Bernoulli Speculator and Trading Strategy Risk

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In a continuous-time model of a complete information economy, we examine the case of a "pure" speculator who chooses to trade only on forward or futures contracts written on interest-rate-sensitive instruments. Assuming logarithmic utility, we assess whether his strategy exhibits the same structure as when he uses primitive assets only. It turns out that when interest rates follow stochastic processes, as in the model of Heath, Jarrow, and Morton (1992), where the instantaneous forward rate is driven by an arbitrary number of factors, the speculative trading strategy involving forwards exhibits an extra term vis-a-vis the one using futures or primitive assets. This extra term, different from a Merton–Breeden dynamic hedge, is novel and can be interpreted as a hedge against an "endogenous risk," namely the interest-rate risk brought about by the optimal trading strategy itself. Thus, only the strategy using futures (or the cash assets themselves) involves a single speculative term, even for the Bernoulli speculator. This result illustrates another major aspect of the marking to market feature that differentiates futures and forwards, and thus has some bearing on the issue of the optimal design of financial contracts. Real financial markets being, in fact, incomplete, the additional "endoge-

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nous" risk associated with forwards cannot be hedged perfectly. Since using futures eliminates the latter, risk-averse agents will find them attractive in relation to forward contracts, other things being equal.
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INTRODUCTION

In standard dynamic portfolio theory without intermediate consumption, the investor maximizes expected utility of terminal wealth by making continuous or periodic revisions of a portfolio comprising primitive (cash) assets. Extensive previous research has revealed that this multiperiod problem in general simplifies significantly when decisions are myopic. A portfolio strategy is said to be myopic when each period decision is treated as if it were the last one, using no information regarding future investment opportunities. As intuition suggests, long-horizon myopic and nonmyopic portfolio strategies can be very different, as convincingly demonstrated in a recent paper by Kim and Omberg (1996). It has been established that, in a complete information economy where all economic state variables are fully observable, myopia results from any of the three following assumptions: (1) a constant or deterministic opportunity set, (2) asset returns uncorrelated with a stochastic opportunity set, or (3) logarithmic (Bernoulli) utility. The latter case can be traced back to Arrow (1964), Mossin (1968), and Samuelson (1969). It has been studied extensively by Adler and Detemple (1988), Breeden (1979), Cass and Stiglitz (1970), Cox, Ingersoll, and Ross (1985), Hakansson (1970, 1971), Kraus and Litzenberger (1975), Merton (1971, 1973), and Rubinstein (1976).[1] Under this assumption, all investors' asset demands will exhibit only one component, a speculative term that reflects the tradeoff between the expected return and the risk offered by the cash asset or portfolio. Under more general preferences, however, the optimal portfolio strategy will comprise a so-called Merton–Breeden hedging component, which is dynamic and informationally based, and which hedges the investor's wealth against unfavorable shifts of the state variable(s).

The purpose of this paper is to investigate whether an investor still follows the standard portfolio strategy when derivatives only are traded, in the context of a complete information economy with stochastic interest rates. In order to eliminate the influence of the random changes of the state variable(s) on the investor's optimal strategy and emphasize the role

[1]A departure from this result was found by Feldman (1992), who discovered a nonmyopic behavior on the part of Bernoulli investors in an incomplete information economy, where the state variables defining the stochastic opportunity set are unobservable. The exception occurs when the information structure available to investors is conditionally Gaussian.

of the stochastic nature of interest rates, we retain the simplifying assumption of a logarithmic investor.[2] In addition, as is classical in the financial literature devoted to portfolio theory, we abstract from complications that arise when intermediate consumption is allowed and consider only terminal wealth. We focus our analysis on (nonoptional) derivative interest-rate instruments.[3] This choice is dictated partly by the fact that the (standard) case of cash assets is solved completely, and partly by the gigantic success of both futures and forward markets on interest rates. Their explosive expansion was a consequence of the overall outburst of interest-rate volatility that plagued most financial markets in the late 1970s and early 1980s. Aimed in particular at providing new hedging techniques to asset-and-liability managers, their sheer size in terms of both liquidity and volume of transactions long since have dwarfed their underlying cash markets. Therefore, a thorough analysis of their specificities seems advocated. We thus consider a "pure" speculator who chooses to intervene only in the bond forward or futures market. In the same manner that "it takes two to tango," it is well known that for derivative markets to be active, they need to attract not only hedgers, but also speculators. In our context, a pure speculator is defined as an expected (log)-utility maximizer who has no initial position in the underlying cash bonds, and who chooses not to trade on the latter. Rather, the decision is made to participate in the derivative market only, typically because of differential information and/or transaction costs.[4] Hence, although we will postulate that markets are free of arbitrage opportunities and market participants are price takers at these no-arbitrage prices, the market open to the pure speculators will be incomplete.[5]

