Modes of Fluctuation in Metal Futures Prices

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This article examines the stochastic structure of metal futures prices. First, this article presents a stationary multi-factor model of fluctuations in the futures price curve. Next, the model is extended to allow for time variation in the factors or "modes" of fluctuation. The model is estimated using futures price data for three very different metals: copper, which is an industrial metal; gold, which is a precious metal; and silver, which is in transition from a precious metal to an industrial metal. The estimation results show that the shapes and importance of the various modes of fluctuation for gold and silver are much different from those for copper. Gold and silver futures price curves can be adequately modeled as a time-varying one-factor model. Copper, however, has a more complicated structure and should be modeled as a time-varying two- or three-factor model. © 2000 John Wiley & Sons, Inc. Jrl Fut Mark 20:219–241, 2000.

INTRODUCTION

The stochastic behavior of spot and futures prices for a commodity is of paramount importance to analysts interested in hedging, risk assessment, or modeling contingent claims on the underlying commodity. Analysts make explicit assumptions about the stochastic behavior of the spot and futures prices when they model the underlying assets. Moreover, the accuracy of these models depends on the accuracy of the assumptions. This article examines the stochastic behavior of metal futures prices.

Many of the earlier studies on commodity prices assumed that interest rates and convenience yields were constant and that all uncertainty

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arose from the spot price of the commodity. See for example, Schwartz (1982) and Brennan and Schwartz (1985). Examination of the data utilized in these earlier studies, however, clearly revealed that the spreads between spot and futures prices varied substantially over time, thereby indicating that a single factor was inadequate to properly model commodity prices. Recognition of the importance of the variability of the spreads between spot and futures prices led to the development of several multi-factor models of commodity prices. For example, Gibson and Schwartz (1990) introduced a two-factor model where the spot price of the commodity and the convenience yield followed a joint stochastic process. Also, Schwartz (1997) presented a three-factor model where the logarithm of the spot price and the convenience yield followed mean reverting processes and interest rates were stochastic.

In addition, early studies on commodity prices typically assumed that the volatility of prices was stationary. Recent studies that have examined the volatility of commodity prices, however, have found substantial evidence indicating that volatility varied over time. For example, Bracker and Smith (1999) examined the volatility of copper futures prices and concluded that this volatility was more properly modeled as a Generalized Autoregressive Conditional Heteroscedastic (GARCH) process of time-varying volatility.

Employing a different approach, Cortazar and Schwartz (1994) utilized the information in copper futures prices to estimate a stationary multi-factor model of movements in the term structure of copper futures prices. First, using futures contracts maturing up to 21 months in the future, they computed daily returns on copper futures for seven quarterly maturity sectors by averaging the returns of the contracts that fell in each maturity sector. Next, they estimated factor loadings for the seven quarterly maturities from the return data using principal components analysis. Then, using the factor loadings for the seven quarterly maturities, they extrapolated factor loadings for 20 additional quarterly maturities. These factor loadings were then scaled so that the resulting factor volatilities matched the historical volatilities for the various maturities. Finally, the factor volatilities were used to price long-dated copper interest-indexed notes. This approach is similar to the arbitrage-free term structure of interest rate movement models developed by Ho and Lee (1986) and Heath, Jarrow, and Morton (1990, 1992).

There were several potential data problems that affected the Cortazar and Schwartz analysis. These problems were due to data limitations associated with pricing long-dated claims, as well as problems with the copper futures data in the sample interval. First, most copper futures contracts with more than roughly nine months to maturity do not normally trade every day (or at the end of the day) and the futures exchange sets settlement prices based on estimates of where the exchange predicts that the contract should trade. Consequently, the prices of copper futures contracts with maturities greater than nine months are frequently only an extrapolation of the prices of shorter maturity contracts computed from actual trades.

Second, although computing quarterly returns by averaging the data eliminates problems resulting from missing observations, this averaging results in the loss of information about the shape of the futures curve. Third, the need for data from January 1978 to December 1990 forced Cortazar and Schwartz to use data for two different Commodity Exchange (COMEX) copper futures contracts, neither of which tracked the physical copper market very well in the data interval.[1]

The current article extends the Cortazar and Schwartz analysis of fluctuations in futures prices in several important ways and avoids the data problems discussed above. First, a stationary multi-factor model of fluctuations in futures prices is presented where the price fluctuations are modeled as fluctuations in a curve instead of fluctuations in a set of discrete points. The model assumes that any change in the futures price curve can be described by a set of independent shift functions. For example, assume that changes in the futures price curve can be explained by two shift functions where the first shift function represents changes in the level of the curve and the second shift function represents changes in the slope of the curve. In this example, any change in the futures price curve would be represented as a linear combination of the two shift functions, i.e., a combination of a change in the level and a change in the slope of the curve. These shift functions are also referred to in this article as the "modes of fluctuation" because they describe the way that the curve shifts.

Second, the model is extended to allow for time variation in the modes of fluctuation. Third, the model is estimated using futures price data for three very different metals: copper, which is an industrial metal; gold, which is a precious metal; and silver, which is in transition from a precious metal to an industrial metal.[2]

[1]See Manser (1986) for a discussion of problems with the COMEX copper futures contract in the 1980s.

[2]Today, silver is more of an industrial metal and less of an investment commodity or a store of value than it was in the 1970s and 1980s. While there is still a substantial stock of silver held for investment purposes or as a store of value, there has been an apparent net disinvestment of silver holdings for these purposes over the last few years.

This article reaches the following conclusions: First, examination of the futures data suggests that representing copper, gold, and silver futures prices as curves appears reasonable inasmuch as futures contracts with contiguous maturities have similar standard deviations and are highly correlated.

Second, estimation of the stationary modes of fluctuation for changes in the logarithms of the futures price curves shows that the modes of fluctuation for copper are quite different from those for gold and silver. Copper is well represented by two or three modes of fluctuation. The first mode for copper is associated with changes in both the level and slope of the futures price curve, i.e., a positive realization of the first mode will increase the level of the curve and decrease the slope of the curve. The second mode is associated with changes in the slope of the curve and the third mode is associated with changes in the curvature of the curve. Gold and silver are well represented by one mode of fluctuation. The first mode for gold and silver is associated with an equal percentage change for all maturities of the futures price curve. Because the futures price curves for gold and silver are always positively sloped in the sample interval, a positive realization of the first mode will increase both the level and the slope of the curve.

