Hedging Downside Risk under Asymmetric Taxation

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INTRODUCTION

One main function of a futures market is the opportunity it provides for decision makers to transfer risk through hedging. A futures hedge consists of two positions, a nontradable spot position and a tradable futures position. Typically, the two positions are the opposite of each other. A long spot position is matched with a short futures position or vice versa. For effective risk reduction, the gains from one position must be offset (at least partially) by the losses from the other position. While spot gains or losses are unequivocally "ordinary" according to tax rules, futures gains or losses may be classified as "capital" or "ordinary."

Under the Corn Products (CP) rule (1955), the gains and losses from hedging transactions initiated in connection with ordinary business purposes (as opposed to investment purposes) were classified as ordinary income. Consequently, futures and spot gains or losses were allowed to offset each other when determining the tax liability. After 1988, the Arkansas Best (AB) rule restricted CP arguing that business intention was not sufficient to justify the ordinary classification. After this ruling, futures gains or losses that didn't fall into a narrowly defined class of trans-

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actions would be treated as "capital." As offsetting income and loss is not allowed between ordinary and capital assets, the risk reduction function of the futures market would be jeopardized. Arkansas Best was reversed in 1993 with the Federal National Mortgage Association ruling, which allowed that gains or losses from hedging to be ordinary income or loss as long as the items being hedged were not capital assets. Although the loudest opponents to the Arkansas Best decision have been satisfied, the economic analysis of the ruling remains an interesting problem.

Purcell (1995) argued that the Arkansas Best tax policies discourage cattle feeders from participating in the futures trading, and hence adversely affected the price discovery function assumed by the futures market. Fox-Andrews (1995) provides details on tax treatment in the United States and the United Kingdom. Within an analytical framework, Pirrong (1995) compared the AB and CP tax rules and showed that futures trading volume is reduced when the AB rule replaces the CP rule. In the analysis, he carefully distinguished between risk-neutral and risk-averse hedgers, analyzing both cases though arguing that risk aversion was a more justifiable behavioral assumption. To characterize the welfare loss under the Arkansas Best ruling, he adopted a mean-variance approach. While some firms hedge to reduce variability of profits, others hedge to avoid financial distress. In fact, the latter objective seems to be more in line with Pirrong's reasoning for risk aversion.

Specifically, when arguing why a firm may be risk averse, Pirrong called upon the probability of financial distress and the associated cost. He then suggested that firms would engage in futures trading to reduce this probability, that is, this risk. While the risk of financial distress is one-sided, the notion of "variance" is two-sided. That is, if one is concerned with financial distress, only "downside risk" is involved. Any larger gains are deemed welcome. Using variance to measure risk implies both larger gains and larger losses are undesirable. A hedger who aims to minimize the variance avoids both financial bust but also avoids financial boom. Although Pirrong (1995) provides an excellent mean-variance analysis for the problem, in this note we attempt to extend his work by investigating the effects of tax rules on downside risk hedging.

BASIC FRAMEWORK

We adopt the framework in Pirrong (1995). Consider a firm with a stochastic business profit, $\pi = I + y$, where I is the deterministic part and y is the stochastic part. That is, $I = E(\pi)$ and $y = \pi - E(\pi)$. There exists a futures contract that serves as a perfect hedging instrument. The payoff from holding a unit of the futures contract is y. Assume the firm holds k units of the futures contract. The total pretax profit is $\pi + ky$. The IRS tax rules will determine the after-tax profit.

Under the CP rule, derivative gains or losses are ordinary. The after-tax profit is:

$$\pi^C = I + y + ky - t^c \max(0, I + y + ky),$$

where t^c is the tax rate. In the calculation, gains and losses from business operations and futures trading are aggregated as offsetting the gains and losses is allowed for tax purposes. If the AB rule prevails, futures gains and losses are treated as capital and taxed separately from business income. Thus, the after-tax profit becomes

$$\pi^A = I + y + ky - t^a \max(0, I + y) - t^a \max(0, ky)$$

where t^a is the tax rate under the AB rule. Clearly there is no offset. If the tax rate remains the same across the two rules, then $\pi^A \leq \pi^C$ as $\max(0, I + y) + \max(0, ky) \geq \max(0, I + y + ky)$.

