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Estimating Time-Varying Optimal Hedge Ratios on Futures Markets

Robert J. Myers

An optimal hedge ratio is usually defined as the proportion of a cash position that should be covered with an opposite position on a futures market. Under certain simplifying assumptions discussed below, optimal hedge ratios can be characterized by a simple rule: set the hedge ratio equal to the ratio of the covariance between cash and futures prices to the variance of the futures price (see Anderson and Danthine (1981) and Benninga, Eldor, and Zilcha (1984)). The conventional approach to implementing this rule is to regress historical cash prices, price changes, or returns on futures prices, price changes, or returns. The resulting slope coefficient is then used as the estimated optimal hedge ratio (see Ederington, (1979) and Kahl (1983)).

There are two problems with the conventional regression approach to optimal hedge ratio estimation. First, it generally fails to take proper account of all of the relevant conditioning information available to hedgers when they make their hedging decision (see Myers and Thompson (1989)). Second, it implicitly assumes that the covariance matrix of cash and futures prices, and hence optimal hedge ratios, are constant over time. There is evidence, however, that commodity price volatility changes as markets move through cycles of high and low uncertainty about future economic conditions (see Anderson (1985) and Fackler (1986)). For example, there was a clear jump in commodity price volatility during the boom of 1973, and during the recent 1988 drought. This suggests that the conditional covariance matrix of cash and futures prices, and hence optimal hedge ratios, may vary substantially over time. A recent article by Cecchetti, Cumby, and Figlewski (1988) estimates time-varying optimal hedge ratios for Treasury bonds using the autoregressive conditional heteroscedastic (ARCH) framework of Engle (1982). They find substantial fluctuations in the time path of optimal hedge ratios.

This article outlines and compares two approaches for estimating time-varying optimal hedge ratios on futures markets. Both methods take account of relevant conditioning information but they differ in their degree of sophistication and ease of estimation. The first method involves calculating moving sample variances and covariances of past prediction errors for cash and futures prices. This method is simple and easy to apply, but is also *ad hoc* and imposes questionable restrictions on the time pattern of commodity price volatility. The second method is the general-

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ized autoregressive conditional heteroscedastic (GARCH) model of Bollerslev (1986). This model provides a flexible and consistent framework for estimating time-varying optimal hedge ratios, but requires a nonlinear maximum likelihood estimator. The GARCH model generalizes the univariate ARCH framework, used by Cecchetti, Cumby, and Figlewski (1988) to estimate time-varying optimal bond hedges, by allowing for much more flexible patterns of persistence in the conditional covariance matrix of cash and futures prices.

These time-varying methods are applied to an example of wheat storage hedging and results are compared with no hedge and constant hedge outcomes using both in-sample and post-sample performance evaluations. There is strong evidence supporting time-varying optimal hedge ratios for wheat but it is found that time-varying optimal hedge ratio estimates computed from the GARCH model perform only marginally better than constant estimates obtained using conventional regression techniques. Thus, in this application the extra expense and complexity of the GARCH model do not appear to be warranted. Of course, this is not a general result and the GARCH model may provide significantly better performance in other applications.

A MODEL WITH TIME-VARYING OPTIMAL HEDGE RATIOS

Hedging behavior is modeled in the one-period portfolio selection framework outlined in Merton (1982). A consumption based intertemporal portfolio model could have been used (e.g., Stulz (1984) and Adler and Detemple (1988)). However, the one-period model is simpler and, in the context of the present article, provides virtually identical hedging rules as the intertemporal model.

An individual investor wants to determine the optimal allocation of initial wealth among two investment opportunities: purchase of a risky asset whose return at the end of the period is stochastic; and purchase of a risk-free bond. There is a futures market in the risky asset and the investor can therefore hedge by selling contracts which mature at or after the end of the period. The margin requirements on futures contracts can be satisfied with interest bearing government securities so that taking out futures positions requires no initial outlay of wealth. An example of this type of investment problem is a commodity trader who allocates initial wealth between purchasing a commodity for storage and later resale, and investing in a risk free bond.

The investor has a von Neumann-Morgenstern utility function, U, defined over end-of-period wealth. Utility is assumed to be increasing, strictly concave and twice differentiable. Given this setup, the portfolio selection problem can be formally stated as

$$\max_{q_{t-1},b_{t-1}} E[U(W_t) \mid \Omega_{t-1}] \tag{1}$$

subject to

$$W_t = (1 + r)[W_{t-1} - p_{t-1}q_{t-1}] + p_t q_{t-1} + (f_t - f_{t-1})b_{t-1} \tag{2}$$

where W_t is end-of-period wealth; W_{t-1} is initial wealth; Ω_{t-1} is the set of information available at the beginning of the period; r is the risk-free interest rate; p_{t-1} is the initial price of the risky asset; p_t is the (stochastic) end-of-period price of the risky asset; q_{t-1} is the quantity of the risky asset purchased; f_{t-1} is the initial futures price; f_t is the (stochastic) end-of-period futures price; and b_{t-1} is the quantity of futures purchased (sold if negative). A dividend (or storage cost) on the risky asset could be incorporated into the model without changing the substance of the hedging analysis.