Third, estimates of the time-varying modes of fluctuation model clearly imply that the modes of fluctuation for the three metals are not stationary over the sample interval. For copper, there is substantial variation in the level and slope of the first mode indicating that the way that the curve fluctuates varies over the sample interval. For gold and silver, there is substantial variation in the level of the first mode of fluctuation. The slope of the first mode, however, remains close to zero throughout the sample interval indicating that the relative percentage impact of the first mode remains equal for all maturities of the futures price curve. The second and third modes for all three metals remain relatively unchanged over the sample interval.

These results suggest that fluctuations in copper, gold, and silver futures prices can be well represented using a small number of factors. These factors can be used to hedge price fluctuations, assess the risk of a position, or model contingent claims on the underlying commodity.

The first section of this article, A Model of Fluctuations in the Futures Price Curve, presents a model of fluctuations in the futures price curve. The second section, The Data, contains a description of the data. Finally, the third section, Estimates of the Modes of Flucuation, presents the results of estimating the model.

A MODEL OF FLUCTUATIONS IN THE FUTURES PRICE CURVE

This section describes the methodology used for separating changes in the futures price curve into a set of independent shift functions using principal components analysis. As discussed earlier in the article, these shift functions can be referred to as the modes of fluctuation of the futures price curve. The first part of this section presents a model with stationary modes of fluctuation. The model is then extended to allow the modes to vary over time. This analysis closely follows the methodology developed by Garbade (1996) to analyze the U.S. Treasury yield curve.

The Futures Price Curve

The futures price curve describes how the price of a futures contract, F, varies with the time remaining to maturity of the contract, y. The model assumes that at any point in time the futures price curve can be expressed in the form

$$F(y,w) = F_0 + \sum_{j=1}^{J} w_j \cdot f_j(y) \tag{1}$$

where F_0 and the $f_j(y)s$ are time-invariant functions of maturity, the w_js are scalar coefficients, and the vector of w_j coefficients is denoted w. The futures price curve in eq. (1) is constructed as a baseline function F_0 plus a linear combination of the f_j shift functions, where the weight on shift function j is the coefficient w_j. The shape and level of futures curves vary over time as a result of variations in the w_j coefficients. The model assumes that each coefficient evolves as a Gaussian random walk with zero drift and a variance of unity per year, and the random walks are statistically independent of each other.

The change in the futures price curve from time t_0 to time t_1 at a maturity of y years, $F(y,w(t_1)) - F(y,w(t_0))$, can be written as

$$\begin{aligned} F(y,w(t_1)) - F(y,w(t_0)) &= \left[F_0 + \sum_{j=1}^{J} w_j(t_1) \cdot f_j(y)\right] \\ &\quad - \left[F_0 + \sum_{j=1}^{J} w_j(t_0) \cdot f_j(y)\right] \\ &= \sum_{j=1}^{J} (w_j(t_1) - w_j(t_0)) \cdot f_j(y). \end{aligned} \tag{2}$$

Equation (2) shows that the change in the futures curve from time t_0 to time t_1 is a linear combination of the f_j shift functions, where the coefficients in the linear combination are the changes in the w_j coefficients from time t_0 to time t_1.

From the assumptions regarding the distribution of the w_j coefficients, the changes in the coefficients from time t_0 to time t_1 are normally distributed as

$$E[w_j(t_1) - w_j(t_0)] = 0 \qquad j > 1, \ldots, J \tag{3}$$

$$\text{var}[w_j(t_1) - w_j(t_0)] = t_1 - t_0 \qquad j = 1, \ldots, J \tag{4}$$

$$\text{cov}[w_j(t_1) - w_j(t_0), w_k(t_1) - w_k(t_0)] = 0, \quad j = 1, \ldots, J \quad k = 1, \ldots, J \; k \neq j \tag{5}$$

Stationary Modes of Fluctuation

This section develops the modes of fluctuation under the hypothesis that the modes of fluctuation are stationary. Each of the f_j shift functions is assumed to be a polynomial function of maturity with I coefficients, which can be written as

$$f_j(y) = \sum_{i=1}^{I} b_{ij} \cdot y^{i-1} \tag{6}$$

where the b_{ij}s are coefficients of the shift functions. Substituting the polynomial in eq. (6) in the shift functions in eq. (2) yields

$$\begin{aligned} F(y,w(t_1)) - F(y,w(t_0)) &= \sum_{j=1}^{J} \left[(w_j(t_1) - w_j(t_0)) \cdot \sum_{i=1}^{I} b_{ij} \cdot y^{i-1} \right] \\ &= \sum_{i=1}^{I} \left[\sum_{j=1}^{J} b_{ij}(w_j(t_1) - w_j(t_0)) \right] \cdot y^{i-1} \end{aligned} \tag{7}$$

Equation (7) shows that the shift in the futures curve from time t_0 to time t_1 is a polynomial with I coefficients.

Given that there are data on H futures prices at monthly maturities expressed in fractions of a year (denoted $y_1, y_2, \ldots, y_H$) for each date in a sequence of $K + 1$ daily dates (denoted $t_1, t_2, \ldots, t_K$), eq. (7) can be written as

$$\Delta F(y_h,t_k) = \sum_{i=1}^{I} \left[\sum_{j=1}^{J} b_{ij} \cdot \Delta w_j(t_k) \right] \cdot y_h^{i-1} \tag{8}$$

where $\Delta F(y_h,t_k)$ is the observed change in the futures price curve from time t_{k-1} to time t_k at a term of y_h years to maturity and where $\Delta\, w_j(t_k)$

is the unobserved random change in the weighting coefficient on shift function j from time t_{k-1} to time t_k.

Defining $\Delta a_i(t_k)$ as

$$\Delta a_i(t_k) = \sum_{j=1}^{J} b_{ij} \cdot \Delta w_j(t_k) \quad i = 1, \ldots, H \quad k = 1, \ldots, K \tag{9}$$

Equation (8) may be rewritten as

$$\Delta F(y_h, t_k) = \sum_{i=1}^{I} \Delta a_i(t_k) \cdot y_h^{i-1} \tag{10}$$

Equation (10) expresses the change in futures prices from time t_{k-1} to time t_k at a term of y_h years as a polynomial function of y_h where the $\Delta a_i(t_k)$ terms are the coefficients of the polynomial, which vary over time. Garbade (1996) calls the $\Delta a_i(t_k)$ composite coefficients "reduced shift" coefficients. The next part of this section describes how the b_{ij} shift function coefficients are estimated from the reduced shift coefficients.