Downside risk occurs when there is a positive probability where the after-tax profit falls below a threshold level $\underline{\pi} > 0$. A firm will choose the optimal futures position k^* to minimize the downside risk (i.e., the probability that $\underline{\pi}$ exceeds the after-tax profit). That is, k^* is chosen to minimize $\Pr(\pi^C \leq \underline{\pi})$ and $\Pr(\pi^A \leq \underline{\pi})$ under the CP and AB rules, respectively; this objective function was also considered in Chowdhry (1995), Bigman (1996), and Basak and Shapiro (1999). The welfare loss associated with a change from the CP rule to the AB rule is then measured by the increase in the downside risk.

HEDGING UNDER THE CP RULE

Under the CP rule, the probability of financial distress, P^C, can be written as follows:

$$P^C = \Pr(\pi^C \leq \underline{\pi}) = \Pr[I + y + ky \leq (1 - t^c)^{-1}\underline{\pi}]. \tag{1}$$

Suppose that $1 + k \geq 0$. The probability becomes $\Pr\{y \leq \theta\}$ where $\theta = (1 + k)^{-1}[(1 - t^c)^{-1}\underline{\pi} - I]$. Let $g(.)$ denote the probability density function of y. Then

$$\partial P^C/\partial k = -(1 + k)^{-1}\theta g(\theta). \tag{2}$$

The above expression is positive when $\theta < 0$ and negative when $\theta > 0$ regardless of the value of k.

On the other hand, when $1 + k \leq 0$, the probability of financial distress is $P^C = \mathrm{PR}\{y \geq \theta\}$. Consequently,

$$\partial P^C/\partial k = (1 + k)^{-1}\theta g(\theta). \tag{3}$$

Thus, it is always positive when $\theta < 0$ and negative when $\theta > 0$. Upon combining the two cases, Proposition 1 follows immediately.

Proposition 1: Suppose that the threshold level $\underline{\pi}$ is smaller than $(1 - t^c)I$. Then the optimal futures position is established at the full hedge, that is, $k^* = -1$.

Under the assumption prescribed in Proposition 1, ($\theta < 0$ when $1 + k > 0$. Hence $\partial P^C/\partial k > 0$. To minimize the downside risk, the hedger should reduce k making it closer to -1. On the other hand, $\theta > 0$ when $1 + k < 0$. Consequently, $\partial k^C/\partial k < 0$ and the hedger should increase k toward -1 to minimize the downside risk. Upon combining the two cases, the optimal futures position is then $k^* = -1$. In fact, at the optimal position $\Pr(\pi^C \leq \underline{\pi}) = \Pr[I \leq (1 - t^c)^{-1}\underline{\pi}] = 0$. Downside risk is completely eliminated.

Note that the after-tax profit is $(1 - t^c)I$ when there is no uncertainty (i.e., $y = 0$ with certainty). The condition prescribed in Proposition 1 requires the threshold profit level be smaller than the after-tax profit in absence of uncertainty. Not only does this assumption seem reasonable, inspection of equations (2) and (3) reveals that there is no solution to the hedger's problem if the condition fails to hold. Therefore, the assumption is maintained throughout this note.

HEDGING UNDER THE AB RULE

The problem becomes much more complicated when the AB rule applies. Proposition 1 becomes implies $\underline{\pi} < (1 - t^a)I$ as the tax rate on capital income is smaller than that on ordinary income (i.e., $t^a < t^a$). Milevsky and Prisman (1999) assume that $t^a = 0.30$ and $t^c = 0.45$. Proposition 2 considers the case that $\underline{\pi} < (1 - t^a)^2 I$.

