−2.81%	0.01%	−2.50%	0.20%	−3.06%	0.06%
MAE	7.36%	2.71%	5.29%	2.25%	5.94%	3.90%	6.14%	2.89%
RMSE	9.46%	3.93%	6.81%	3.05%	7.71%	5.39%	7.99%	4.14%

The in-sample test result is split into two panels. Panel A shows the index points error in each subgroup as well as subtotals and grand totals. Panel B shows the percentage error based on actual price. Model prices are calculated using the estimated parameters using the sample data in the same day.

Table VI shows the out-of-sample test results. Unlike the in-sample test, where the LBA dominates in all categories, the results are somewhat mixed in the out-sample test. The LBA still displays much smaller bias and better fit. However, in some of the short time-to-maturity and out-of-the-money categories, the results are mixed in terms of MAE and RMSE.

Bias. The CRR model still shows moneyness bias, i.e., increasing degree of overpricing (under pricing) with increasing degree of in the money (out of money). The level of mispricing is approximately at the same level as the in-sample test. There is some inconsistency of the mean-error measure in index points and percentage. For example, the CRR model underprices the overall call options by 0.01 index points, while it is overpricing the call options by 2.3%. This is not surprising because the index-points measure puts more weight on in-the-money options, which are usually more expensive, and the percentage measure favors out-of-the-money options. From the difference in the two measures, we also can

TABLE VI

Out-of-Sample Test Result

	$T \geq 15$		$35 > T > 15$		$15 \geq T$		All Maturity	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
				Panel A: Index-Points Error				
<0.98								
Mean	−0.168	−0.001	−0.111	0.084	0.016	0.057	−0.158	0.013
MAE	0.217	0.133	0.148	0.107	0.040	0.069	0.205	0.128
RMSE	0.249	0.167	0.168	0.128	0.068	0.101	0.237	0.161
0.98–0.99								
Mean	−0.085	−0.027	−0.156	0.024	−0.067	0.038	−0.107	0.011
MAE	0.191	0.152	0.182	0.110	0.127	0.098	0.169	0.120
RMSE	0.229	0.186	0.212	0.146	0.148	0.126	0.202	0.155
0.99–1.00								
Mean	0.098	−0.037	−0.090	0.001	−0.046	0.015	−0.035	−0.001
MAE	0.175	0.056	0.177	0.147	0.174	0.166	0.175	0.156
RMSE	0.230	0.205	0.216	0.190	0.202	0.206	0.214	0.199
1.00–1.01								
Mean	0.276	−0.074	0.163	0.016	0.189	0.052	0.188	0.019
MAE	0.299	0.231	0.231	0.195	0.282	0.218	0.261	0.209
RMSE	0.328	0.301	0.284	0.240	0.354	0.278	0.320	0.264
1.01–1.02								
Mean	0.597	−0.081	0.372	−0.052	0.306	−0.067	0.402	−0.063
MAE	0.597	0.228	0.378	0.170	0.373	0.259	0.425	0.209
RMSE	0.631	0.304	0.438	0.211	0.453	0.323	0.491	0.270
≥1.02								
Mean	NaN	NaN	−0.536	−0.041	0.374	−0.041	0.099	−0.010
MAE	NaN	NaN	0.536	0.153	0.385	0.132	0.501	0.148
RMSE	NaN	NaN	0.565	0.186	0.439	0.171	0.539	0.182
All Moneyness								
Mean	−0.023	−0.025	0.024	0.011	0.039	0.023	0.014	0.004
MAE	0.225	0.155	0.227	0.148	0.205	0.167	0.220	0.156
RMSE	0.276	0.201	0.286	0.191	0.266	0.219	0.277	0.203
				Panel B: Percentage Error				
<0.98								
Mean	−10.26%	0.40%	−6.75%	8.74%	4.27%	13.96%	−9.55%	1.81%
MAE	12.31%	8.29%	11.69%	10.60%	8.00%	15.93%	12.16%	8.74%
RMSE	14.76%	10.83%	13.11%	12.98%	14.60%	26.02%	14.52%	11.51%
0.98–0.99								
Mean	−3.39%	−0.68%	−7.75%	1.24%	−4.66%	5.90%	−5.44%	1.96%
MAE	6.28%	4.99%	9.35%	6.08%	12.13%	11.89%	9.16%	7.40%
RMSE	7.51%	6.03%	11.05%	8.21%	14.01%	16.15%	11.05%	10.64%
0.99–1.00								
Mean	1.97%	−0.62%	−2.88%	0.02%	−2.22%	0.81%	−1.65%	0.20%
MAE	3.77%	3.35%	5.19%	4.35%	8.16%	8.10%	6.08%	5.64%
RMSE	5.05%	4.43%	6.49%	5.71%	9.87%	10.64%	7.79%	7.87%