Switching to matrix notation, $\Delta a(t_k)$ is the I by 1 vector of Δa_i coefficients at time t_k, B is the I by J matrix of b_{ij} coefficients, and $\Delta w(t_k)$ is the J by 1 vector of Δw_j coefficients at time t_k. From the assumption that each of the w_j coefficients follows a Gaussian random walk with zero drift and a variance of unity per year, and that the random walks are statistically independent of each other, it follows that, for daily data, $\Delta w(t_k)$ has an expected value of zero and a covariance matrix of $250^{-1} \cdot \Omega_J$ where Ω_J is the J by J identity matrix. It also follows that the covariance matrix of $\Delta a(t_k)$, denoted $\Omega_{\Delta a}$, is

$$\Omega_{\Delta a} = 250^{-1} \cdot B \cdot B' \tag{11}$$

where $\Omega_{\Delta a}$ is stationary as long as the B matrix of coefficients of the shift function polynomials is stationary.

The reduced shift coefficients in eq. (10) can be estimated directly from the futures price curve data and the covariance matrix $\Omega_{\Delta a}$ can be estimated with the time series estimates of the reduced shift coefficients as

$$\hat{\Omega}_{\Delta a} = K^{-1} \sum_{k=1}^{K} \Delta \hat{a}(t_k) \cdot \Delta \hat{a}(t_k)' \tag{12}$$

The estimate $\hat{\Omega}_{\Delta a}$ of the covariance matrix of the reduced shift coefficients can be used to estimate the B matrix of the coefficients of the shift function polynomials.

Define

$$\Delta F(t_k) = \begin{bmatrix} \Delta F(y_1,t_k) \\ \cdot \\ \cdot \\ \cdot \\ \Delta F(y_H,t_k) \end{bmatrix} \quad k = 1, \ldots, K \tag{13}$$

$$Y = \begin{bmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{I-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & y_H & y_H^2 & \cdots & y_H^{I-1} \end{bmatrix}. \tag{14}$$

Equation (10) can be rewritten as

$$\Delta F(t_k) = Y \cdot \Delta a(t_k) \quad k = 1, \ldots, K \tag{15}$$

The covariance matrix of the futures price changes denoted $\Omega_{\Delta F}$ is

$$\Omega_{\Delta F} = Y \cdot \Omega_{\Delta a} \cdot Y'. \tag{16}$$

Estimates of $\Omega_{\Delta F}$ can be decomposed as

$$\hat{\Omega}_{\Delta F} = \tilde{V}_{\Delta F} \cdot \tilde{D}_{\Delta F} \cdot \tilde{V}'_{\Delta F} \tag{17}$$

where $\tilde{D}_{\Delta F}$ is an I by I diagonal matrix of the I nonzero eigenvalues of $\tilde{\Omega}_{\Delta F}$ and $\tilde{V}_{\Delta F}$ is an H by I matrix of the associated eigenvectors. The estimator of the B matrix, the coefficients of the shift function polynomials, for $J = I$ modes of fluctuation is

$$\hat{B} = (250)^{1/2} \cdot (Y' \cdot Y)^{-1} \cdot Y' \cdot \tilde{V}_{\Delta F} \cdot \tilde{D}_{\Delta F}^{1/2} \tag{18}$$

The $\hat{B}$ matrix coefficients are used to calculate the shift functions in eq. (6), which are the weighting coefficients of the principal components of the covariance matrix $\Omega_{\Delta F}$.

Time-Varying Modes of Fluctuation

Equation (11) shows that the covariance matrix of reduced shift coefficients, $\Omega_{\Delta a}$, is stationary if the B matrix of the shift function coefficients is stationary. Time variation of the modes of fluctuation implies time variation in $\Omega_{\Delta a}$. To estimate time-varying modes of fluctuation, $\Omega_{\Delta a}$ is replaced with the time-varying estimator

$$\hat{\Omega}_{\Delta a}(t_i) = \frac{\sum_{k=1}^{K} \Delta\hat{a}(t_k) \cdot \Delta\hat{a}(t_k)' \cdot \omega^{|i-k|}}{\sum_{k=1}^{K} \omega^{|i-k|}} \quad i = 1, \ldots, K \qquad (19)$$

where $\hat{\Omega}_{\Delta a}(t_i)$ is an estimator of the covariance matrix at time t_i of the reduced shift coefficients. The estimator is a two-sided geometrically weighted average, where the weighting parameter ω takes on a value between 0 and 1. Values of the weighting coefficient closer to unity imply that the covariance matrix is less volatile over time. In the case where ω takes the value of 1, the covariance matrix is stationary. Baron (1989) proposed the time-varying estimator of $\hat{\Omega}_{\Delta a}$ in eq. (19) and a derivation of the estimator is contained in Garbade (1996).

Once the time series for $\hat{\Omega}_{\Delta a}(t_i)$ has been estimated, a time series of covariance matrices for changes in the futures price curve can be computed as

$$\hat{\Omega}_{\Delta F}(t_i) = Y \cdot \hat{\Omega}_{\Delta a}(t_i) \cdot Y'. \qquad (20)$$

Following eq. (17), each $\hat{\Omega}_{\Delta F}(t_i)$ can be decomposed as

$$\hat{\Omega}_{\Delta F}(t_i) = \tilde{V}_{\Delta F}(t_i) \cdot \tilde{D}_{\Delta F}(t_i) \cdot \tilde{V}_{\Delta F}(t_i)' \qquad (21)$$

and coefficients of the shift function polynomials can be estimated as

$$\hat{B}(t_i) = 250^{1/2} \cdot (Y' \cdot Y)^{-1} \cdot Y' \cdot \tilde{V}_{\Delta F}(t_i) \cdot \tilde{D}_{\Delta F}(t_i)^{1/2} \qquad (22)$$

THE DATA

The data are daily prices for the COMEX futures contracts for copper, gold, and silver from January 3, 1990 to December 31, 1996.