A crucial ingredient to our analysis is the distinction between forward and futures contracts. Under the common assumption that financial

[2]It would be only marginally more difficult to assume, for instance, isoelastic utility. The optimal strategy would contain the same terms as shown in this paper (except that the coefficient of relative risk aversion, instead of being one, would be a constant smaller than one) plus some extra Merton–Breeden terms to hedge against the random fluctuations of each and every state variable.

[3]As long as interest rates are stochastic, little does it matter on what exact underlying instruments are written the derivatives. For instance, we could have selected forward and futures contracts written on a stock index or on an exchange rate between two currencies. We chose fixed-income derivatives because i) we want to focus on interest rate risk, and ii) the major economic significance of these markets is undisputed.

[4]Since forward contracts (and also futures when interest rates are deterministic) are redundant instruments in perfect markets, the only meaningful way to introduce them without explicitly modeling imperfections is to assume that *implicit* differential transaction or information costs provide *some* investors (the speculators defined below) an incentive to trade on derivative markets rather than on primitive ones. Indeed, many investors face higher trading costs for cash instruments than for futures or forwards, information-search costs to find default-free cash assets, differential taxation problems, and various constraints imposed on their balance sheets.

[5]Because of this incompleteness, the technical derivation of the "pure" speculator's optimal strategy is somewhat involved and will be left to the Appendix.

markets are perfect, futures and forwards essentially are equivalent instruments. Yet marking to market futures positions by settling contract gains and losses on a continuous (daily) basis brings about under stochastic interest rates a potentially important difference between these two derivatives, even in perfect markets. For simplicity, we distinguish forwards and futures by this marking to market feature only. As was shown in such a context by Cox, Ingersoll, and Ross (1981) and Richard and Sundaresan (1981), and more recently under various assumptions by Duffie and Stanton (1992), forward and futures prices differ when interest rates are stochastic. We do not contribute to this well-known pricing problem. We adopt the continuous-time model pioneered by Heath et al. (1992), where any number of factors drives the instantaneous forward rate. This makes arbitrary the shape of the yield curve and provides a convenient framework for deriving the respective optimal investment strategies using interest-rate forwards or futures.

Since interest rates are stochastic, the opportunity sets spanned by the forward or futures contracts, respectively, will be different. In other words, the risk–return relationship offered to speculators in the derivative market will differ according to whether they can trade futures or forwards. It is precisely the objective of this paper to analyze Bernoulli investors' behavior when forward or futures contracts written on interest rates are used as speculative vehicles. It turns out that the results are strikingly different according to which derivative is utilized. While strategies using futures exhibit the usual structure with only one (speculative) term, strategies involving forwards possess an extra term. The latter can be interpreted, not as a Merton–Breeden hedging component, but as a hedge against the interest-rate risk brought about by the forward strategy itself.

The remainder of this paper is organized as follows. The Framework presents the model's framework and main assumptions. Speculation in the Interest Rate Derivatives Market examines the Bernoulli speculator's optimal strategy using forwards, and then futures. We offer some concluding remarks in the final section. A mathematical appendix explains the methodology developed by He and Pearson (1991) and Karatzas, Lehoczky, Shreve, and Xu (1991) that we have adopted and presents the technical derivations of the results.

THE FRAMEWORK

In our framework, individuals can trade continuously on the financial market until time τ, the horizon of the economy. A locally riskless asset and an arbitrary number n of pure default-free discount bonds sufficient to complete the market are traded. The latter pay one dollar at their respective maturities, respectively T_j, $j = 1, \ldots n$, with $T_{j-1} < T_j < \tau$. Also, either a forward contract or a futures contract written on the discount bond of maturity T_2 is available for trade, whose maturity, for simplicity and without loss of generality, is $T_F = T_1$. The following assumptions provide the necessary details.