Proposition 2: Assume that $\underline{\pi}$ is smaller than $(1 - t^a)^2 I$. The optimal futures position is within the interval $[-1, t^a - 1]$. If there is an interior solution k^* then it must satisfy the following equation:

$$g\left(\frac{\underline{\pi} - I}{1 + k^* - t^a k^*}\right)\frac{(1 - t^a)(I - \underline{\pi})}{(1 + k^* - t^a k^*)^2}$$

$$= g\left(\frac{\underline{\pi} - (1 - t^a)I}{1 + k^* - t^a}\right)\frac{(1 - t^a)I - \underline{\pi}}{(1 + k^* - t^a)^2}. \tag{4}$$

Otherwise, the optimal futures position is located at the lower boundary point, -1. The proof of the proposition is provided in the Appendix.

The condition that $\underline{\pi} < (1 - t^a)^2 I$ is not demanding. Recall that Proposition 1 requires $\underline{\pi} < (1 - t^c)I$. Using the tax rates from Milevsky and Prisman (1999), the former requires $\underline{\pi} < 0.49I$, while the latter requires $\underline{\pi} < 0.55I$. Also note that equation (4) merely provides the first order condition for downside risk minimization. There is also a second order condition that an interior solution must satisfy if it exists. Proposition 3 provides a sufficient condition to ensure the existence of an interior solution. The proof is provided in Appendix I.

Proposition 3: Suppose that the probability density function of y, $g(y)$, is symmetric at zero. If $g(y)$ declines quickly when y moves away from the origin, then an interior solution exists, which is also the optimal futures position.

As an example, the uniform distribution function will not ensure that the optimal futures position be characterized by equation (4). In our numerical experiments, both normal and double exponential density functions conform to Proposition 3.

NUMERICAL RESULTS

To analyze the effects of various parameter values (i.e., $\underline{\pi}$, I, and t^a) on the optimal futures position, the usual approach would apply the comparative static analysis to equation (4). Due to the complexity of the expression and to the possibility of a dominant boundary solution, analytical results are not available. Instead, numerical simulations are adopted. Let σ denote the standard deviation of y. If k^* satisfies equation (4) for ($\underline{\pi}$, I, t^a), then it is also the solution for the parameter vector ($\underline{\pi}/\sigma$, I/σ, t^a). Our numerical experiments therefore adopt the latter as the parameter vector. Two density functions are considered: normal and double exponential. Let $y = \sigma w$. If y is normally distributed, then the density function of w is $g_w(w) = (2\pi)^{-1/2}\exp(-w^2/2)$ where π takes its standard numerical meaning. For the double exponential density function, $g_w(w) = (1/2)\exp(-|w|)$, where $|w|$ is the absolute value of y. Normal density function is popular in simulation study. The double exponential density is chosen because its statistical properties are in sharp contrast to the normal density. By examining the two densities, some degree of robustness of the simulation results can be obtained.

Table I shows the optimal futures positions for various parameter values when the distribution of y is either normal or double exponential. For example, suppose that $I/\sigma = 1$, $\underline{\pi}/\sigma = 0.3$, and $t^a = 0.3$. Here the optimal futures position under the AB rule will be -0.859 if the return

TABLE 1

Minimum Downside-Risk Hedge Ratios

t^a	I/σ	π/σ	Normal	Double Exponential
0.3	1.0	0.4	−0.818	−0.759
		0.3	−0.859	−0.786
		0.2	−0.889	−0.813
		0.1	−0.909	−0.839
	0.9	0.4	−0.787	−0.742
		0.3	−0.833	−0.767
		0.2	−0.872	−0.795
		0.1	−0.900	−0.823
	1.5	0.6	−0.858	−0.866
		0.45	−0.889	−0.894
		0.3	−0.909	−0.914
		0.15	−0.923	−0.928
0.25	1.0	0.4	−0.891	−0.828
		0.3	−0.918	−0.857
		0.2	−0.934	−0.882
		0.1	−0.945	−0.903
	0.9	0.4	−0.863	−0.807
		0.3	−0.902	−0.836
		0.2	−0.926	−0.865
		0.1	−0.941	−0.891
0.10	1.0	0.4	−0.990	−0.986
		0.3	−0.991	−0.989
		0.2	−0.992	−0.991
		0.1	−0.994	−0.992
	0.9	0.4	−0.989	−0.984
		0.3	−0.991	−0.988
		0.2	−0.992	−0.990
		0.1	−0.994	−0.991