The copper data are settlement prices (approximately 2:00 p.m. New York time) for the COMEX futures contract for Grade 1 electrolytic cathode copper quoted in cents per pound. The Grade 1 electrolytic cathode copper contract started trading on July 28, 1988, but it took approximately one year for that contract to become liquid, especially in the intermediate and longer maturities. Consequently, the data interval is started in 1990. For the data interval used in this study, trading is conducted in (1) every current calendar month; (2) the immediately following eleven calendar months; and (3) every January, March, May, July, September, and December in a 23 month period from the current calendar month. The phrase "trading is conducted in" is used by the exchange to mean that market participants can trade in these contract months if there is interest in doing so. The January, March, May, July, September, and December contracts are referred to as the "normal" cycle month contracts for copper. The normal cycle month contracts are typically more liquid than the non-normal cycle month contracts, especially as the time to maturity of a contract increases.

The gold data are settlement prices (approximately 2:30 p.m. New York time) for the COMEX futures contract for 99.5% pure gold quoted in dollars per troy ounce. Trading is conducted in (1) every current calendar month; (2) the immediately following two calendar months; (3) every February, April, August, and October in a 23 month period from the current calendar month; and (4) every June and December in a 60 month period from the current calendar month.

The silver data are settlement prices (approximately 2:25 p.m. New York time) for the COMEX futures contract for 99.9% pure silver quoted in dollars per troy ounce. Trading is conducted in (1) every current calendar month; (2) the immediately following two calendar months; (3) every January, March, May, and September in a 23 month period from the current calendar month; and (4) every July and December in a 60 month period from the current calendar month.

Note that the exchange does not normally set a settlement price for a contract that has not started trading, even though the contract is officially open for trading. Consequently, many of the long-dated contract months that are open for trading have no open interest or prices because market participants have not begun trading the contract. Also, whenever there is any open interest in a contract, the exchange must provide a daily settlement price for the contract to allow market participants to calculate normal variation margin payments, even when there is no trading in the contract.

A seller of a COMEX metal futures contract has the option to deliver the underlying metal any time during the settlement month and the seller must notify the exchange of his or her intention to deliver two business days before the delivery day. The first day on which the seller of a futures contract can notify the exchange of his or her intention to deliver, which is referred to as the first "Date of Presentation," is the next to last business day of the calendar month preceding the delivery month.

For copper, eight futures contracts are followed at all times, the first four nearby contracts (the contracts with the shortest time to maturity) and the next four normal cycle month contracts. For gold and silver, the first eight nearby futures contracts are followed. Longer maturity futures contracts for these metals do not typically trade on a regular basis. A contract is dropped from the data set two business days before the contract's first Date of Presentation to avoid pricing anomalies associated with delivery against the contract.

Time to maturity of a futures contract is computed as the length of the interval from the date of the observation to the first Date of Presentation for the contract. Accordingly, the maturities of the copper futures contracts in the data sample range from roughly zero to just over one-month for the first contract to 9 to 10 months for the eighth contract. For gold and silver, the time to maturity of the futures contracts range from roughly zero to one-month for the first contract to 12 to 14 months for the eighth contract.

Table I presents the standard deviations and correlations of daily changes in the logarithms of the futures prices for copper, gold, and silver. The table shows that copper futures contracts with similar maturities have similar standard deviations and are highly correlated. Also, standard deviations decrease as time to maturity increases. For example, the first shortest maturity contract (denoted contract 1 in the table) and the second shortest maturity contract (denoted contract 2 in the table) have a correlation of .9878 and standard deviations of .235 and .228, respectively. Futures contracts with dissimilar maturities are less highly correlated and have much different standard deviations. The first and the eighth contracts have a correlation of .8281 and standard deviations of .235 and .173, respectively. For gold, the standard deviations are almost identical for all eight contracts. These contracts are highly correlated and the correlation remains high for futures contracts with dissimilar maturities. The standard deviations for silver decline slightly as maturity increases. The silver contracts, however, are also highly correlated and the correlation remains high for contracts with dissimilar maturities.

The gold and silver data suggest that futures contracts for a metal with different maturities are excellent substitutes for one another. This is probably due to both the constant ample supply of the underlying metals and the low carrying costs relative to the value of the metal. Copper futures contracts with different maturities are poorer substitutes for one another than gold and silver. The lower correlations between copper contracts with different delivery dates are probably primarily due to both the poor substitutability of copper for different delivery dates when there are

TABLE I

Standard Deviations and Correlations of Daily Changes in the Logarithms of Futures Prices[1]

Contract	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
				Copper				
Standard Deviation	.2353	.2281	.2165	.2075	.1944	.1842	.1791	.1732
				Correlation				
1	1.0000	.9878	.9774	.9682	.9444	.9170	.8822	.8281
2	.9878	1.0000	.9905	.9835	.9627	.9372	.9041	.8502
3	.9774	.9905	1.0000	.9922	.9763	.9536	.9230	.8713
4	.9682	.9835	.9922	1.0000	.9854	.9674	.9401	.8915
5	.9444	.9627	.9763	.9854	1.0000	.9859	.9665	.9250
6	.9170	.9372	.9536	.9674	.9859	1.0000	.9879	.9546
7	.8822	.9041	.9230	.9401	.9665	.9879	1.0000	.9762
8	.8281	.8502	.8713	.8915	.9250	.9546	.9762	1.0000
				Gold				
Standard Deviation	.1379	.1378	.1380	.1380	.1380	.1379	.1378	.1377
				Correlation				
1	1.0000	.9996	.9993	.9988	.9981	.9971	.9958	.9945
2	.9996	1.0000	.9997	.9993	.9987	.9978	.9966	.9953
3	.9993	.9997	1.0000	.9997	.9993	.9986	.9975	.9964
4	.9988	.9993	.9997	1.0000	.9998	.9993	.9985	.9975
5	.9981	.9987	.9993	.9998	1.0000	.9998	.9992	.9984
6	.9971	.9978	.9986	.9993	.9998	1.0000	.9996	.9991
7	.9958	.9966	.9975	.9985	.9992	.9996	1.0000	.9996
8	.9945	.9953	.9964	.9975	.9984	.9991	.9996	1.0000
				Silver				
Standard Deviation	.2516	.2514	.2501	.2491	.2484	.2479	.2473	.2464
				Correlation				
1	1.0000	.9983	.9970	.9959	.9952	.9943	.9933	.9922
2	.9983	1.0000	.9990	.9981	.9976	.9970	.9962	.9954
3	.9970	.9990	1.0000	.9995	.9992	.9987	.9981	.9974
4	.9959	.9981	.9995	1.0000	.9999	.9996	.9992	.9987
5	.9952	.9976	.9992	.9999	1.0000	.9998	.9995	.9992
6	.9943	.9970	.9987	.9996	.9998	1.0000	.9998	.9996
7	.9933	.9962	.9981	.9992	.9995	.9998	1.0000	.9999
8	.9922	.9954	.9974	.9987	.9992	.9996	.9999	1.0000

[1]The shortest maturity contract for each metal is denoted contract 1, the next shortest maturity contract is denoted contract 2 and so forth.

low inventories of the underlying metal, as well as the higher carrying costs of copper relative to its value.