Assumption 1: Dynamics of the Primitive Assets

At each time t, the price $P(t,T_j)$ of a discount bond whose maturity is T_j, $j = 1, \ldots, n$, is given by:

$$P(t,T_j) = \exp\left[-\int_t^{T_j} f(t,s)ds\right] \tag{1}$$

where $f(t,s)$ is the instantaneous forward rate for s at t, with $s \leq \tau$, the instantaneous spot rate is $r(t) = f(t,t)$, and the price of the locally riskless asset (money market account) is equal to:

$$B(t) = \exp\left[\int_0^t r(u)du\right] \tag{2}$$

To give our model some additional structure, we assume, following Heath et al. (1992), that the instantaneous forward rate (hereafter the forward rate) is the solution to the following stochastic differential equation:

$$df(t,s) = \mu(t,s)dt + \Sigma(t,s)^{\nabla}dZ(t) \tag{3}$$

where $\mu(.)$, the drift term, and $\Sigma(t,s)$, the K-dimensional vector of diffusion parameters (volatilities), are assumed to satisfy the usual conditions[6] such that (3) has a unique solution, and "∇" denotes a transpose. Z(.) is a K-dimensional Brownian motion defined on the complete probability space (Ω,F,Q), where Q is the actual (historical) probability. The forward rate is adapted to the augmented filtration generated by this Brownian motion. This filtration is denoted $F = \{F_t\}_{t\in[0,\tau]}$ and is assumed to satisfy the usual conditions.[7] The initial value of the forward rate, $f(0,s)$, is observable and given by the initial yield curve prevailing on the market.

[6]See Heath et al. (1992).

[7]The filtration contains all the events whose probability with respect to Q is null. See, for instance, Karatzas and Shreve (1991).

Assumption 2: Absence of Frictions and of Arbitrage Opportunities

The assumption of absence-of-arbitrage opportunities in a frictionless financial market is well known, since Harrison and Kreps (1979), to be tantamount to assuming the existence of a probability measure defined with respect to a given numeraire and equivalent to Q, such that the relative prices of the assets with respect to the numeraire are martingales.[8] We will use this property in the Appendix.

Now, applying Ito's lemma to (1), given (3); yields the stochastic differential equation satisfied by the discount bonds: [9]

$$dP(t,T_j) = P(t,T_j)[\mu_P(t,T_j)dt + \Sigma_P(t,T_j)^{\nabla}dZ(t)] \; j = 1, \ldots, n. \quad (4)$$

where $\Sigma_P(t,T_j)$ is the K-dimensional vector of the volatilities associated with $P(t,T_j)$ and $\mu_P(t,T_j)$ is of no interest here, but could easily be computed as the sum of the riskless rate $r(t)$ plus a risk premium that depends on the bond maturity date T_j.

Assumption 3: Admissible Strategies

Each economic agent who can freely trade on the continuously open financial market adopts a portfolio strategy that consists in choosing the appropriate number of units of the locally riskless asset and of each of the n discount bonds. Such strategies are assumed to be admissible,[10] and, in particular, to be self-financing.

Assumption 4: Derivatives Markets

Lastly, the economy is characterized by the presence of a derivative instrument written on the discount bond of maturity T_2, and whose maturity date, without loss of generality, is $T_F = T_1$.

When the derivative asset is a *forward* contract, its price at date $t \leq T_F$ is noted $G(t,T_F,T_2) \equiv G(t)$ for simplicity, and, given Assumption 2, is equal to:

[8]This equivalence result holds only for simple strategies, i.e. strategies that need a portfolio reallocation only a finite number of times (see Harrison & Kreps, 1979; Harrison & Pliska, 1981).
[9]The complicated part of the drift term, $b(t,T_j)$, is of no interest here, but could be computed easily given the forward rate (5).
[10]To save space, we do not specify the properties of admissible strategies. (See Cox & Huang, 1989; Harrison & Kreps, 1979; Harrison & Pliska, 1981; and Heath et al., 1992). The complete definition given by Cox and Huang (1989) is the one adopted.

$$G(t) = \frac{P(t,T_2)}{P(t,T_F)}. \tag{5}$$

When the derivative instrument is a *futures* contract, its price at date $t \leq T_F$ is noted $H(t,T_F,T_2) \equiv H(t)$. Since we do not need to know this price explicitly, we just assume that it follows a diffusion process.[11]

We are now equipped fully to examine the problem faced by a speculator investing in interest rates derivatives.