TABLE VI (Continued)

Out-of-Sample Test Result

	$T \geq 15$		$35 > T > 15$		$15 \geq T$		*All Maturity*	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
			Panel B: Percentage Error (Continued)					
1.00–1.01								
Mean	4.10%	−0.92%	2.82%	0.55%	3.71%	0.93%	3.35%	0.51%
MAE	4.47%	3.34%	4.08%	3.52%	6.24%	4.96%	5.00%	4.08%
RMSE	4.95%	4.36%	5.06%	4.61%	7.73%	6.40%	6.26%	5.38%
1.01–1.02								
Mean	6.15%	−0.83%	4.16%	−0.61%	3.81%	−1.07%	4.50%	−0.79%
MAE	6.15%	2.29%	4.23%	1.89%	4.78%	3.37%	4.82%	2.42%
RMSE	6.51%	3.10%	4.83%	2.37%	5.73%	4.27%	5.51%	3.20%
≥1.02								
Mean	NaN	NaN	4.31%	0.00%	3.34%	−0.36%	4.09%	−0.08%
MAE	NaN	NaN	4.31%	1.19%	3.43%	1.12%	4.11%	1.17%
RMSE	NaN	NaN	4.55%	1.44%	3.92%	1.45%	4.41%	1.44%
All Moneyness								
Mean	−4.08%	−0.25%	−2.04%	0.88%	−2.81%	2.00%	−2.30%	0.87%
MAE	8.08%	5.75%	6.21%	4.56%	8.32%	7.83%	7.42%	5.91%
RMSE	10.62%	7.96%	7.92%	6.51%	10.33%	11.23%	9.56%	8.61%

The out-of-sample test result is split into two panels. Panel A shows the index points error in each subgroup as well as subtotals and grand totals. Panel B shows the percentage error based on actual price. Model prices are calculated using the estimated parameters using the sample data in the same day.

discover that the CRR underprices in-the-money options and overprices out-of-the-money ones.

The LBA generally produces very small bias regardless of different categories. It outperforms the CRR model in every category. However, the mean error in percentage terms increased from 0.06% in the in-sample test to 0.87% in this out-of-sample test.

Goodness of Fit. The goodness of fit for CRR only decreases marginally from the in-sample test. However, the LBA overall error is more than doubled according to all measures. This is due partly to the fact that the parameter estimates are less stable for the LBA model.

The LBA out-of-sample fit or prediction performance outperforms the CRR in every major moneyness category, as well as in the overall category. However, a detailed look at each subcategory reveals that the CRR model performs slightly better with out-of-the-money and short time-to-maturity options. The CRR performs better than the LBA for calls with moneyness less than 0.98 and time to maturity shorter than 15 days. However, we should notice that there are only 6 calls in this category. In terms of RMSE of percentage error, the CRR performs slightly better with calls of moneyness 0.98 to 0.99 and 0.99 to 1.00 and with time to maturity of less than 15 days. However, the converse is true in terms of MAE in percentage terms. In addition, the measure in terms of index points shows that the LBA actually dominates the CRR for options with $T \geq 15$ and S/X between 0.98 and 0.99. This inconsistency may be due to the fact that the percentage error uses the model price as 100%. We can see from the mean percentage error in these two categories that the CRR produces values of -4.7 and -2.2%, i.e., it overprices the options, while the LBA produces positive values, or underprices the options. Thus the percentage measure is more favorable to the CRR model. The RMSE punishes extreme values more, so it shows that the CRR actually performs better.