of investment is normally distributed, and −0.786 if it is instead double exponentially distributed. Several properties for the optimal futures position can be drawn from the numerical results.

First, the hedger becomes more active in futures trading (i.e., $|k^*|$ increases) as the mean of the business profit, I, increases. For example, suppose that I/σ increases from 0.9 to 1.0 while $t^a = 0.3$ and $\underline{\pi}/\sigma = 0.2$. The hedger increases his short position from 0.872 to 0.889 under normal distribution assumption. When I is large, financial distress occurs only when there is a severe stochastic loss (i.e., $y << 0$). In this case, a large short position would generate great profits from futures trading.

Second, the hedger holds a smaller short position when the tax rate t^a increases. Suppose that normal distribution prevails and that $I/\sigma = 0.9$ and $\underline{\pi}/\sigma = 0.3$. The hedger has 0.902 and 0.833 units of short position when $t^a = 0.25$ and 0.30, respectively. Higher tax rates reduce the hedging effectiveness of the futures contract and depress futures trading activities. Third, the hedger holds a larger short position when the financial distress threshold, $\underline{\pi}$ is small. For example, when $t^a = 0.25$ and $I/\sigma = 1.0$, the short position increases from 0.891 to 0.945 (for normal distribution) when $\underline{\pi}/\sigma$ declines from 0.4 to 0.1. That is, the futures contract is more useful in eliminating extreme adverse outcomes.

Finally, we evaluate the effect of the variation of business profit, σ. As σ increases, both I/σ and $\underline{\pi}/\sigma$ decrease proportionally. Table I shows that the hedger responds with a smaller futures position. When $t^a = 0.3$, a simultaneous increase of I/σ from 1.0 to 1.5 and $\underline{\pi}/\sigma$ from 0.4 to 0.6 (i.e., σ is reduced to two-thirds of its original value) increases the short position from 0.818 to 0.858 under the normal distribution assumption. Note that the futures contract incurs the same volatility as the underlying business profit. When the business profit becomes more volatile, a larger futures position produces larger fluctuations in profits. Consequently, it is more likely to encounter financial distress.

TAX-LOSS CARRYFORWARD

In the basic model, a company pays tax when income is positive. In case of a negative income, there is of course no tax; but there is also no gain. Under the current tax code, losses can be carried forward. Thus, a loss creates benefits to be recognized in the future. This is especially important under the AB rule. To capture this effect, a smaller tax rate is considered: $t^a = 0.1$. The resulting optimal hedge ratios are displayed in the bottom panel of Table I. As expected, the firm takes on a larger futures position when tax rates decrease. In fact, for the parameter values considered in this paper, the firm will engage in a near perfect hedge strategy when the tax rate is 0.1.

MINIMUM VARIANCE HEDGE RATIOS

The conventional approach to measuring risk assumes that firms engage in futures trading to minimize the variance of their profits. As mentioned, Pirrong adopted this method to compare the welfare effects of the AB and CP rules. Under this modeling strategy the firm chooses k to minimize $\mathrm{Var}(\pi^A)$. An illustration for analytically obtaining the minimum variance hedge ratios can be found in Appendix II.