SPECULATION IN THE INTEREST RATE DERIVATIVES MARKET

Consider a speculator endowed with an arbitrary initial wealth $W(0)$. His investment horizon is $T_I \leq T_F < T_2$. He chooses to trade on the forward (or futures) market, not the bond market,[12] to maximize the expected log-utility of his terminal wealth $W(T_I)$. We want to stress the theoretical and practical importance of the dynamic aspect of the speculative strategy.[13] This is because the (continuous) arrival of information regarding market prices makes it necessary, in general, to revise continuously the position to achieve end-of-period optimality.

The Bernoulli speculator's problem is to choose the portfolio strategy that solves:

$$\begin{cases} \mathrm{Max}_{\Delta_i,\Gamma_i} \mathrm{E}_Q[Ln W_i(T_I)] \\ \text{s.t. the portfolio strategy is admissible and satisfies} \\ \text{ the budget constraint} \end{cases} \tag{6}$$

where $i = G, H$ and the budget constraint writes either:

$$W_G(t) = P(t,T_F)\int_0^t \Delta_G(u)dG(u) + \Gamma_G(t)B(t) \tag{7}$$

or

$$W_H(t) = \int_0^t \exp\left[\int_u^t r(u)du\right]\Delta_H(u)dH(u) + \Gamma_H(t)B(t) \tag{8}$$

[11]Relevant references for the pricing of futures contracts include Duffie and Stanton (1992), Flesaker (1993), and Jarrow (1988).

[12]That is why we call this investor a pure speculator. For instance, the investor may find that transaction and/or information costs in the bond market are prohibitive to him/her. We also might involve reasons related to differential taxation. For a discussion on this point, see Poncet and Portait (1993).

[13]We use continuous time (dt) for mathematical ease. Nothing essential would change if discrete time (Δt) was used.

where the subscript G (resp. H) indicates the use of forward (resp. futures) contracts, $\Delta_i(t)$ is the number of contracts *held* (not traded) at date t, and $\Gamma_i(t)$ is the number of units of the money market account held at instant t. The first term on the RHS of (7) is the current (at date t) value of the profits and losses incurred from the forward position. Since these cumulative "flows" are cashed-in or -out at date T_F only, the discount factor $P(t, T_F)$ is required. The first term on the RHS of (8) is the margin account associated with the speculator's position in futures.

The presence of the locally riskless asset is required. The logarithmic investor is known to choose the optimum-growth portfolio associated with Long (1990). In order to replicate this portfolio by using derivatives instead of bonds, the investor also must make use of the riskless asset.[14] The main reason is that, except for margin calls, trading on forwards (or futures) involves no investment up to time T_I. Thus, wealth is essentially invested in the riskless asset.

Here we face a technical problem when trying to solve (6), since the financial market is incomplete for the pure speculator who trades only the riskless asset and the forward (or futures) contract. Before turning to the solution of this problem, we must characterize the no-arbitrage assumption on the market to which the speculator has access. When markets are incomplete, it is well known that the martingale measure associated with a given numeraire is not unique. The set of martingale measures, however, can be characterized by using the diffusion matrix of the forward (or futures) price.[15] To solve for the speculator's problem, we use the approach developed by Karatzas et al. (1991) and He and Pearson (1991). Since this technique is more involved than the intended general tone of this article, we chose to leave the full explanation of the approach and all technical derivations to the Appendix.

Our results are gathered in the following.

Proposition a: *The pure Bernoulli speculator's optimal dynamic strategy using futures is given by*

$$\hat{\delta}_H(t) = \kappa_H(t)^{\triangledown}\Sigma_H(t)(\Sigma_H(t)^{\triangledown}\Sigma_H(t))^{-1} \tag{9}$$

where the hat, "^", denotes an optimal solution, $\kappa_H(t)$ is the (stochastic) market price of the risk associated with the futures, $\Sigma_H(t)$is the K-dimensional volatility of the futures price, and

$$\hat{\delta}_H(t) \equiv \hat{\Delta}_H(t)H(t)/\hat{W}_H(t)$$

[14]This is analogous to replicating the option with its underlying asset and the risk-free asset in the Black–Scholes world.

[15]See He and Pearson (1991) for a technical discussion and relevant references.

is the "value" of the futures position relative to that of total wealth.