Inasmuch as futures contracts with contiguous maturities have similar standard deviations and are highly correlated, representing copper, gold, and silver futures prices as curves appears reasonable.

ESTIMATES OF THE MODES OF FLUCTUATION

The stationary b_{ij} shift function coefficients are estimated using the following steps. First, the futures price curve is estimated for every day in the sample interval using the futures contract data. Second, the daily estimates of the futures price curve are used to estimate the Δa_i reduced shift coefficients for every day in the sample interval as in eq. (15). Third, the covariance matrix of the reduced shift coefficients $\Omega_{\Delta a}$ is estimated using the $\Delta \hat{a}_i$ coefficients as in eq. (12). Fourth, the covariance matrix of the reduced shift coefficients is used to estimate the B matrix of shift function coefficients as in eqs. (16), (17), and (18).

The time-varying b_{ij} shift function coefficients are then estimated by replacing steps three and four with their time-varying counterparts in eqs. (19) to (22).

Estimates of the Futures Price Curve

It is assumed that the baseline function F_0 in eq. (1) can be expressed as a polynomial function of maturity. Given the assumptions that the baseline function and the shift functions are polynomial functions of maturity, it follows that the futures price curve in eq. (1) is also a polynomial function of maturity. This point is discussed in greater detail below. It is assumed that at any point in time the futures price curve can be estimated as

$$F = Y \cdot \beta + e \tag{23}$$

where F is a vector of the H observed futures prices

$$\mathrm{F} = \begin{bmatrix} \mathrm{F}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{F}_H \end{bmatrix}$$

Y is a matrix of the times to maturity of the respective futures contracts of the form

$$Y = \begin{bmatrix} 1 & y_1 & Y_1^2 & \cdots & y_1^{N-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & y_H & y_H^2 & \cdots & y_H^{N-1} \end{bmatrix}$$

β is a matrix of the polynomial coefficients to be estimated

$$\beta = \begin{bmatrix} \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_N \end{bmatrix}$$

and e is an H by 1 vector of error terms.

Daily futures price curves are estimated for copper, gold, and silver using a sixth order polynomial that allows for the consistent estimation of 3 modes of fluctuation for each metal. Garbade (1996) shows the consistency conditions that are necessary to provide an arbitrage-free representation of the futures curve. For the case where the baseline function and the shift functions are both polynomial functions of maturity, consistency between the shape of the curve and its shift functions requires that, to estimate $J = 3$ modes of fluctuation, the futures price curve in eq. (23) must be a sixth order polynomial with 7 coefficients and the shift functions in eq. (7) must be second order polynomials with $I = 3$ coefficients.

Gold and silver consistently have positively sloped futures price curves over the sample interval, which result is widely observed for metals when there are ample supplies of the underlying metal. Gold and silver are good examples of metals with consistently ample supply, which supply is not closely tied to current production.[3] Newly mined gold adds only 1 to 2 percent per year to the total stock of gold. The supply of newly mined gold is also not closely linked to the current price.[4] The copper futures curve took on both a positive and a negative slope over the sample interval. Copper supply, as measured by copper inventories in the London Metal Exchange and the COMEX approved warehouses, has varied over time. The level of copper inventories has been negatively correlated with the level of the futures price curve and positively correlated with the slope of the futures price curve. Typically, a high level of the futures price curve

[3]Occasionally, however, supplies of gold and silver in a suitable form for delivery on a futures contract can be tight, temporarily causing the slope of the short maturity segment of the curve to invert. For example, on November 28, 1995, the price of the November 1995 contract traded above the price of the December 1995 contract. At that time (i.e., in the fall of 1995), there was strong demand for gold for use in industry, and, at the same time, gold in COMEX warehouses reached an 18-year low. By the end of November 1995, spot gold supplies were tight and overnight gold lease rates hit 11.5% per annum. The high gold lending rate and inversion of the curve drew gold into the market and the short-term portion of the curve quickly reverted to a positive slope.

[4]For example, the spot gold prices dropped from roughly \$350 an ounce at the beginning of 1997 to below \$300 an ounce (an 18-year low) by year-end. At the same time, gold production in 1997 rose 5% from 1996 due to projects that had been in the pipeline for two to three years. Despite the continued low gold prices, it was forecast in 1998 that gold production would rise a further 1% per year over the next several years.

is associated with a negative slope and a low level of the futures price curve is associated with a relatively small positive slope.

Periodically, the most distant contract for each metal appears to be mispriced, leading to spurious results for the long end of the futures price curves. The apparent mispricing is probably due to a lack of active trading in the last futures contracts. Consequently, this article limits its analysis to maximum maturities of the futures price curves of 8 months for copper and 10 months for gold and silver. Finally, differences in logarithms of the futures price data are used in the foregoing analysis to facilitate comparability.

Estimates of the Reduced Shift Coefficients

Examination of the data shows low levels of statistically significant serial correlation in the futures price data. To correct for the serial correlation, estimates of the futures price curve are computed for H monthly maturities where H = 8 for copper and H = 10 for gold and silver, using the coefficient estimates of eq. (23). Daily differences in the natural logarithms of the futures prices are computed for the H monthly maturities and first or second order moving-average models are estimated for the monthly maturities using the differenced data.

The reduced shift coefficients in eq. (10) are estimated for each metal for I = 3, a second order polynomial which, as mentioned earlier, is required to consistently estimate 3 modes of fluctuation. Table II reports the standard deviations and correlations for the estimates of the reduced shift coefficients. For copper, the standard deviations of all three coefficients are relatively large and roughly equal in size. Δa_1 is moderately negatively correlated with Δa_2 and weakly positively correlated with Δa_3, while Δa_2 is strongly negatively correlated with changes in Δa_3. These results reflect substantial changes in the shape and level of the copper futures price curve. For gold and silver, the standard deviation of Δa_1 is much larger than the standard deviations of Δa_2 and Δa_3, indicating that there is much less variation in the shape of the futures curve for gold and silver than for copper.