For some density functions, analytical solutions may be unavailable. Instead, simulation methods must be called upon to find the variance-minimizing hedge ratios. For example, in case of double exponential den-

TABLE 2

Minimum Variance Hedge Ratios

t^a	I/σ	Normal	Double Exponential
0.3	1.0	−0.865	−0.871
0.25	1.0	−0.892	−0.898
0.1	1.0	−0.961	−0.967

sity function, draw ten thousand random samples from a uniform random variable with support [0, 1]. Let z_j be the jth sample. Define $w_j = \ln(2z_j)$ when $z_j \leq 0.5$ and $w_j = -\ln[2(1 - z_j)]$ when $z_j \geq 0.5$. Then w_j can be treated as a sample from the standard double exponential random variable. For a given k, ten thousand random samples of π^A are derived each using a w_j value to replace y in equation (2). The sample variance of π^A is then calculated. Finally, a grid search method is applied to derive the minimum variance hedge ratio, h_d. For the normal density, ten thousand random samples can be directly generated and no transformation is required.

Some numerical results are provided in Table II. When $I/\sigma = 1$ and $t^a = 0.3$, the minimum variance hedge ratio is -0.865 for the normal density function. Thus, it is larger than the minimum downside-risk hedge ratio when $\underline{\pi}/\sigma \leq 0.2$. If $\underline{\pi}/\sigma \geq 0.3$, the minimum variance hedge ratio is smaller. The greatest difference (5.43%) occurs at the case $\underline{\pi}/\sigma = 0.4$. For the double exponential density, the minimum variance hedge ratio is -0.871 when $I/\sigma = 1$ and $t^a = 0.3$. It is smaller than all the minimum downside-risk hedge ratios. Again, the greatest difference (12.86%) occurs when $\underline{\pi}/\sigma = 0.4$.

In sum, the difference between the two ratios is greater in the double exponential case and when the threshold is large. Similar observations can be made for the case $t^a = 0.25$. When $t^a = 0.1$, all hedge ratios cluster near one. However, for both densities, the minimum variance hedge ratio is always smaller than the minimum downside-risk hedge ratio. The difference is less than 3%.

DIFFERENT TAX TREATMENTS

Under the AB rule the gains/losses from futures trading are treated as capital income and may not be aggregated with ordinary income when calculating the tax burden. The tax rate for capital income, t^a, is less than

TABLE 3

Minimum Downside-Risk Hedge Ratios ($t^a = 0.25$, $t^c = 0.3$)

I/σ	π/σ	Normal	Double Exponential
1.0	0.4	−0.820	−0.762
	0.3	−0.855	−0.790
	0.2	−0.880	−0.817
	0.1	−0.896	−0.842
0.9	0.4	−0.790	−0.746
	0.3	−0.835	−0.770
	0.2	−0.868	−0.799
	0.1	−0.890	−0.827

that for ordinary income, t^c. Accordingly, the firm's profit under the AB rule should be modified as follows:

$$\pi^D = I + y + ky - t^c \max(0, I + y) - t^a \max(0, ky) \tag{5}$$

The firm will choose the optimal hedge ratio to minimize the probability that the after-tax income falls below the threshold, $\Pr(\pi^D \leq \underline{\pi})$. The solution is characterized in the following proposition.

Proposition 4: Assume that $\underline{\pi}$ is smaller than $(1 - t^a)(1 - t^c)I$. The optimal futures position is within the interval $[-1, t^c - 1]$. If there is an interior solution k^*, then it must satisfy the following equation:

$$g\left(\frac{\underline{\pi} - I}{1 + k^* - t^a k^*}\right)\frac{(1 - t^a)(I - \underline{\pi})}{(1 + k^* - t^a k^*)^2} = g\left(\frac{\underline{\pi} - (1 - t^c)I}{1 + k^* - t^c}\right)\frac{(1 - t^c)I - \underline{\pi}}{(1 + k^* - t^c)^2}. \tag{6}$$

Otherwise, the optimal futures position is located at the lower boundary point, -1. The proof of Proposition 4 is similar to that of Proposition 2.