Choosing how much to invest in the risky asset implicity defines the optimal investment in the risk free bond because of the wealth constraint. The two first-order conditions can therefore be written as

$$E[U'(W_t)\,|\,\Omega_{t-1}]\mu_t^p + \mathrm{Cov}[U'(W_t), p_t\,|\,\Omega_{t-1}] = 0 \tag{3}$$

$$E[U'(W_t)\,|\,\Omega_{t-1}]\mu_t^f + \mathrm{Cov}[U'(W_t), f_t\,|\,\Omega_{t-1}] = 0 \tag{4}$$

where μ_t^p and μ_t^f are the conditional expectations of returns to holding cash and futures positions, respectively:

$$\mu_t^p = E(p_t\,|\,\Omega_{t-1}) - (1 + r)p_t \tag{5}$$

$$\mu_t^f = E(f_t\,|\,\Omega_{t-1}) - f_{t-1}. \tag{6}$$

A result proven in Rubinstein (1976) is helpful in analyzing these first-order conditions further. Rubinstein shows that if two random variables X and Y are joint normally distributed, and g is a differentiable function, then

$$\mathrm{Cov}[g(X), Y] = E[g'(X)]\mathrm{Cov}(X, Y).$$

Thus, assuming the joint distribution of (W_t, p_t, f_t) conditional on Ω_{t-1} is multivariate normal then (3) and (4) can be expressed[1]

$$\mu_t^p + \frac{U''}{U'}\,\mathrm{Cov}(W_t, p_t\,|\,\Omega_{t-1}) = 0 \tag{7}$$

$$\mu_t^f + \frac{U''}{U'}\,\mathrm{Cov}(W_t, f_t\,|\,\Omega_{t-1}) = 0 \tag{8}$$

where it is understood that $U'' = E[U''(W_t)\,|\,\Omega_{t-1}]$ and $U' = E[U'(W_t)\,|\,\Omega_{t-1}]$. The covariance terms can be evaluated easily using the budget constraint (2). This leads to the pair of equations

$$H_t\begin{bmatrix} q_{t-1} \\ b_{t-1} \end{bmatrix} = -\frac{U'}{U''}\begin{bmatrix} \mu_t^p \\ \mu_t^f \end{bmatrix} \tag{9}$$

where H_t is the covariance matrix of $(p_t, f_t)'$ conditional on the information set Ω_{t-1}.

Premultiplying both sides of (9) by the inverse of H_t gives the asset and futures demand functions

$$\begin{bmatrix} q_{t-1} \\ b_{t-1} \end{bmatrix} = -\frac{U'}{U''}H_t^{-1}\begin{bmatrix} \mu_t^p \\ \mu_t^f \end{bmatrix}. \tag{10}$$

The usual assumption is that the conditional covariance matrix H_t is constant over time. Here, however, the time subscript indicates that time-varying volatility is allowed for explicitly in the model.

The optimal hedge ratio is the proportion of the long cash position which should be covered by futures selling. Dividing $-b_{t-1}$ by q_{t-1} gives

$$\frac{-b_{t-1}}{q_{t-1}} = \frac{h_t^{11}\mu_t^f - h_t^{21}\mu_t^p}{h_t^{21}\mu_t^f - h_t^{22}\mu_t^p} \tag{11}$$

where h_t^{ij} is the element in the ith row and jth column of H_t. In general, asset demands depend on investor risk preferences as well as on the probability distribution

[1]While there is evidence that the unconditional distributions of asset and commodity prices are not normal, conditional normality is a more reasonable assumption (see Baillie and Myers (1989)). Conditional normality combined with time-varying volatility can lead to unconditional distributions with unstable variances and fatter tails than the normal. These are precisely the characteristics which have been found in studies of the unconditional distribution of asset prices.

of asset prices. However, the optimal hedge ratio is preference free under the assumptions of this model.

It is often assumed that the expected return to trading futures is approximately zero, $\mu_t^f = 0$ for all t, so that (11) reduces to

$$\frac{-b_{t-1}}{q_{t-1}} = \frac{h_t^{21}}{h_t^{22}} \tag{12}$$

and the optimal hedge ratio equals a simple ratio of the conditional covariance between cash and futures prices to the conditional variance of futures. Equation (12) is also the hedge ratio which minimizes the variance of the investors total return (see Kahl (1983)). The remainder of the article focuses on estimation of this simplified hedging rule.