Estimates of the Stationary Modes of Fluctuation

The shift function coefficients are estimated using the estimates of the reduced shift coefficients as described in eqs. (16), (17), and (18). Table III presents estimates of the shift functions for the three metals and Fig-

TABLE II

Standard Deviations and Correlations of the Reduced Shift Coefficients[1]

$$\Delta F(y,t_k) = \sum_{i=1}^{I} \Delta a_i(t_k) \cdot y^{i-1}$$

	Δa_1	Δa_2	Δa_3
	Copper		
Standard Deviation	.242	.313	.300
Correlation with			
Δa_1	1.000	−.564	.247
Δa_2	−.564	1.000	−.857
Δa_3	.247	−.857	1.000
	Gold		
Standard Deviation	.116	.019	.021
Correlation with			
Δa_1	1.000	.059	−.150
Δa_2	.059	1.000	−.821
Δa_3	−.150	−.821	1.000
	Silver		
Standard Deviation	.227	.030	.034
Correlation with			
Δa_1	1.000	−.054	−.112
Δa_2	−.054	1.000	−.874
Δa_3	−.112	−.874	1.000

[1]Equations (10) and (15) show that the change in the futures price curve from time t_{k-1} to time t_k, $\Delta F(y,t_k)$, can be expressed as a polynomial function of time to maturity y where the Δa_i coefficients are referred to as the "reduced shift" coefficients.

ure 1 shows plots of the shift functions computed using the data in Table III.

Equations (6), (7), and (8) show that $\Delta F(y_h,t_k)$, the observed change in the futures price curve from time t_{k-1} to time t_k at a term of y_h years to maturity, can be rewritten as a weighted sum of J statistically independent random variables

$$\begin{aligned} \Delta F(y_h,t_k) &= \sum_{i=1}^{I} \left[\sum_{j=1}^{J} b_{ij} \cdot \Delta w_j(t_k) \right] \cdot y_h^{i-1} \\ &= \sum_{j=1}^{J} \Delta w_j(t_k) \cdot \sum_{i=1}^{I} b_{ij} \cdot y_h^{i-1} \\ &= \sum_{j=1}^{J} \Delta w_j(t_k) \cdot f_j(y_h) \qquad (24) \end{aligned}$$

TABLE III

Stationary Estimates of the First Three Shift Functions[1]

$$f_j(y) = \sum_{i=1}^{I} b_{ij} \cdot y^{i-1}$$

Copper

$f_1(y) = .234 - .111 \cdot y + .025 \cdot y^2$
$f_2(y) = .060 - .213 \cdot y + .095 \cdot y^2$
$f_3(y) = .027 - .210 \cdot y + .287 \cdot y^2$

Gold

$f_1(y) = .116 + .002 \cdot y - .003 \cdot y^2$
$f_2(y) = .005 - .008 \cdot y - .005 \cdot y^2$
$f_3(y) = .003 - .019 \cdot y + .021 \cdot y^3$

Silver

$f_1(y) = .223 - .002 \cdot y - .005 \cdot y^2$
$f_2(y) = .005 - .003 \cdot y - .011 \cdot y^2$
$f_3(y) = .005 - .030 \cdot y + .032 \cdot y^2$

[1]The shift functions are the principal component weights for the changes in the natural logarithm of the futures price curve. The coefficients of the shift functions are estimated using equation (18).

where b_{ij} is coefficient i of shift function j, $\Delta w_j(t_k)$, is the unobserved change in the random weighting coefficient on shift function j from time t_{k-1} to time t_k, and $f_j(y_h)$ is shift function j evaluated at maturity y_h. For example, using the estimates of the copper shift functions in Table III, the change in the logarithm of the price of a 3 month, $y_h = .25$ year, copper futures can be written as the sum of the three shift functions multiplied by their respective weighting coefficients

$$\begin{aligned} \Delta F(.25, t_k) &= (.234 - .111 \cdot .25 + .025 \cdot .25^2) \cdot \Delta w_1 \\ &+ (.060 - .213 \cdot .25 + .095 \cdot .25^2) \cdot \Delta w_2 \\ &+ (.027 - .210 \cdot .25 + .287 \cdot .25^2) \cdot \Delta w_3 \\ &= (.208) \cdot \Delta w_1 + (.013) \cdot \Delta w_2 - (.008) \cdot \Delta w_3 \end{aligned}$$

This means that, for example, a realization of the first component of $\Delta w_1 = +.1$ will lead to an increase in the price of the futures curve of approximately 2.1% ($.0208 = (.208) \cdot (.1)$) at a maturity of three months. Again at a maturity of three months, a realization of the second component of $\Delta w_2 = -.5$ will lead to a decrease in the price of the futures curve of approximately .7% ($-.0065 = (.013) \cdot (-.5)$) and an increase in the third component of $\Delta w_3 = +.5$ will lead to a decrease in the futures price curve of approximately .4% ($-.0040 = (-.008) \cdot (.5)$).