Implications. The better out-of-sample test result of the LBA again verifies that the stochastic-volatility model is a better specification. However, the fact that the out-of-sample errors of the LBA doubled while the CRR only increased marginally indicates that the LBA model's parameter is less stable than that of the CRR. Nevertheless, we expect that better parameter estimation procedures may improve the parameter stability of LBA.

Dynamic-Intraday-Hedging Test

Often the hedging effectiveness makes option-pricing models useful to investors. To assess the models' hedging effectiveness, we followed the dynamic-intraday-hedging-performance method of Chang et al. (1998). In their study, they assumed that the hedge portfolio was continuously rebalanced throughout the day. The hedging error is defined as the per-tick change in actual price versus change in the model price. This hedge test is different from the traditional daily-rebalanced delta-neutral hedge in the sense that it can capture the intraday dynamics of the prices. The hedging results are shown in Table VII. Both models show reasonably good hedging results without any significant bias. In terms of hedging effectiveness, the two models produce very close errors in terms of MAE, as well as RMSE, in most categories. However, the CRR slightly outperforms the LBA in some categories.

Bias. The hedging bias is measured by the mean hedging errors. As can be seen in Table VII, the bias is below 0.01 in terms of index points and below 1% in terms of percentage for most categories. The largest error occurs for options with less than 15 days to maturity and with moneyness less than 0.98. The large error is most likely because there

TABLE VII

Dynamic-Intrady-Hedging Test Result

	T ≥ 15		*35 > T > 15*		*15 ≥ T*		*All Maturity*	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
				Panel A: Index-Points Error				
<0.98								
Mean	−0.002	−0.011	−0.014	−0.015	−0.051	−0.036	−0.005	−0.012
MAE	0.073	0.089	0.064	0.064	0.052	0.038	0.071	0.084
RMSE	0.095	0.120	0.091	0.090	0.074	0.048	0.094	0.115
0.98–0.99								
Mean	0.013	0.014	0.000	−0.002	−0.007	−0.008	0.002	0.002
MAE	0.083	0.100	0.074	0.075	0.062	0.059	0.074	0.078
RMSE	0.107	0.130	0.093	0.095	0.080	0.077	0.094	0.103
0.99–1.00								
Mean	−0.002	0.009	−0.003	−0.007	0.002	0.001	−0.001	−0.001
MAE	0.096	0.112	0.073	0.081	0.092	0.097	0.084	0.093
RMSE	0.123	0.143	0.091	0.102	0.115	0.121	0.107	0.118
1.00–1.01								
Mean	0.002	−0.009	0.004	0.009	0.005	0.008	0.004	0.006
MAE	0.098	0.118	0.100	0.114	0.122	0.139	0.108	0.124
RMSE	0.148	0.170	0.127	0.145	0.151	0.171	0.139	0.158
1.01–1.02								
Mean	−0.027	0.001	−0.004	−0.004	−0.010	−0.003	−0.011	−0.002
MAE	0.147	0.146	0.140	0.145	0.145	0.147	0.143	0.146
RMSE	0.216	0.227	0.184	0.194	0.183	0.183	0.193	0.200
≥1.02								
Mean	NaN	NaN	0.020	0.021	0.034	0.035	0.023	0.024
MAE	NaN	NaN	0.141	0.144	0.123	0.122	0.138	0.141
RMSE	NaN	NaN	0.192	0.194	0.161	0.161	0.188	0.189
All Moneyness								
Mean	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000
MAE	0.086	0.102	0.088	0.094	0.094	0.099	0.089	0.098
RMSE	0.117	0.138	0.117	0.126	0.122	0.129	0.118	0.131
				Panel B: Percentage Error				
<0.98								
Mean	−0.25%	−0.72%	−1.28%	−1.50%	−10.01%	−7.82%	−0.56%	−0.95%
MAE	4.01%	5.29%	5.09%	5.93%	10.26%	8.17%	4.28%	5.43%
RMSE	5.38%	7.03%	7.06%	8.14%	15.55%	10.93%	5.96%	7.29%
0.98–0.99								
Mean	0.39%	0.39%	−0.02%	−0.12%	−0.74%	−0.91%	−0.09%	−0.18%
MAE	2.75%	3.30%	3.73%	4.06%	6.03%	6.37%	4.06%	4.47%
RMSE	3.57%	4.29%	4.66%	5.09%	7.74%	8.15%	5.44%	5.90%
0.99–1.00								
Mean	−0.09%	0.10%	−0.07%	−0.17%	0.15%	0.11%	0.00%	−0.02%
MAE	2.09%	2.37%	2.07%	2.37%	3.89%	4.22%	2.72%	3.03%
RMSE	2.73%	3.03%	2.61%	2.99%	5.10%	5.52%	3.72%	4.09%