ESTIMATION

To operationalize the hedging rule, the conditional covariance matrix of cash and futures prices must be estimated. To begin, price realizations are separated into two components: an expectation conditioned on information available at time $t - 1$ and a random shock that is unpredictable. Formally, this can be expressed

$$p_t = E(p_t|\Omega_{t-1}) + u_t \tag{13}$$

$$f_t = E(f_t|\Omega_{t-1}) + v_t \tag{14}$$

where u_t is the prediction error for the cash price; and v_t is the prediction error for the futures price. The prediction errors are assumed to be serially uncorrelated with expected value zero (conditional on the information set). However, the conditional covariance matrix of the prediction errors is allowed to vary over time in response to changing market conditions.

Estimation requires specifying a model for the conditional means of cash and futures prices. If the expected return to holding futures is zero then the conditional mean of the futures price can be specified $E(f_t\,|\,\Omega_{t-1}) = f_{t-1}$. Despite its simplicity, there is considerable evidence supporting this model for commodity futures price data (see Gordon (1985) and Hudson, Leuthold, and Sarassoro (1987)). A model for the conditional mean of cash prices is $E(p_t\,|\,\Omega_{t-1}) = \alpha + p_{t-1}$. For storable seasonally-produced commodities, the constant term is a drift reflecting carrying charges and convenience yield over the course of the crop year. These are very simple models for the conditional means of cash and futures prices but they seem to fit the data well in the application to wheat storage hedging below. Extending the approach to more general models of the conditional mean for cash prices is straightforward. But when the expected return on futures is nonzero and/or cash commodity holdings are stochastic, then (12) is no longer the expected utility-maximizing hedge ratio and a different framework is required.[2]

These conditional mean models imply that (13) and (14) can be re-written

$$\Delta p_t = \alpha + u_t \tag{15}$$

$$\Delta f_t = v_t \tag{16}$$

where Δ is the first difference operator. The prediction errors have a time-varying covariance matrix

[2]See Rolfo (1980) and Cecchetti, Cumby, and Figlewski (1988) for examples of how to proceed in these circumstances.

$$H_t = E(\epsilon_t \epsilon_t' \mid \Omega_{t-1}) \tag{17}$$

where $\epsilon_t = (u_t, v_t)'$. Operationalizing the hedging rule (12) requires a model for H_t. Three alternative conditional covariance models are considered here.

Constant Conditional Covariance Matrix

The usual assumption is that the conditional covariance matrix is constant over time,

$$H_t = C \quad \text{for all } t. \tag{18}$$

In this case, an estimate of the conditional covariance matrix, C, is easy to obtain using least squares methods (see Myers and Thompson, (1989)). Once C has been estimated, the constant optimal hedge ratio can be computed as the ratio (12). This approach is very easy to apply but takes no account of the time-varying volatility that appears to characterize many cash and futures price series.

Moving Sample Variances and Covariances

Time-varying volatility can be incorporated into the estimation procedure in a naive way using moving sample variances and covariances of past price prediction errors. Suppose that the conditional covariance matrix at t depends on the past n prediction errors according to

$$H_t = \frac{1}{n} \sum_{i=1}^{n} \epsilon_{t-i} \epsilon_{t-i}'. \tag{19}$$

This implies that squared forecast errors are autocorrelated and the conditional covariance matrix therefore changes over time.

Once n is chosen there are no unknown parameters in the model for H_t. Therefore, the model can be estimated by first applying ordinary least squares to (15). This provides an estimate of the conditional mean parameter, α. Using the regression residuals and data on futures price changes, then compute moving sample variances and covariances of past prediction errors using some arbitrary value for n. Once H_t is estimated over the sample in this way, time-varying optimal hedge ratios can be estimated by computing the ratio (12) at each period.

Moving sample variances and covariances are easy to estimate but have two serious disadvantages. First, even if (19) is the appropriate model for H_t the simple two-step estimation procedure is inefficient. Least squares estimation of the conditional mean parameter, α, does not provide optimal inference when the conditional covariance matrix changes over time. Thus, improved inference generally results from maximum likelihood estimation. Second, (19) is a very restrictive model for H_t. It allows for autocorrelation in the squared prediction errors, so that price volatility may change over time, but the weights on past squared prediction errors are equal through lags n and zero thereafter. This imposes an extremely inflexible structure on the relationship between past volatility and current volatility. The GARCH model provides a more flexible mechanism for accounting for time-varying volatility in commodity prices.