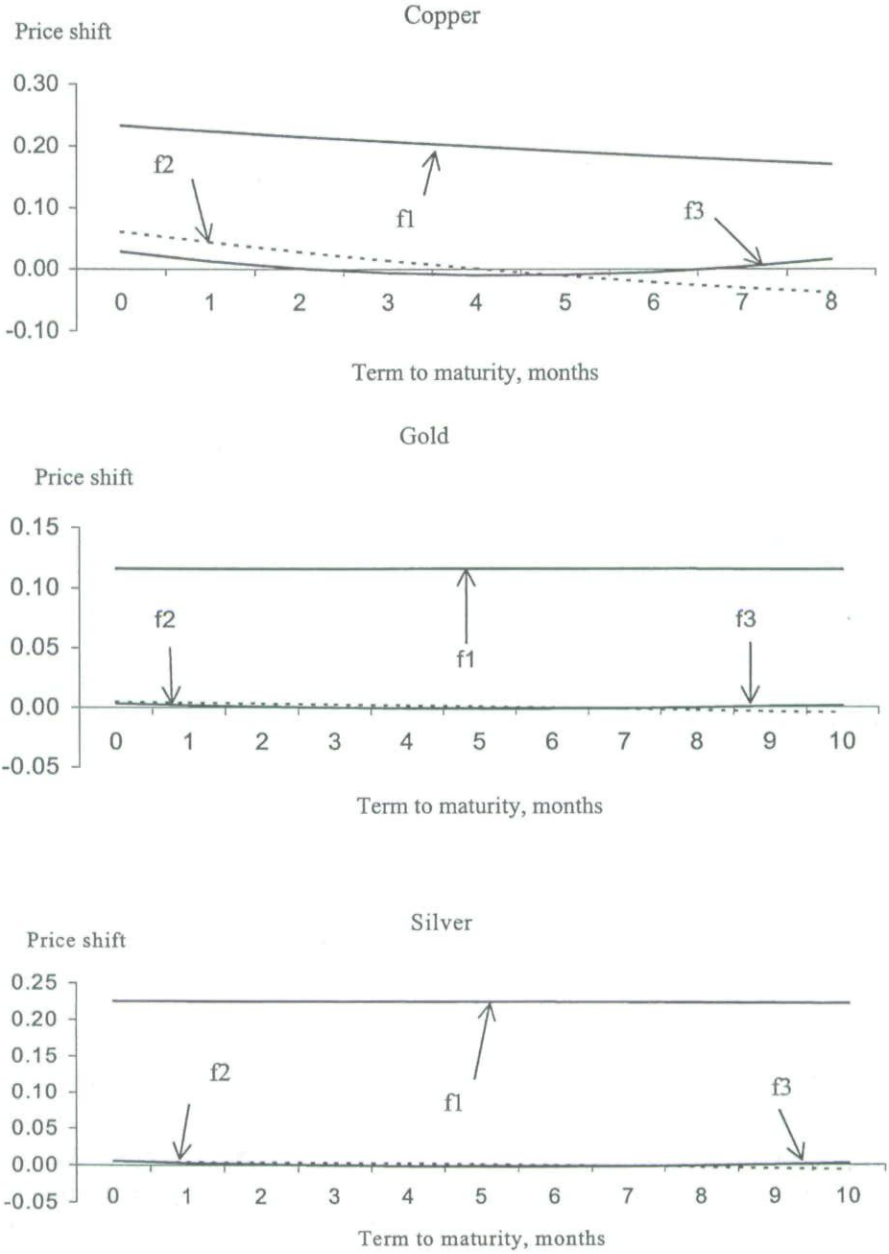


FIGURE 1
Stationary estimates of the first three shift functions. (Change in the natural logarithm of futures prices.)

Figure 1 shows how the three shift functions (or modes of fluctuation) for each metal affect the futures price curves. Note that the shift functions are constructed using principal components analysis so that the first shift function explains the highest proportion of the variance of changes in the futures price curve, the seconds shift function explains the next highest proportion of the variance of changes in the futures curve, and so on. For copper, the first mode is positive for all maturities, with larger values for shorter maturities than for longer maturities. Thus, a positive realization of the first mode for copper leads to both an increase in the level of all prices and a decrease in the slope of the curve, i.e., the prices of short maturities increase more than longer maturities. A negative realization of the first mode leads to a decrease in the level of all prices and an increase in the slope of the curve.

A positive realization of the second mode for copper leads to an increase in short maturity prices and a decrease in long maturity prices, and leaves the intermediate maturity prices relatively unchanged. Thus, a positive realization of the second mode leads to a decrease in the slope of the curve, and a negative realization of the second mode leads to an increase in the slope of the curve.

The third mode is associated with changes in the curvature of the curve. A positive realization of the third mode leads to an increase in short and long maturity prices and leaves the intermediate maturity prices relatively unchanged, and a negative realization of the third mode leads to a decline in short and long maturity prices while leaving the intermediate maturities unchanged.

The modes of fluctuation for gold and silver look very similar to each other. For gold and silver, the first mode is positive for all maturities and the slope of the first mode is close to zero, i.e., the value of the first mode is positive and almost identical for all maturities. Thus, a positive realization of the first mode leads to an approximately equal percentage increase in the level of all prices. Inasmuch as the slopes of the gold and silver futures price curves are always positive in the data interval, a positive realization of the first mode leads to an increase in the slope of the curve as well as an increase in the level of the curve. Likewise, a negative realization of the first mode will lead to an equal percentage decrease in all prices.

The interpretation of the second and third modes of fluctuation for gold and silver are the same as for copper. The second mode is associated with changes in the slope of the curve and the third mode is associated with changes in the curvature of the curve. The relative magnitudes of the effects of the second and third modes, however, are much smaller for gold and silver than for copper.

TABLE IV

Cumulative Percentage of Total Variance Explained by First Three Principal Components

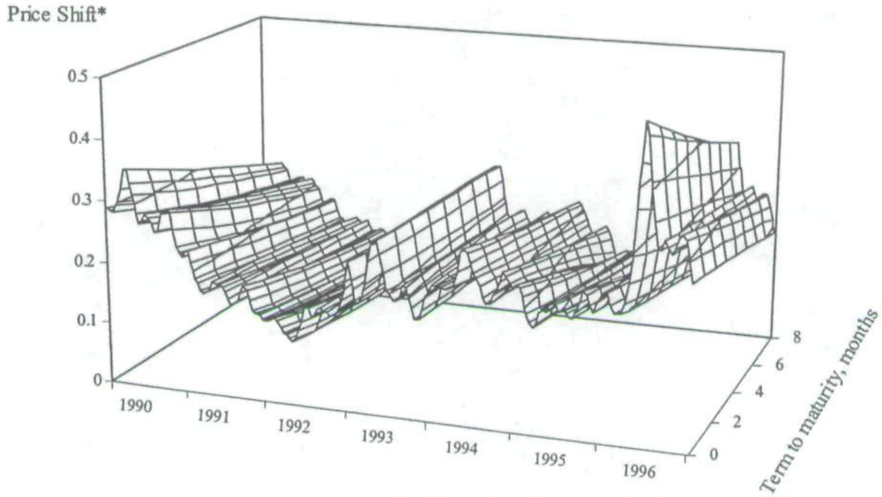
	Principal Component		
Component	1	1 and 2	1, 2, and 3
Copper	.9794	.9979	1.0000
Gold	.9993	.9999	1.0000
Silver	.9997	1.0000	1.0000

Table IV presents the cumulative percentage of the total variance explained by the first three principal components. For copper, the first component explains approximately 98% of the total variance in the copper futures price curve. The second component explains an additional 1.8% of the variance and the third component explains roughly an additional .2% of the variance. Thus, components of an order higher than 3 are relatively unimportant. For gold and silver, the first principal component explains more than 99.9% of the variance in the futures price curve. Thus, components of an order higher than one for gold and silver are relatively unimportant.