TABLE VII (Continued)

Dynamic-Intrady-Hedging Test Result

	$T \geq 15$		$35 > T > 15$		$15 \geq T$		*All Maturity*	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
	Panel B: Percentage Error (Continued)							
1.00–1.01								
Mean	0.01%	−0.18%	0.06%	0.12%	0.09%	0.10%	0.07%	0.07%
MAE	1.49%	1.70%	1.72%	1.97%	2.62%	2.90%	2.02%	2.27%
RMSE	2.28%	2.50%	2.17%	2.54%	3.27%	3.56%	2.63%	2.95%
1.01–1.02								
Mean	−0.30%	0.00%	0.00%	0.01%	−0.12%	−0.01%	−0.10%	0.00%
MAE	1.52%	1.41%	1.61%	1.59%	1.86%	1.80%	1.64%	1.59%
RMSE	2.27%	2.22%	2.15%	2.15%	2.37%	2.27%	2.23%	2.19%
≥1.02								
Mean	NaN	NaN	0.16%	0.16%	0.28%	0.27%	0.18%	0.18%
MAE	NaN	NaN	1.12%	1.10%	1.09%	1.04%	1.11%	1.09%
RMSE	NaN	NaN	1.51%	1.46%	1.42%	1.36%	1.49%	1.44%
All Moneyness								
Mean	−0.03%	−0.17%	−0.08%	−0.13%	−0.19%	−0.23%	−0.09%	−0.17%
MAE	2.95%	3.67%	2.48%	2.77%	4.07%	4.34%	3.06%	3.48%
RMSE	4.13%	5.20%	3.47%	3.88%	5.64%	5.90%	4.36%	4.92%

The dynamic-intraday-hedging-test result is split into two panels. The index-point hedging error is defined as Δ actual option price − Δ model option price, while the percentage hedging error is the index-point hedging error as a percentage of model-option price. Panel A shows the index-points error in each subgroup as well as subtotals and grand totals. Panel B shows the percentage error based on actual price. Model prices are calculated using the estimated parameters using the sample data of the previous trading day.

are only six options in this category; hence, the category has small weight in the parameter estimation.

Hedging Effectiveness. The CRR and the LBA produce similar results in most categories in MAD and RMSE. Both models produce reasonably good hedge results. In terms of index-point error, the CRR seems to outperform slightly the LBA in many categories, as well as the overall category in Table VII, although there are a few cases where the LBA performs slightly better. In terms of percentage, the CRR performs better with out-of-the-money calls and the LBA performs better with in-the-money calls.

Implications. It is surprising that, although the LBA outperforms the CRR in both the in-sample and out-of-sample test, their hedging performance is close, and the CRR even slightly outperforms the LBA in some categories. This result may imply that, as long as the hedging is rebalanced frequently enough, the hedging effectiveness will not be very sensitive to model misspecifications.

Profitability Test

It is interesting to know whether the models can produce abnormal profits. The dynamic intraday hedge will not be feasible in the real world due to transaction costs, monitoring costs, etc. The daily-rebalanced hedging is more realistic for the true purpose of testing the profitability of the models.

Consider the traditional delta-neutral hedge. For the CRR model, the only source of uncertainty arises from the underlying asset price changes, so a delta-neutral-hedge portfolio is constructed using the underlying asset and one option. However, when volatility is stochastic, two options, together with the underlying asset, are needed to hedge fully the risks. To ensure that the hedging performance was measured correctly, we could have selected samples where both options, as well as the underlying asset, were traded concurrently in two consecutive days. However, this would have lead to a very small sample and insignificant test results. It also was not practicable because the additional option required may not exist or does not trade frequently.