The GARCH Model

A more formal way to introduce time-varying volatility is through the GARCH model of Engle (1982) and Bollerslev (1986). Suppose that H_t can be specified

$$\text{vech}(H_t) = C + \sum_{i=1}^{m} A_i \text{vech}(\epsilon_{t-i}\epsilon'_{t-i}) + \sum_{j=1}^{q} B_j \text{vech}(H_{t-j}) \tag{20}$$

where C is a (2×1) vector of parameters; the A_i are (3×3) matrices of parameters for $i = 1, 2, \ldots, m$; the B_j are (3×3) matrices of parameters for $j = 1, 2, \ldots, q$; and vech is the column stacking operator that stacks the lower triangular portion of a symmetric matrix. This is a bivariate GARCH(m, q) model and it allows autocorrelation in the squared prediction errors (i.e., volatility) to be modeled flexibly, in much the same way that an ARIMA specification provides a flexible means of modeling the autocorrelation in the levels of time series.

Assuming that the conditional distributions of the prediction errors are normal, the log-likelihood function for a sample of T observations on cash and futures prices is

$$L(\Theta) = -T \log 2\pi - 0.5\sum_{t=1}^{T} \log|H_t(\Theta)| - 0.5\sum_{t=1}^{T} \epsilon_t(\Theta)' H_t^{-1}(\Theta)\epsilon_t(\Theta) \tag{21}$$

where $\Theta = \{\alpha, C, A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_q\}$ is the set of all conditional mean and variance parameters. Notice that H_t and ϵ_t are functions of the sample data on cash and futures prices and the parameter vector, Θ. Thus, given a sample, estimation proceeds by maximizing the log-likelihood with respect to the unknown parameters.

Estimating Θ by maximum likelihood involves a complex nonlinear optimization problem. Once this has been accomplished, however, the time path of optimal hedge ratios can be computed easily by taking the ratio of h_t^{21} to h_t^{22} at each time period in the sample. The algorithm used for the estimation results in this article is based on Berndt, Hall, Hall, and Hausman (1974).

The GARCH model represents a flexible specification for modeling time-varying volatility in asset prices, and maximum likelihood is an optimal approach to inference. Thus, the GARCH model has significant theoretical advantages over moving sample variances and covariances. On the other hand, the GARCH model is much more difficult and demanding to estimate. A natural question is whether the additional effort required to estimate the GARCH model provides a significantly improved hedging performance, compared to simpler approaches. This question is investigated with an example because results will invariably depend on the particular application under study.

AN APPLICATION

Optimal hedge ratios are estimated for storage hedging of wheat to illustrate and test the time-varying methods. Wheat is an important commodity in many parts of the U.S. and wheat futures contracts maturing in various months are traded at the Chicago Board of Trade, Kansas City, and Minneapolis. Here, attention is focused on the May and December contracts at the Chicago Board of Trade.

Consider an investor that takes out long positions in wheat. That is, the investor buys and stores wheat for later resale at a price which is unknown at the time of purchase. The investor can hedge the long cash position by selling futures. Hedging in two contract months, May and December, is allowed in the analysis. It is assumed that the investor takes out futures positions in one of these contracts and reevaluates his or her portfolio on a weekly basis. Each week the portfolio may be adjusted to reflect changing information and economic conditions. When a contract matures, futures positions are rolled over into the same contract month of the next year.

Because portfolio adjustment is assumed to occur on a weekly basis, weekly data are used in the empirical analysis. Cash prices are for the Saginaw market in Michigan and are from Mid-States Terminals, Toledo, Ohio. Futures prices are for the May and December contracts on the Chicago Board of Trade and are from various issues of the Chicago Board of Trade Statistical Annual. All data are the mid-week (Wednesday) closing price and the sample period runs from June 1977 to May 1983, a total of six years. However, the data are split into two segments, the first for estimation and in-sample performance evaluation and the second for post-sample testing. The estimation period runs from June 1977 to May 1983, leaving two years of data for post-sample tests.

Separate bivariate GARCH models are estimated for cash and May futures prices, and for cash and December futures prices. Preliminary results suggest that a GARCH(1, 1) model, with one lag on the squared prediction errors and one lag on past conditional covariance matrices, provides an adequate representation of wheat price volatility. This is consistent with the findings of a number of other studies on the conditional distribution of asset prices (see Engle and Bollerslev (1986)). Even in the GARCH(1, 1) model, however, there are 21 conditional covariance matrix parameters to estimate, suggesting that the model may be over-parameterized. Following Bollerslev, Engle, and Wooldridge (1988), a more parsimonious representation can be obtained by assuming that the A and B matrices are diagonal. This is a natural simplification because it implies that each variance and covariance depends only on its own past values and prediction errors. The diagonal restrictions reduce the number of conditional covariance matrix parameters to nine.