Estimates of the Time-Varying Modes of Fluctuation

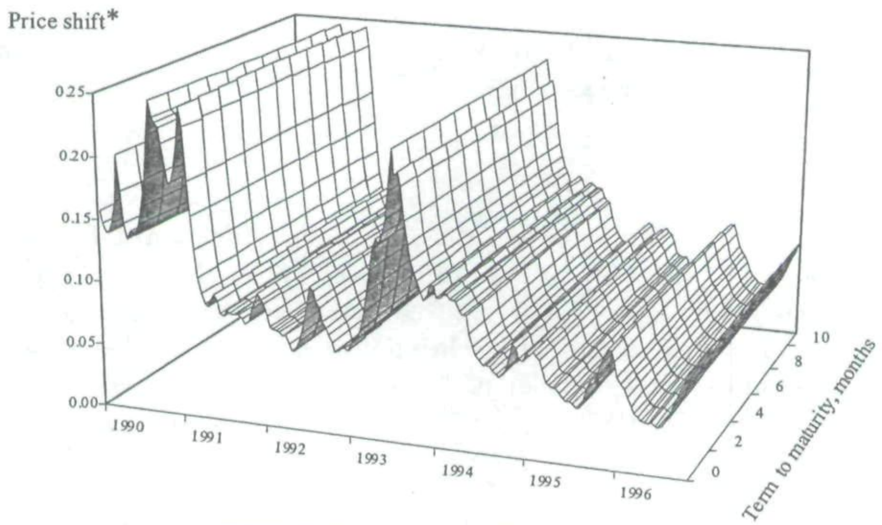
Estimation of $\Omega_{\Delta a}$ in eq. (19) produces a value of $\omega = .94$ for copper and gold and a value of $\omega = .98$ for silver, which implies that the modes of fluctuation for the three metals are not stationary over the sample interval. Figure 2 presents the time-varying first mode of fluctuation for copper. The figure clearly shows that there is substantial variation in the slope and level of the first mode over the sample interval. The first mode is steeply negatively sloped (shorter maturities have higher values than longer maturities) and its level is high in 1990 and in 1996 and is flat and relatively low in 1992. The first mode is also flat and relatively high at times, e.g., 1993, and steeply sloped and low at other times, e.g., 1994. However, the first mode is always positive for all maturities, with the values of shorter maturities larger than longer maturities. Thus, a positive realization of the first mode will increase the level of the whole curve. However, the impact of the first mode on the slope of the futures curve varies.

Figures 3 and 4 show the time-varying first modes of fluctuation for gold and silver, respectively. The first modes of fluctuation for gold and



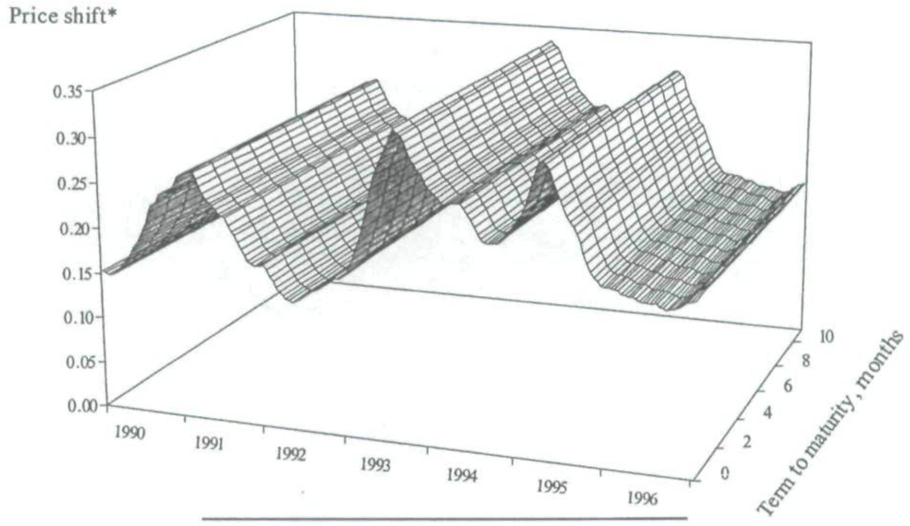
*Change in the natural logarithm of futures prices.

FIGURE 2
Time varying first mode fluctuation for copper.



*Change in the natural logarithm of futures prices.

FIGURE 3
Time varying first mode fluctuation for gold.



*Change in the natural logarithm of futures prices.

FIGURE 4
Time varying first mode fluctuation for silver.

silver are very similar to each other. There are substantial changes in the level of the first mode, but the slope of the first mode remains close to zero over the sample interval, indicating that the relative percentage impact of the first mode on all maturities of the futures price curve remains equal over the sample interval.

The second and third modes for all three metals remain relatively unchanged over the sample interval.

CONCLUSION

The main conclusions of the article are: First, estimation of the stationary modes of fluctuation for the futures price curves shows that the modes of fluctuation for copper are much different from the modes for gold and silver. The copper data are well represented by two or three modes of fluctuation. The first mode of fluctuation for copper is associated with changes in both the level and slope of the futures curve, i.e., a positive realization of the first mode leads to an increase in the level of the curve and a decrease in the slope of the curve. The second mode is associated with changes in the slope of the curve and the third mode is associated with changes in the curvature of the curve. For gold and silver, the data are well represented by one mode of fluctuation. For gold and silver, the first mode of fluctuation is associated with equal percentage changes in all maturities of the futures price curve, i.e., a positive realization of the first mode leads to an increase in the level and the slope of the curve.

Second, estimates of the time-varying modes clearly imply that the modes of fluctuation for the three metals are not stationary over the sample interval. For copper, there is substantial variation in the shape and level of the first mode over the sample interval. For gold and silver, there is substantial variation in the level of the first mode. The slope of the first mode, however, remains close to zero throughout the sample interval. The second and third modes for all three metals remain relatively unchanged over the sample interval.

These results show that fluctuations in copper, gold, and silver futures prices can be well represented using a small number of factors. These factors can be used to hedge price fluctuations, assess the risk of a position or model contingent claim contracts on the underlying commodity.

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