Alternatively, we could have used the underlying asset to hedge each option to achieve minimum variance in the hedged- portfolio value. This minimum-variance hedging approach also is common in the literature.[7] The minimum-variance hedge ratio x is given by

$$x = \frac{\partial C}{\partial S} + \frac{\partial C}{\partial V}\frac{\rho b}{S}. \tag{11}$$

The derivation of the minimum variance hedge ratio is shown in Appendix B. Minimum-variance portfolios were constructed for both models to assess their ability to produce profits. The minimum-variance hedge ratio was calculated for every option in day t using parameters implied through day $t-1$'s option prices. The hedged portfolio consists of x short positions in the underlying futures, that requires zero initial cash outlay, and one long position in the option financed through shorting the risk-free asset. Thus, it is a zero-cost portfolio at day t. The portfolio is liquidated on day $t+1$, so the investor obtains the option value at day t, the difference of the futures prices of the two days, and covers the short position of the risk-free asset. A detailed representation of the process is shown in Table VIII.

The hedge is repeated for each traded price on each day as long as there is a transaction of the contract on the following day. This method

[7]See Nandi (1996) and Bakshi et al. (1997).

TABLE VIII

Cash Flow of the Hedged Portfolio

Time	Call	Futures	Risk-Free Asset	Value
t	$-C_t$	$x \cdot 0$	$+C_t$	0
$t+1$	C_{t+1}	$x \cdot (F_t - F_{t+1})$	$-C_t(1 + r/365)$	$C_{t+1} + x \cdot (F_t - F_{t+1}) - C_t \cdot (1 + r/365)$

C_t is the call price on day t, x is the hedge ratio, F_t is the futures price on dat t, and r is the annualized risk-free rate.

effectively rolls over the hedged portfolio daily. Whether too long this zero-cost portfolio or too short it is decided by checking whether the model-call price is greater or less than the actual price. The hedged portfolio is valued at the last transaction price of day $t + 1$ of the same contract. The profits then are measured by the average portfolio values. The profit is reported for different categories. The simple Student T-test is used to test for the significance of the profits.

Hedge Ratios. The minimum-variance hedge ratios of both models are shown in Table IX. We find that $\partial C/\partial S$ of the CRR is generally smaller than that of the LBA. However, the relative large $\partial C/\partial V$ and the negative-correlation coefficient make the actual hedge ratio of the LBA smaller than that of the CRR in every case. In other words, the CRR model overhedges when there is negative correlation between the volatility and the underlying asset. In addition, we find that in the 2795 minimum-variance hedged-portfolios constructed, the LBA takes long positions in 1318 portfolios and short positions in the remaining 1477. This is distributed quite evenly because the pricing error between the LBA model and market price is distributed evenly around zero. On the contrary, the CRR takes 2484 long positions and 311 short positions. This is due partly to the fact that CRR tends to overprice out-of-the-money options, and our sample has more out-of-the-money calls, which is typical with American options, since in-the-money options sometimes are exercised before the maturity and out-of-the-money options always remain in the market.

Profitability. The profitability test result in Table X shows that the LBA outperforms the CRR in every category. The LBA produces profit in all categories except for two, while the CRR produces loss in most cases. In terms of index points, the LBA results in profits in all cases except for two. For options with moneyness between 1.01 and 1.02 and time to maturity between 15 and 35 days, the LBA produces an insignificant loss of -0.022, and for options with less than 15 days to maturity and with moneyness between 0.99 and 1.00, it produces a statistically significant loss. There are 11 cases out of the 28 categories where the profits pro-