Estimation results for the two diagonal GARCH(1, 1) models are shown in Table I (May Contract) and Table II (December Contract). For purposes of comparison, results from estimating constant conditional covariance matrix models are also included in the tables. The only unknown parameter in the conditional covariance matrix of the moving sample variances and covariances model is n, the number of periods to include in the moving sample calculation. In the analysis which follows, $n = 10$ is used.

The GARCH parameter estimates in Tables I and II are all highly significant, and a likelihood ratio test of the constant covariance matrix model versus the GARCH model rejects the former in favor of the latter at essentially any significance level. For the May (December) contract the $\chi^2(6)$ statistic is 31.98 (69.53) and the critical value for the test at the 0.005 significant level is 18.55. Thus, there is strong evidence of time-varying volatility in wheat prices.

Optimal hedge ratio estimates from the constant conditional covariance matrix and moving sample variances and covariances models are computed using standard techniques. For the GARCH model, in-sample hedge ratios are constructed using the in-sample estimates of the time-varying conditional covariance matrices $\hat{H}_t$. Post-sample hedge ratios are computed using the parameter estimates in Tables I and II, along with realized values of cash and futures prices available up to the time the portfolio is being adjusted. These post-sample estimates are therefore based only on information that is available at the time each hedging decision is made. More efficient use of available information can be achieved by updating the GARCH model's parameter estimates as each new observation becomes available. However, this is very costly and is not attempted here.

Table I
ESTIMATION RESULTS FOR THE MAY CONTRACT

Constant Conditional Covariance Matrix Model

$$\Delta p_t = \underset{(2.228)}{0.009} + \hat{u}_t;\ \Delta f_t = \hat{v}_t$$

$$\hat{H}_t = \begin{bmatrix} \underset{(13.131)}{0.018} & \underset{(12.827)}{0.013} \\ \underset{(12.827)}{0.013} & \underset{(15.757)}{0.014} \end{bmatrix} \quad \text{for all } t.$$

Log-likelihood = 884.176

GARCH(1, 1) Model

$$\Delta p_t = \underset{(2.142)}{0.009} + \hat{u}_t;\ \Delta f_t = \hat{v}_t$$

$$\mathrm{vech}(\hat{H}_t) = \begin{bmatrix} \underset{(3.680)}{0.002} \\ \underset{(3.462)}{0.001} \\ \underset{(2.566)}{0.001} \end{bmatrix} + \begin{bmatrix} \underset{(4.486)}{0.128} & 0.0 & 0.0 \\ 0.0 & \underset{(3.311)}{0.073} & 0.0 \\ 0.0 & 0.0 & \underset{(3.476)}{0.075} \end{bmatrix} \begin{bmatrix} \hat{u}^2_{t-1} \\ \hat{v}_{t-1}\hat{u}_{t-1} \\ \hat{v}^2_{t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{(18.523)}{0.762} & 0.0 & 0.0 \\ 0.0 & \underset{(25.700)}{0.851} & \underset{0.0}{0.0} \\ 0.0 & 0.0 & \underset{(23.479)}{0.861} \end{bmatrix} \begin{bmatrix} \hat{h}^{11}_{t-1} \\ \hat{h}^{21}_{t-1} \\ \hat{h}^{22}_{t-1} \end{bmatrix}$$

Log-likelihood = 900.164

Note: Values in parentheses are asymptotic t-ratios.

Estimated optimal hedge ratio paths for hedging in the May and December contracts are illustrated in Figures 1 and 2, respectively. In each figure, the upper graph shows the hedge ratio path estimated with moving sample variances and covariances, and also displays the constant optimal hedge ratio estimates of 0.93 (May contract) and 0.92 (December contract) as a reference point. The lower graph shows the hedge ratio path estimated with the GARCH model, again compared to the constant estimates as a reference point. Estimated hedge ratios from the two time-varying methods have a tendency to move together, and they both fluctuate consid-

Table II
ESTIMATION RESULTS FOR THE DECEMBER CONTRACT

Constant Conditional Covariance Matrix Model

$$\Delta p_t = \underset{(1.500)}{0.007} + \hat{u}_t;\ \Delta f_t = \hat{v}_t$$

$$\hat{H}_t = \begin{bmatrix} \underset{(12.793)}{0.017} & \underset{(12.633)}{0.012} \\ \underset{(12.633)}{0.012} & \underset{(15.868)}{0.013} \end{bmatrix}$$