Propostition b: *The pure Bernoulli speculator's optimal dynamic strategy using forward contracts is given by:*

$$\hat{\delta}_G(t) = \kappa_G(t)^{\triangledown}\Sigma_G(t)(\Sigma_G(t)^{\triangledown}\Sigma_G(t))^{-1} - (1 - \hat{\gamma}_G(t))\Sigma_P(t,T_F)^{\triangledown}\Sigma_G(t)(\Sigma_G(t)^{\triangledown}\Sigma_G(t))^{-1} \tag{10}$$

where $\kappa_G(t)$ is the (stochastic) market price of the risk associated with the forward contract, $\Sigma_P(t,T_F)$ is the K-dimensional volatility of the cash bond of maturity T_F, $\Sigma_G(t)$ is the K-dimensional volatility of the forward price,

$$\hat{\delta}_G(t) \equiv \hat{\Delta}_G(t)P(t,T_2)/\hat{W}_G(t)$$

is the "value" of the forward position relative to that of total wealth, and

$$\hat{\gamma}_G(t) \equiv \hat{\Gamma}_G(t)B(t)/\hat{W}_G(t)$$

is the proportion of wealth invested in the riskless asset.

We comment on this proposition by examining first the case of futures, for which our results conform to standard portfolio theory when only primitive (cash) assets are involved. The logarithmic speculator always has a long position in the futures market, together with a long position in the riskless asset, provided the market price of risk $\kappa_H(t)$ is positive. The optimal number of contracts the investor holds is strictly proportional to: (1) the future value of the investor's optimal wealth, and (2) the market price of the risk associated with the derivative asset. Conspicuously, the optimal strategy involves only a mean-variance term and does not include a Merton–Breeden component that would hedge the investor's wealth against unexpected shifts in the locally riskless interest rate. Thus, we recover the Bernoulli speculator's myopic behavior.

In addition, it is noteworthy that, due to this unique myopic feature of the logarithmic utility, the fictitious risky assets that complete the market are (see Appendix) ones that offer a zero-risk premium. This is the reason why eq (9) has its familiar aspect, although the market is incomplete.

Now, the result obtained with forward contracts strikingly differs from that achieved with futures. Primarily, there is an extra term in (10) that involves the fraction of wealth noninvested in the riskless asset, $(1 - \hat{\gamma}_G(t))$, times a usual covariance/variance ratio. This could be reminiscent of a Merton–Breeden informationally based component that is used by a multiperiod, nonmyopic investor to hedge unfavorable shifts in the investment opportunity set.[16] Recall that this component comprises the two multiplicative terms: J_{WX}/J_{WW} Cov(X,Y)/Var(Y) where J is the investor's value function, $J_{WX}(\equiv \partial J_W/\partial X)$, is the cross-partial derivative that represents the effect of the economic state variable X on the marginal value of wealth W, J_{WW} is the second partial derivative with respect to wealth, and Y is the financial asset held. Thus, the first term obviously is preference dependent. As discussed in the introduction, this Merton–Breeden component vanishes either if (1) X is not stochastic, or (2) asset Y return is uncorrelated with the stochastic state variable X, or (3) utility is logarithmic because, as shown by Adler and Detemple (1988), the cross partial derivative J_{WX} vanishes.

Here, however, our extra term does not qualify as a Merton–Breeden component to the extent that a) it is not a hedge against future levels of a state variable, and b) its first component, $(1 - \hat{\gamma}_G(t))$, is preference free, and depends only on wealth. Rather, it stems from the fact that since the forward position is not marked to market, it faces an additional interest-rate risk (that obviously disappears in the case of futures) on the cumulative paper profits or losses that have accrued so far.

Indeed, in our framework, two sources of randomness can be distinguished. The first one, which is classical, is that the *market price of risk* associated with the forward price, $\kappa_G(t)$, itself fluctuates randomly. All investors but the logarithmic investor hedge against this risk (and, since the opportunity set is affected, the hedge is preference dependent). The second source is the *interest-rate risk induced by the forward trading strategy itself.* Because the investor anticipates that one period ahead, the current value of the investor's cumulative forward position (also) will have changed; even the Bernoulli speculator will hedge optimally against unfavorable interest-rate moves brought about by the investment strategy. This hedge is preference-free (since the opportunity set is not affected by the strategy per se), but, as intuition suggests, depends on the fraction of wealth $(1 - \hat{\gamma}_G(t))$ that is generated by the forward strategy. To the best of our knowledge, this paper is the first to exhibit this type of hedge. As it is elicited by the trading strategy itself, one could refer to it as an "endogenous risk" hedge.