To analyze the effects of different tax treatments, consider the case that $t^a = 0.25$, $t^c = 0.3$, and $I/\sigma = 1$. Table III presents the minimum downside-risk hedge ratios in this case. As expected, the hedge ratio increases (i.e., a smaller futures position) as $\underline{\pi}/\sigma$ increases. When compared to the case $t^a = t^c = 0.25$ (symmetric tax treatment), the firm always assumes a smaller futures position under asymmetric tax treatment. An increase in the operating income tax (from 0.25 to 0.3) is similar to a reduction in I and thus the firm assumes a smaller short position. When compared to the case $t^a = t^c = 0.3$, the firm always hedges more under an asymmetric tax treatment when y has a double exponential density. However, it may hedge more or less in the presence of a normal density. That is, the effect of an increase in the capital income tax on futures hedging may be positive or negative. Similar results are obtained when $I/\sigma = 0.9$.

CONCLUSIONS

In this paper we consider the hedging decision to reduce downside risk under different tax rules. When the Corn Products rule prevails, the full hedge is optimal and the downside risk can be completely eliminated. If, instead, the Arkansas Best rule is applied, then a partial hedge is most likely the optimal policy. In this latter case there is always residual downside risk.

Numerical results indicate that under the AB rule, firms that minimize downside risk will lower their use of futures markets to spread risk when the threshold of financial distress, the tax rate, and the volatility of business profits increase. Higher expected business profit has a positive effect. Allowing for tax loss carryforward induces the firm to assume a larger short position.

The minimum downside risk hedge ratio differs from the conventional minimum variance hedge ratio particularly when the threshold is large and when the stochastic income has a double exponential distribution. A final scenario considers different capital income and operating income taxes. It is found that, although an increase in the operating income tax always discourages futures trading, an increase in capital income tax may promote futures trading under certain circumstances.

APPENDIX I

Proof of Proposition 2

Under asymmetric taxation, the downside risk is:

$$\Pr[\pi^A \leq \underline{\pi}] = \Pr[I + y + ky - t^a \max(0, I + y) - t^a \max(0, ky) \leq \underline{\pi}].$$

Suppose that $-1 \leq k \leq 0$. There are three possible regimes for y: (1) $y \leq -I$, (2) $-I \leq y \leq 0$, and (3) $y \geq 0$. For Case (1), the condition $\pi^A \leq \underline{\pi}$ implies $y \leq (1 + k - t^a k)^{-1}(\underline{\pi} - I)$. To characterize the probability, one needs to compare $(1 + k - t^a k)^{-1}(\underline{\pi} - I)$ and $-I$. After algebraic manipulation, it is easily found

$$(1 + k - t^a k)^{-1}(\underline{\pi} - I) \geq -I \text{ if and only if } k \geq -[(1 - t^a)I]^{-1}\underline{\pi}.$$

Note that $-[(1 - t^a)I]^{-1}\pi \geq -1$. Thus, the downside risk becomes

$$\Pr[\pi^A \leq \underline{\pi}] = \Pr[y \leq (1 + k - t^a k)^{-1}(\underline{\pi} - I)]$$

when $k \leq -[(1 - t^a)I]^{-1}\underline{\pi}$; otherwise, $\Pr[\pi^A \leq \underline{\pi}] = \Pr[y \leq -I]$.

For Case (2), the condition $\pi^A \leq \underline{\pi}$ implies $y \leq (1 + k)^{-1}[(1 - t^a)^{-1}\underline{\pi} - I] \leq 0$. The relationship between $(1 + k)^{-1}[(1 - t^a)^{-1}\underline{\pi} - I]$ and $-I$ is as follows:

$$(1 + \mathrm{k})^{-1}[(1 - t^a)^{-1}\underline{\pi} - I] \geq -I \text{ if and only if } k \geq -[(1 - t^a)\mathrm{I}]^{-1}\underline{\pi}.$$

The downside risk is

$$\Pr\{\pi^A \leq \underline{\pi}\} = \Pr\{-I \leq y \leq (1 + k)^{-1}[(1 - t^a)^{-1}\underline{\pi} - I]\}$$

when $k \geq -[(1 - t^a)I]^{-1}\underline{\pi} \geq -1$; otherwise, $\Pr[\pi^A \leq \underline{\pi}] = 0$.