TABLE IX

Minimum-Variance Hedge Ratios

	T ≥ 15		*35 > T > 15*		*15 ≥ T*		*All Maturity*	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
<0.98								
Hedge Ratio	0.212	0.109	0.190	0.105	0.133	0.076	0.208	0.108
$\partial C/\partial S$	0.212	0.232	0.190	0.179	0.133	0.125	0.208	0.222
$\partial C/\partial V$		167.986		139.669		71.953		162.430
0.98–0.99								
Hedge Ratio	0.317	0.205	0.279	0.186	0.219	0.131	0.275	0.177
$\partial C/\partial S$	0.317	0.361	0.279	0.281	0.219	0.207	0.275	0.288
$\partial C/\partial V$		227.339		178.433		100.212		173.250
0.99–1.00								
Hedge Ratio	0.433	0.331	0.421	0.340	0.405	0.316	0.418	0.330
$\partial C/\partial S$	0.433	0.504	0.421	0.456	0.405	0.430	0.418	0.457
$\partial C/\partial V$		260.002		217.288		147.739		201.708
1.00–1.01								
Hedge Ratio	0.565	0.484	0.577	0.526	0.600	0.555	0.584	0.530
$\partial C/\partial S$	0.565	0.640	0.577	0.640	0.600	0.667	0.584	0.650
$\partial C/\partial V$		257.346		211.454		147.383		194.718
1.01–1.02								
Hedge Ratio	0.694	0.610	0.708	0.656	0.786	0.733	0.722	0.662
$\partial C/\partial S$	0.694	0.746	0.708	0.752	0.786	0.811	0.722	0.764
$\partial C/\partial V$		229.540		181.381		103.003		176.184
≥1.02								
Hedge Ratio	NaN	NaN	0.837	0.777	0.875	0.815	0.844	0.784
$\partial C/\partial S$	NaN	NaN	0.837	0.848	0.875	0.873	0.844	0.852
$\partial C/\partial V$		NaN		130.042		90.639		123.037
All Moneyness								
Hedge Ratio	0.335	0.233	0.456	0.381	0.429	0.354	0.410	0.326
$\partial C/\partial S$	0.335	0.378	0.456	0.485	0.429	0.454	0.410	0.442
$\partial C/\partial V$		212.676		193.620		131.219		183.010

The minimum hedge ratio for CRR is $\partial C/\partial S$ only, while the LBA, both $\partial C/\partial S$ and $\partial C/\partial V$, as shown in eq (11).

duced by the LBA are statistically significant at 1%. The CRR produces two significant profits for options with $T \geq 35$ and S/X of 0.98 to 0.99 and 0.99 to 1.00. However, there are 11 categories where the CRR produces significant losses. Overall, the LBA produces a significant profit of 0.046, while the CRR produces a significant loss of -0.032.

The findings are similar in terms of percentage profits. The LBA produces 11 significant profits and one significant loss, while the CRR produces 12 significant losses and two significant profits. There is some inconsistency between the index point and percentage measure here as well. For example, the LBA produces a significant 0.067 profit for options

TABLE X

Profitability Test Result

	$T \geq 15$		$35 > T > 15$		$15 \geq T$		*All Maturity*	
Moneyness S/X	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*	*CRR*	*LBA*
<0.98								
Index Point	−0.028*	0.007	−0.075*	0.025	−0.057	0.090	−0.035*	0.011
Percentage	−1.74%*	1.14%	−0.06%*	0.82%	−8.11%	14.65%	−2.47%*	1.28%
0.98–0.99								
Index Point	0.067*	0.150*	−0.060*	0.020	−0.037	0.067*	−0.011	0.076*
Percentage	2.83%*	5.47%*	−3.68%*	1.23%	−3.24%	8.14%	−1.39%	4.54%*
0.99–1.00								
Index Point	0.059*	0.069	−0.023	0.072*	−0.127*	−0.052*	−0.042*	0.027*
Percentage	1.39%*	1.49%*	−0.91%*	2.14%*	−5.34%*	−2.39%*	−1.99%*	0.40%
1.00–1.01								
Index Point	−0.012	0.084	−0.035	0.080*	−0.086*	0.042	−0.050*	0.067*
Percentage	−0.20%	1.11%	−0.63%	1.57%*	−1.83%*	0.97%*	−1.01%*	1.29%*
1.01–1.02								
Index Point	−0.025	0.025	−0.048	−0.022	−0.003	0.080	−0.032	0.012
Percentage	−0.22%	0.21%	−0.60%	−0.34%	0.05%	0.93%	−0.36%	0.08%
≥1.02								
Index Point	NaN	NaN	0.000	0.098*	0.039	0.126	0.007	0.103
Percentage	NaN	NaN	0.00%	0.67%	0.40%	1.07%	0.07%	0.74%*
All Moneyness								
Index Point	0.017	0.065*	−0.038*	0.053*	−0.083*	0.013	−0.032*	0.046*
Percentage	0.32%	2.32%*	−1.74%*	1.43%*	−3.59%*	1.55%	−1.57%*	1.75%*