Log-likelihood = 884.737

GARCH(1, 1) Model

$$\Delta p_t = \underset{(2.447)}{0.009} + \hat{u}_t;\ \Delta f_t = \hat{v}_t$$

$$\mathrm{vech}(\hat{H}_t) = \begin{bmatrix} \underset{(5.187)}{0.002} \\ \underset{(4.525)}{0.001} \\ \underset{(3.421)}{0.001} \end{bmatrix} + \begin{bmatrix} \underset{(7.489)}{0.150} & 0.0 & 0.0 \\ 0.0 & \underset{(7.243)}{0.086} & 0.0 \\ 0.0 & 0.0 & \underset{(5.490)}{0.106} \end{bmatrix} \begin{bmatrix} \hat{u}^2_{t-1} \\ \hat{v}_{t-1}\hat{u}_{t-1} \\ \hat{v}^2_{t-1} \end{bmatrix}$$

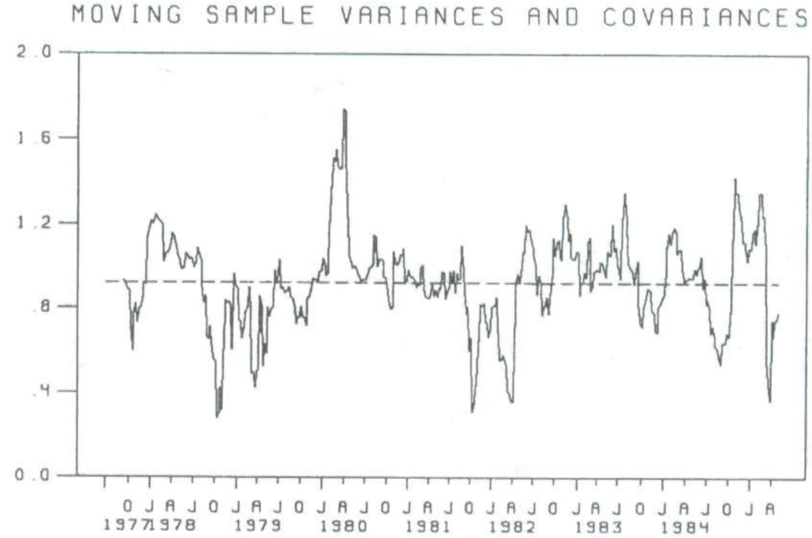
$$+ \begin{bmatrix} \underset{(33.994)}{0.736} & 0.0 & 0.0 \\ 0.0 & \underset{(92.489)}{0.834} & 0.0 \\ 0.0 & 0.0 & \underset{(30.555)}{0.810} \end{bmatrix} \begin{bmatrix} \hat{h}^{11}_{t-1} \\ \hat{h}^{21}_{t-1} \\ \hat{h}^{22}_{t-1} \end{bmatrix}$$

Log-likelihood = 919.502

Note: Values in parentheses are asymptotic t-ratios.

erably over time. However, fluctuations in the moving sample variances and covariances hedge ratios are clearly more pronounced than in the GARCH hedge ratios. This suggests that futures positions would have to be adjusted by much greater amounts when using the moving sample variances and covariances model compared to the GARCH model.

Figures 1 and 2 demonstrate that in many periods there is a wide divergence between hedge ratios estimated assuming a constant conditional covariance matrix for cash and futures prices, and hedge ratios estimated with the GARCH model. As



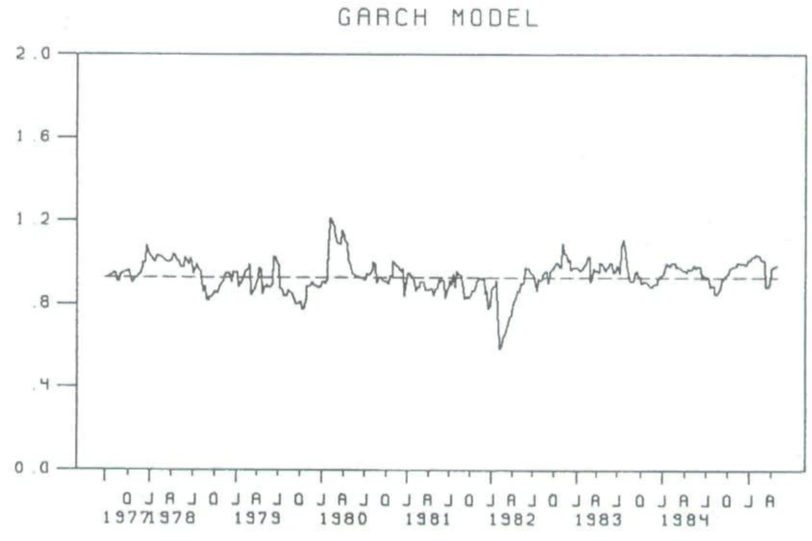
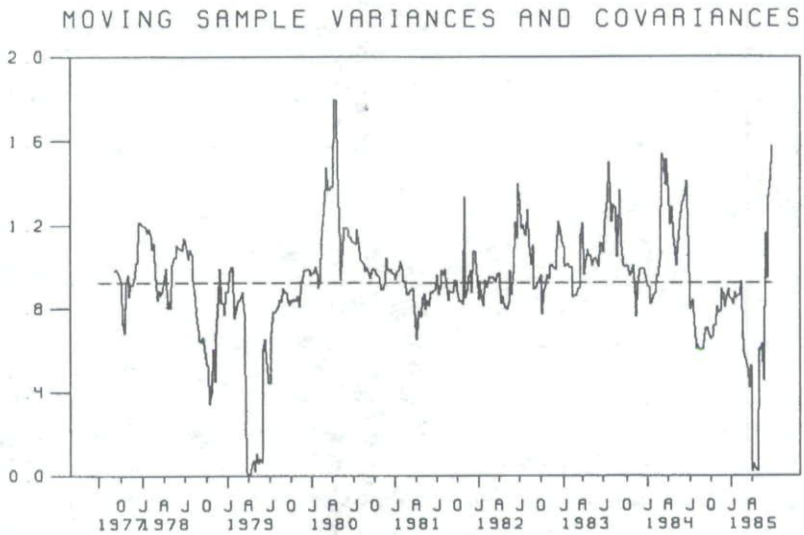