Finally, in Case (3), the condition $\pi^A \leq \underline{\pi}$ implies $(1 + k - t^a)y \leq \underline{\pi} - (1 - t^a)I$. Thus, the downside risk is

$$\Pr\{\pi^A \leq \underline{\pi}\} = \Pr\{y \geq (1 + k - t^a)^{-1}[\underline{\pi} - (1 - t^a)I]\}$$

when $1 + k - t^a \leq 0$ and zero otherwise.

There are two critical values for k when calculating the downside risk: $k_1 = t^a - 1$ or $k_2 = -[(1 - t^a)I]^{-1}\underline{\pi}$. The condition in Proposition 2 implies $-1 \leq k_1 < k_2 < 0$. By examining the downside risk, it can be found that $\Pr\{\pi^A \leq \underline{\pi}\}$ is an increasing function of k when $k_1 \leq k < 0$. Suppose that $-1 \leq k \leq k_1$. The downside risk is

$$P^A = Pr\{\pi^A \leq \underline{\pi}\} = \Pr[y \leq (1 + k - t^a k)^{-1}(\underline{\pi} - I)]$$
$$+ \Pr\{y \geq (1 + k - t^a)^{-1}[\underline{\pi} - (1 - t^a)I]\}.$$

Upon taking the partial derivative with respect to k, the condition $\partial P^A/\partial k = 0$ is reduced to equation (4). While $\partial P^A/\partial k$ may be positive or negative at $k = -1$ depending upon the density function of y, it is always positive at $k = k_1$. In other words, assuming $k \geq -1$, the optimal futures position k^* lies in the interval $[-1, k_1]$. If there is an interior solution, then equation (4) must be satisfied. Otherwise, the solution is at the boundary point, $k^* = -1$.

The remaining task is to show that the optimal futures position must exceed -1. Following a similar analysis to the above, one can find that the downside risk is a decreasing function of k when $k \leq -1$. This result completes the proof.

Proof of Proposition 3:

To ensure an interior solution, a sufficient condition is that $\partial P^A/\partial k < 0$ at $k = -1$. Thus, one requires

$$g\left(\frac{\underline{\pi} - I}{t^a}\right)\frac{(1 - t^a)(I - \underline{\pi})}{(t^a)^2} - g\left(\frac{\underline{\pi} - (1 - t^a)I}{-t^a}\right)\frac{(1 - t^a)I - \underline{\pi}}{(t^a)^2} < 0.$$

Note that $(1 - t^a)(I - \underline{\pi}) > (1 - t^a)I - \underline{\pi}$. The above inequality implies that the probability density at $(\underline{\pi} - I)/t^a$ must be smaller than at $[\underline{\pi} - (1 - t^a)I]/(-t^a)$. If $g(\cdot)$ is symmetric, then the following condition is required:

$$g\left(\frac{\underline{\pi} - I}{t^a}\right) = g\left(\frac{\underline{\pi} - I}{-t^a}\right) \leq g\left(\frac{\underline{\pi} - (1 - t^a)I}{-t^a}\right).$$

That is, the density function must decline quickly when moving away from the origin. Because $\partial P^A/\partial k < 0$ at $k = -1$ and $\partial P^A/\partial k > 0$ at $k = k_1$, there must be an interior solution at which $\partial P^A/\partial k = 0$.