The profit test result is shown as the index-points profit and percentage profit in each subgroup, as well as subtotals and grand totals. Model prices are calculated using the estimated parameters using the sample data of the previous trading day. If an actual option price is higher than the theoretical price, i.e., the model price, then a zero-cost, delta-neutral portfolio is constructed to short the option, as in Table IV, and vice versa. The profit then is defined as the portfolio value at the next day closing prices. Percentage profit is not based on investment cost, because that is zero. Instead, we use the option price when we construct the portfolio as 100%.

with $T \geq 15$ and S/X between 0.98 and 0.99, but at the same time, it produces an insignificant 8.14% profit for the same category.

The findings here for the CRR model are consistent with Chang et al. (1998), who reported negative profits for the Black–Scholes model with S&P 500 futures options during the period of June 1994 to December 1994. This result is in contrast with Whaley (1986), who reported significant profits. Chang et al. (1998) explained that the Black–Scholes model's ability of capturing abnormal profits may have been absorbed by the market over the years. Our test, with more recent data, verifies this point again, though our model shows smaller loss for CRR than theirs for the Black–Scholes model, with quadratic approximation for the early exercise premiums.

Implications. The profits for the LBA and the losses for the CRR show that the LBA actually can predict accurately price, and is even able to capture some opportunities for abnormal returns. On the other hand, the CRR has pricing bias and is not able to produce any profits. However, the trading profits from the LBA are not large enough to cover the transaction costs and the bid-ask spread for normal investors.

CONCLUSIONS

In this paper, we improved the lattice-based approach to price numerically American options under stochastic volatility. We extended the model to allow for mean reversion in the volatility process, correlation between the volatility and the underlying as well as non-zero risk premium of the volatility. The risk-neutral probability and jumps measure derived in this paper is very general and is easily adaptable to other processes. The model is computationally efficient, accurate, and relatively easy to implement.

We empirically tested the LBA model with S&P 500 futures options intraday data. Compared with traditional constant volatility model, the LBA showed better in-sample and out-of-sample test results. The dynamic intraday hedging test showed that there is not much improvement in terms of hedging. However, the profitability test showed that there are opportunities for abnormal profit when trading with the LBA model. The profits are significant at 1% level, although the 0.046 index points or 1.75% profit will disappear when transaction costs are taken into considerations. The CRR model produced losses in this test.

The model can be improved by using better parameter estimation procedures and interpolation and extrapolation methods that are more appropriate. These are essentially calibration problems common in numerical-option pricing. In addition, the model can be extended to include more factors such as stochastic interest rate and jumps.

APPENDIX A: THE DERIVATION OF THE RISK-NEUTRAL PROBABILITIES AND JUMP SIZES

Assuming the underlying asset follows a log-normal process and the squared volatility follows the squared-root process as shown in eq (1) and (2). Further more, assume that at any point of time, both the underlying asset price S and the squared volatility V are followed by either an upward jump or a downward jump (discrete time price movement).

$$
\begin{aligned}
S_{i+1}^{+} &= S_t + u_S \\
S_{i+1}^{-} &= S_t + d_S \\
V_{i+1}^{+} &= V_t + u_V \\
V_{i+1}^{-} &= V_t + d_V,
\end{aligned}
\tag{A-1}
$$

where u_S, d_S, u_V, d_V are discrete time-additive up and down jumps of S and V.