Figure 1
Time-varying optimal hedge ratios using the May contract.

yet, however, no evidence has been presented on the hedging performance of the alternative rules. Performance testing is necessary to determine whether the additional complexity of the GARCH approach is justified.

Performance tests are carried out using a simulation study. An investor is assumed to hold one bushel of wheat continuously over the sample period. The investor hedges fluctuations in his or her wealth (caused by fluctuating cash prices) by selling either May or December futures. The investor's wealth at the end of the



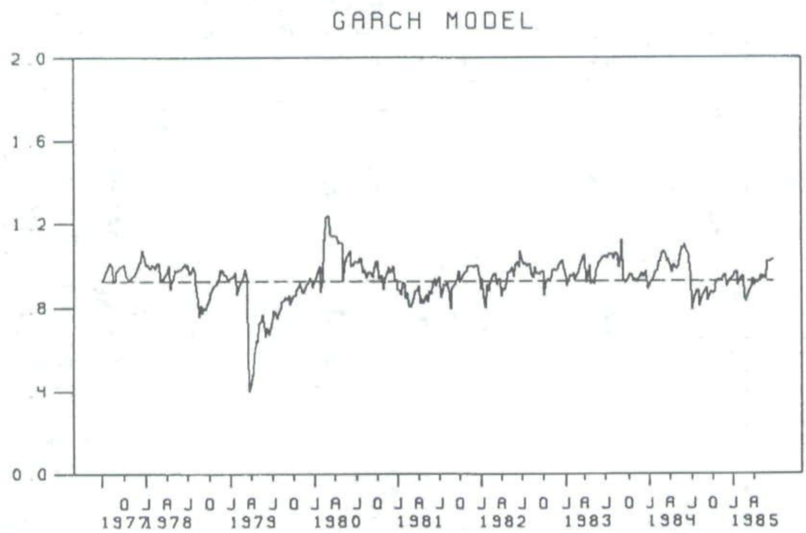


Figure 2
Time-varying optimal hedge ratios using the December contract.

week equals the cash value of the bushel of wheat plus or minus any gain or loss on futures. The futures position can be adjusted on a weekly basis. As the futures contract matures, futures positions are rolled over into the same contract month of the next year. Performance is evaluated in terms of effects on the mean and variance of the investor's wealth position.

Ex-ante evaluation proceeds by assuming that the GARCH specification is the correct probability model for wheat cash and futures prices. Then using this model, the *ex-ante* conditional expectation and conditional standard deviation of wealth are constructed for each date under four different hedging rules, no hedge, the constant hedge, the moving sample variances and covariances hedge, and the GARCH hedge. The results for each date are then averaged over the relevant period to provide a summary measure of hedging performance. *Ex-post* evaluation proceeds by constructing the actual *ex-post* wealth levels that would have been achieved under each hedging rule. The *ex-post* wealth levels for each date are then averaged over the relevant period to provide a summary measure.

Performance test results are reported in Tables III and IV. Table III assumes all hedging is done in the May contract while Table IV assumes all hedging is done in the December contract. Results are expressed in dollars per bushel, as well as percentage deviations from the no hedge outcomes. Furthermore, results are provided for the in-sample and post-sample periods separately as well as for the two periods combined.

Tables III and IV give very similar results, indicating that hedging performance is insensitive to the contract month used to implement the hedge. Within each table, the *ex-ante* conditional means of wealth are the same irrespective of which hedging rule is used, because of the assumption that the expected return from holding futures is zero. But, of course, the *ex-ante* conditional standard deviation of wealth does change with the hedging rule. The GARCH hedge reduced this conditional standard deviation by between 45% and 48%, compared to no hedging, depending on which contract month and sample period are being considered. The GARCH hedge ratio performs marginally better in terms of variance reduction than either the constant hedge or the moving sample variances and covariances hedge.