APPENDIX II

Under this modeling strategy the firm chooses k to minimize $\mathrm{Var}(\pi^A)$. Let Q_1 denote $E[\max(0, I + y)]$ and let Q_2 denote $E[\max(0, ky)]$. When y is normally distributed with zero mean and variance σ^2, it can be shown that

$$Q_1 = I\Phi(I/\sigma) + (\sigma/\sqrt{2\pi})\exp(-I^2/2\sigma^2); \tag{A1}$$

$$Q_2 = -k\sigma/\sqrt{2\pi}. \tag{A2}$$

By expansion, $\mathrm{Var}(\pi^A)$ can be written as sums of several variance and covariance terms. Given the normality assumption, expressions for each component in the expansion can be derived:

$$Var[(1 + k)y] = (1 + k)^2\sigma^2; \tag{A3}$$

$$Var[\max(0, I + y)] = (I^2 + \sigma^2)\Phi(I/\sigma) + (I\sigma/\sqrt{2\pi})\exp(-I^2/2\sigma^2) - (Q_1)^2; \tag{A4}$$

$$Var[\max(0,\ ky)] = (k^2\sigma^2/2) - (Q_2)^2; \tag{A5}$$

$$Cov[y,\ \max(0,\ I + y)] = \sigma^2\Phi(I/\sigma); \tag{A6}$$

$$Cov[y,\ \max(0,\ ky)] = k\sigma^2/2; \tag{A7}$$

$$Cov[\max(0,\ I + y),\ \max(0,\ ky)] \tag{A8}$$

$$= (k\sigma I/\sqrt{2\pi})[1 - 2\exp(-I^2/2\sigma^2)] + k\sigma^2[0.5 - \Phi(-I/\sigma)] - Q_1Q_2.$$

Using these results, the minimum variance hedge ratio can be obtained by setting the partial derivative $\partial\mathrm{Var}(\pi^A)/\partial k = 0$. Let h_n denote minimum variance hedge ratio. Then $h_n = -(R_1 + R_2)/R_3$, where

$$R_1 = (t^a)^2(\sigma I/\sqrt{2\pi})[1 - 2\exp(-I^2/2\sigma^2)] + \sigma^2[0.5 - \Phi(-I/\sigma)]; \tag{A9}$$

$$R_2 = \sigma^2 + t^a\sigma^2[\Phi(I/\sigma) - 0.5]; \tag{A10}$$

$$R_3 = \sigma^2[1 + (t^a)^2(0.5 - (1/2\pi))]. \tag{A11}$$

Suppose that y is a standard double exponential random variable, i.e., $g(y) = (1/2)\exp(-|y|)$. Then $\mathrm{Var}(y) = 2$. Also,

$$E[\max(0,\ I + y)] = I + 0.5e^{-I}; \tag{A12}$$

$$E[\max(0,\ ky)] = -k/2; \tag{A13}$$

$$Var[\max(0,\ I + y)] = 2 - Ie^{-I} - e^{-I} - 0.25e^{-2I}; \tag{A14}$$

$$Var[\max(0,\ ky)] = k^2/4; \tag{A15}$$

$$Cov[y,\ \max(0,\ I + y)] = 2 - e^{-I} - 0.5Ie^{-I}; \tag{A16}$$

$$Cov[y,\ \max(0,\ ky)] = k/4; \tag{A17}$$

$$Cov[\max(0,\ I + y),\ \max(0,\ ky)] = k - 0.5kIe^{-I} - 0.75ke^{-I}. \tag{A18}$$

Again, an explicit expression for $\mathrm{Var}(\pi^A)$ is available. Let h_d denote the minimum variance hedge ratio. Then $h_d = -(R_4 + R_5)/R_6$, where

$$R_4 = \sqrt{2} + t^a\,(4.5 - e^{-I} - 0.5Ie^{-I}); \tag{A19}$$

$$R_5 = 2(t^a)^2(1 - 0.5Ie^{-I} - 0.75e^{-I}); \tag{A20}$$

$$R_6 = \sqrt{2} + t^a + 0.5(t^a)^2. \tag{A21}$$

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