We use p to denote the probability when an upward jump of S occurs, and q to denote the probability when an upward jump of V occurs. p_{11}, p_{12}, p_{21}, p_{22} are the joint probabilities of the respective jumps with the first subscript representing S, second for V, 1 for upward jump, and 2 for a downward jump. Then we have the following constrains for discrete-jumps convergence to continuous process as $\Delta t \to 0$ by matching the first moments and second moments of the processes:

$$
\begin{aligned}
E(\Delta S) &: Sr\Delta t = pu_S + (1 - p)d_S \\
E(\Delta V) &: \kappa(\theta - V)\Delta t = qu_V + (1 - q)d_V \\
E(S^2) - [E(S)]^2 &: S^2V\Delta t = [pu_S^2 + (1 - p)d_S^2] - pu_S + (1 - p)d_S]^2 \\
E(V^2) - [E(V)]^2 &: b^2V\Delta t = [qu_V^2 + (1 - q)d_V^2] \\
&\qquad - [qu_V + (1 - q)d_V]^2 \\
E(SV) - E(S)E(V) &: \rho SbV\Delta t = (p_{11}u_Su_V + p_{12}u_Sd_V + p_{21}d_Su_V + p_{22}d_Sd_V) \\
&\qquad - [pu_S + (1 - p)d_S][qu_V + (1 - q)d_V].
\end{aligned}
\tag{A-2}
$$

We also have some further constraints on the probabilities:

$$
\begin{aligned}
p_{11} + p_{12} &= p \\
p_{11} + p_{12} &= q \\
p_{11} + p_{12} + p_{21} + p_{22} &= 1 \\
p = q &= 0.5.
\end{aligned}
\tag{A-3}
$$

The last condition is arbitrary, since we have more unknowns than constraints. We can also assume $u_S = -d_S$ and $u_V = -d_V$, similar to the Cox, Ross, and Rubenstein's standard binomial-tree setup. However, since the binomial tree is not recombining with stochastic volatility any-way, we prefer to set $p = q = 0.5$ to ensure that the probabilities are well defined. The solution to the above problem is eq (6).

APPENDIX B: THE DERIVATION OF MINIMUM-HEDGE RATIO

The derivation of minimum-hedge ratio x is as follows. It is assumed that the underlying asset and volatility process is given by eq (1) and (2).

A contingent claim on the underlying satisfies the following PDE:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial V}dV + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 C}{\partial V^2}dV^2 + \frac{\partial^2 C}{\partial S \partial V}dSdV. \tag{B-1}$$

Applying Ito's Lemma and ignoring higher orders of dt, the variance and covariance of the derivatives are given by:

$$\begin{aligned}
dS^2 &= S^2Vdt \\
dV^2 &= b^2Vdt \\
dSdV &= \rho SbVdt \\
dC^2 &= \left(\left(\frac{\partial C}{\partial S}\right)^2 S^2V + \left(\frac{\partial C}{\partial V}\right)^2 b^2V + 2\frac{\partial C}{\partial S}\frac{\partial C}{\partial V}\rho SbV\right)dt \\
dCdS &= \left(\frac{\partial C}{\partial S}S^2V + \frac{\partial C}{\partial V}\rho SbV\right)dt.
\end{aligned} \tag{B-2}$$

If we construct a portfolio by longing one option and shorting x underlying, the variance is:

$$\begin{aligned}
Var(dC - xdS) &= Var(dC) + x^2Var(dS) - 2xCov(dC, dS) \\
&= \left(\left(\frac{\partial C}{\partial S}\right)^2 S^2V + \left(\frac{\partial C^2}{\partial V}b^2V\right) + 2\frac{\partial C}{\partial S}\frac{\partial C}{\partial V}\rho SbV\right) \\
&\quad dt + x^2S^2Vdt - 2x\left(\frac{\partial C}{\partial S}S^2V + \frac{\partial C}{\partial V}\rho SbV\right)dt.
\end{aligned} \tag{B-3}$$

The first order condition is:

$$\frac{\partial Var(dC - xdS)}{\partial x} = 2xS^2Vdt - 2\left(\frac{\partial C}{\partial S}S^2V + \frac{\partial C}{\partial V}\rho SbV\right)dt = 0. \tag{B-4}$$

Solve for x from eq (B-4), we obtain the solution of x in eq (11).

If we use opposite positions in the risk-free asset to finance the portfolio, the above solution also is obtained when we assume constant interest rate, or deterministic interest rate that has zero correlation with the futures and options. The minimum-variance hedge ratio for a constant volatility model is simply the first term in the above equation since the correlation coefficient ρ is zero. So for constant volatility model, the minimum-variance hedge is the same as the traditional delta-neutral hedge.

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