Second, as with futures, the logarithmic speculator always will start with a long position in the forward market, along with a positive investment in the money-market account, provided the market price of risk $\kappa_G(t)$ is positive. However, since $\hat{\gamma}_G(t)$, the proportion of wealth invested

[16]Adler and Detemple (1988) provide an excellent discussion of this issue in the context of hedging a given, nontraded cash position. The nature of our problem, however, is very different in that we consider speculation or investment, not hedging.

in the riskless asset, is smaller than one in the case of a winning forward position, and since the covariance between the forward-price changes and the $P(t,T_F)$ bond-price changes is presumably positive, the extra risk involved leads the speculator to reduce, other things being equal, the holdings in the forward contracts, which could even become theoretically negative. Thus, this strategy loses the proportionality in wealth and in the market price of risk that characterized the strategy involving futures.

Lastly, it remains true, as in the futures case, that the fictitious assets that complete the market offer zero-risk premia, because of the unique shape exhibited by the logarithmic utility function.

CONCLUDING REMARKS

In a complete information economy where all state variables are stochastic but fully observable, logarithmic utility is sufficient to produce myopia of portfolio decisions involving primitive (cash) financial assets. Thus, portfolio strategies involve only one speculative term. Retaining the complete information assumption, we showed that logarithmic speculators who decide to trade on interest-rate forwards, but not futures, follow a more complex strategy that does not qualify as nonmyopic.

The debate relative to the respective benefits and drawbacks of over-the-counter forwards and officially organized futures generally focused on transaction costs, safety considerations, counterparty risk, and liquidity-related issues. None of these somewhat important features is relevant in our abstract setting. Even so, precise account of the marking-to-market mechanism allows one to substantiate the claim that using futures on interest-rate instruments differs significantly from using forwards under stochastic interest rates. Indeed, even for a pure Bernoulli speculator, optimal wealth levels and trading strategies differ materially. In particular, when forwards are involved, a new type of hedge, different from a Merton–Breeden hedge, is performed against the risk of the trading strategy itself, while the futures strategy retains the familiar, simpler, structure.

Thus, our findings have some bearing on the issue of optimal design of financial contracts. Given that interest rates indeed are stochastic in the real world, we already know from financial theory that forward and futures prices should be different. Whether this difference is, in practice, economically significant or not depends partly on the instruments underlying the derivatives, and partly on subjective assessments. Empirically, as intuition suggests, the difference is more pronounced for instruments directly related to interest rates than for stocks, for instance, or commodities. The issue examined here is different. It consists in assessing whether an optimal speculative strategy using futures should differ from the one using forwards. It has been shown recently[17] that optimal hedging of a nontraded-cash-bond position using futures also involves a simpler formula than hedging with forwards. Thus, it appears that the optimal strategies followed by most market participants are, in theory, easier to compute, implement, and control when futures contracts are used. At the practical level, it is clear that an additional source of risk must be recognized and properly managed when forward contracts are traded. If markets were complete, the necessary correction would be easy to compute, although it still would be time consuming. Since real financial markets are incomplete, however, the additional risk cannot be hedged perfectly. As using futures eliminates the implied residual risk, risk-averse agents will find them attractive in relation to forward contracts, other things being equal. Although, since the net supply of derivatives is zero, nothing precise can be said on economic welfare at the aggregate level, our results are rather encouraging from the perspective of official markets to the extent that marking to market is (also) an efficient manner to prevent default or counterparty risk.

APPENDIX

This appendix is devoted to the exposition of the martingale approach to portfolio choice in incomplete markets, as developed by Karatzas et al. (1991) and He and Pearson (1991), and to the proof of the Proposition in the text. Since the technique used in the case of futures is the same as that of forwards, we focus on the latter, and solve the former more rapidly.

First apply Ito's lemma to $G(t)$ given by (5) to obtain:

$$dG(t) = G(t)\mu_G(t)dt + G(t)\Sigma_G(t)^{\nabla}dZ(t) \qquad \text{(A1)}$$

where $\mu_G(t)$ could be (to no effect) explicited using (5) and $\Sigma_G(t)$ is the (K-dimensional) volatility of the forward price [see the similar eq (4)].

Since markets are arbitrage free, there exists a stochastic process $\kappa_G(t)$ followed by the market price of risk associated with the forward contract that is assumed to satisfy the Novikov's condition:

$$E^Q\left[\exp\left\{\frac{1}{2}\int_0^t \kappa_G(u)^{\nabla}\kappa_G(u)du\right\}\right] < \infty,\ t \leq T_F,$$

so that the process $\eta^G(T_F)$ defined below is a Q-martingale.[18] A martin-

[17]See Lioui and Poncet (in press). Optimal hedge ratios typically differ from 1 to 15% in absolute value for reasonable parameters.

[18]See Karatzas and Shreve (1991, p. 198).

gale measure associated with the locally riskless asset as the numeraire can be constructed such that:

$$\left.\frac{dQ^G}{dQ}\right|_{F_{T_F}} \equiv \eta^G(T_F)$$

$$= \exp\left\{-\int_0^{T_F} \kappa_G(t)^\nabla dZ(t) - \frac{1}{2}\int_0^{T_F} \kappa_G(t)^\nabla \kappa_G(t)dt\right\} \quad \text{(A2)}$$

Since the market is not complete (for the speculator), the measure Q^G is not unique. Let

$$\text{Ker}(\Sigma_G(t)^\nabla) \equiv [\theta(t) \in R^K | \Sigma_G(t)^\nabla \theta(t) = 0]$$

be the kernel of vector $\Sigma_G(t)^\nabla$. We can construct other martingale measures such that:

$$\left.\frac{dQ_\theta^G}{dQ}\right|_{F_{T_F}} \equiv \eta_\theta^G(T_F) = \exp\left\{-\int_0^{T_F} (\kappa_G(t) + \theta(t))^\nabla dZ(t)\right.$$

$$\left. - \frac{1}{2}\int_0^{T_F} (\kappa_G(t)^\nabla \kappa_G(t) + \theta(t)^\nabla \theta(t))dt\right\} \quad \text{(A3)}$$

The choice of the martingale measure that will be the pricing rule for the Bernoulli investor is part of the solution to the investor's problem.

We solve the problem in three steps. First, we choose an arbitrary vector $\theta(t)$ from $\text{Ker}(\Sigma_G(t)^\nabla)$ and fix a martingale measure Q_θ^G. We find the optimal wealth the investor will reach using this measure, given the investor's budget constraint. This optimal wealth obviously depends upon the selected vector $\theta(t)$. If markets were complete for the speculator, we could be sure that there exists a forward strategy that enables the investor to reach this wealth. However, in incomplete markets, the wealth obtained in the first step may not be reachable. The second step then consists of choosing the particular vector of $\text{Ker}(\Sigma_G(t)^\nabla)$ that ensures that this wealth indeed is reachable. This vector is chosen so as to yield the minimum of the speculator's expected utility to reflect the fact that the speculator is constrained by the market incompleteness. This also characterizes a particular martingale measure. In the third and last step, this martingale measure is used to construct the portfolio strategy that allows replication of the optimal wealth.

Proof of Proposition (Part b).

In the first step, the speculator's program is :

$$\begin{cases} \max\limits_{W_G(T_I)} \; E^Q[\text{Ln } W_G(T_I)] \\ \text{s.t. } E^{Q^G_\theta}\left[\dfrac{W_G(T_I)}{B(T_I)}\right] = W(0) \end{cases}$$

the solution to which is:

$$\hat{W}_G(T_I) = W(0)B(T_T)(\eta^G_\theta(T_I))^{-1} \tag{A4}$$

The value function of the program is $J_G(\theta) \equiv E^Q[\text{Ln}\hat{W}_G(T_I)]$. Using (A3) and (A4), we obtain:

$$J_G(\theta) = \text{Ln}W(0) + E^Q\left[\int_0^{T_I} r(s)ds\right] + \frac{1}{2}E^Q\left[\int_0^{T_I} \kappa_G(t)^\nabla\kappa_G(t)dt\right]$$
$$+ \frac{1}{2}E^Q\left[\int_0^{T_I} \theta(t)^\nabla\theta(t)dt\right]$$

In the second step, we have to find θ such that

$$\begin{cases} \min\limits_{\theta} J_G(\theta) \\ \text{s.t. } \Sigma_G(t)^\nabla\theta(t) = 0 \text{ for } t \in [0,T_I] \end{cases}$$

Solving this program is equivalent to solving

$$\begin{cases} \min\limits_{\theta} \dfrac{1}{2}\theta(t)^\nabla\theta(t) \\ \text{s.t. } \Sigma_G(t)^\nabla\theta(t) = 0 \end{cases}$$

whose solution is obviously the null vector: $\theta(t) = 0$. This surprising result uniquely is valid for the logarithmic utility. Thus, the martingale measure is given by (A2) and the optimal wealth (A4) is equal to:

$$\hat{W}_G(T_I) = W(0)B(T_I)(\eta^G(T_I))^{-1}$$

Its value at each time t is equal to:

$$\frac{\hat{W}_G(t)}{B(t)} = E^{Q^G}\left[\frac{\hat{W}_G(T_I)}{B(T_I)}\middle|F_t\right] = W(0)E^{Q^G}[\eta^G(t))^{-1}|F_t]$$
$$= W(0)(\eta^G(t))^{-1} \tag{A5}$$

where we have used lemma 2.5 on page 43 of Cox and Huang (1989) (Bayes' rule). Applying Ito's lemma to (A5), we get the dynamics of the optimal wealth:

$$d\hat{W}_G(t) = (.)dt + \hat{W}_G(t)\kappa_G(t)^{\triangledown}dZ(t) \tag{A6}$$

From equation (7) in the text, the dynamics of the speculator's wealth also is given by:

$$\begin{aligned} d\hat{W}_G(t) &= (.)dt + \hat{\Delta}_G(t)P(t,T_F)G(t)\Sigma_G(t)^{\triangledown}dZ(t) \\ &\quad + \left(\int_0^t \hat{\Delta}_G(u)dG(u)\right)P(t,T_F)\Sigma_P(t,T_F)^{\triangledown}dZ(t) \end{aligned}$$

Now define:

$$\hat{\gamma}_G(t) \equiv \hat{\Gamma}_G(t)B(t)/\hat{W}_G(t)$$

and

$$\hat{\delta}_G(t) \equiv \hat{\Delta}_G(t)P(t,T_2)/\hat{W}_G(t) = \hat{\Delta}_G(t)P(t,T_F)G(t)/\hat{W}_G(t)$$

[using (5)] and recall, from (7), that

$$P(t,T_F)\int_0^t \hat{\Delta}_G(u)dG(u) = \hat{W}_G(t) - \hat{\Gamma}_G(t)B(t)$$

Hence, $d\hat{W}_G(t)$ writes:

$$\begin{aligned} d\hat{W}_G(t) &= (.)dt + \hat{\delta}_G(t)\hat{W}_G(t)\Sigma_G(t)^{\triangledown}dZ(t) \\ &\quad + (1 - \hat{\gamma}_G(t), \hat{W}_G(t)\Sigma_P(t,T_F)^{\triangledown}dZ(t) \end{aligned} \tag{A7}$$

Identifying the diffusion coefficients in (A6) and (A7) yields the result.

Proof of Proposition (Part a)

We apply the same methodology to derive the speculator's futures strategy. First, apply Ito's lemma to $H(t)$ to obtain:

$$dH(t) = H(t)\mu_H(t)dt + H(t)\Sigma_H(t)^{\triangledown}dZ(t) \tag{A8}$$

with the same interpretation as for $dG(t)$.

The process $\kappa_H(t)$, followed by the market price of risk associated with the futures contract, is assumed to satisfy the Novikov's condition. Following the same steps as above, we derive:

$$d\hat{W}_H(t) = (.)dt + \hat{W}_H(t)\kappa_H(t)^\nabla dZ(t) \tag{A9}$$

From equation (8) in the text, the speculator's wealth dynamics is also given by:

$$d\hat{W}_H(t) = (.)dt + \hat{\Delta}_H(t)H(t)\Sigma_H(t)^\nabla dZ(t) \tag{A10}$$

Defining $\hat{\delta}_H(t) \equiv \hat{\Delta}_H(t)H(t)/\hat{W}_H(t)$ and identifying the diffusion coefficients in (A9) and (A10) yields the desired result.

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