While the *ex-ante* results illustrate what can be expected from using each hedging rule over the long run, actual *ex-post* performance during any particular sample period could be quite different. Therefore, *ex-post* incomes are calculated for each date in the *in-sample* and *post-sample* periods under the same four hedging rules.

Table III
HEDGE RATIO PERFORMANCE EVALUATIONS FOR THE MAY CONTRACT

	In-Sample Average	Post-Sample Average	Combined Sample Average
Ex-Ante Evaluation			
$E(W_t \mid \Omega_{t-1})$	3.4120	3.2689	3.3752
$\sigma(W_t \mid \Omega_{t-1})$			
-No hedge	0.1296 (0%)	0.1095 (0%)	0.1244 (0%)
-Constant hedge	0.0717 (−44.7%)	0.0578 (−47.2%)	0.0681 (−45.3%)
-Moving hedge	0.0731 (−43.6%)	0.0599 (−45.3%)	0.0697 (−44.0%)
-GARCH hedge	0.0710 (−45.2%)	0.0575 (−47.5%)	0.0675 (−45.7%)
Ex-Post Evaluation			
W_t			
-No hedge	3.4089 (0%)	3.2604 (0%)	3.3706 (0%)
-Constant hedge	3.4138 (0.14%)	3.2693 (0.27%)	3.3765 (0.18%)
-Moving hedge	3.4156 (0.20%)	3.2660 (0.17%)	3.3770 (0.19%)
-GARCH hedge	3.4134 (0.13%)	3.2686 (0.25%)	3.3760 (0.16%)

Note: $\sigma(W_t \mid \Omega_{t-1})$ is the conditional standard deviation of wealth and numbers in parentheses are percentage deviations from the no hedge outcome.

Table IV
HEDGE RATIO PERFORMANCE EVALUATIONS FOR THE DECEMBER CONTRACT

	In-Sample Average	Post-Sample Average	Combined Sample Average
Ex-Ante Evaluation			
$E(W_t \mid \Omega_{t-1})$	3.4105	3.2709	3.3735
$\sigma(W_t \mid \Omega_{t-1})$			
-No hedge	0.1293 (0%)	0.1091 (0%)	0.1240 (0%)
-Constant hedge	0.0718 (−44.5%)	0.0599 (−45.1%)	0.0686 (−44.7%)
-Moving hedge	0.0727 (−43.8%)	0.0638 (−41.5%)	0.0704 (−43.2%)
-GARCH hedge	0.0709 (−45.2%)	0.0595 (−45.5%)	0.0679 (−45.2%)
Ex-Post Evaluation			
W_t			
-No hedge	3.4069 (0%)	3.2619 (0%)	3.3684 (0%)
-Constant hedge	3.4087 (0.05%)	3.2686 (0.21%)	3.3715 (0.09%)
-Moving hedge	3.4121 (0.15%)	3.2664 (0.14%)	3.3734 (0.15%)
-GARCH hedge	3.4093 (0.07%)	3.2682 (0.19%)	3.3719 (0.10%)

Note: $\sigma(W_t \mid \Omega_{t-1})$ is the conditional standard deviation of wealth and numbers in parentheses are percentage deviations from the no hedge outcome.

Results are reported in the second part of Tables III and IV and indicate that, on average, *ex-post* income levels are not very sensitive to the hedging rule used. This is exactly what is expected if the average return to holding futures is zero.

A striking feature of these results is that the constant hedge, the moving sample variances and covariances hedge, and the GARCH hedge all provide a remarkably similar hedging performance. Thus, while the GARCH model performs best in terms of *ex-ante* variance reduction, the simple constant optimal hedge ratio estimate performs almost as well. Furthermore, the constant estimate (marginally) outperforms the moving sample variances and covariances estimate. If the hedger is extremely risk averse, then the extra cost of working with a GARCH model may still be warranted. However, since the GARCH model performs only marginally better, is much more complex to estimate, and since the continual futures adjustments that it requires entails extra commission charges, a strong case can be made for using the constant optimal hedge ratio estimate. Of course, this conclusion applies only to the wheat storage hedging example studied here and quite different results may be found in other applications.

CONCLUSIONS

This article outlines two methods for estimating time-varying optimal hedge ratios on futures markets and compares them to traditional regression methods, which implicitly assume hedge ratios are constant over time. It is argued that the GARCH model has distinct theoretical advantages over alternative methods, and that improved optimal hedge ratio estimates should therefore arise from GARCH models of the conditional distributions of cash and futures prices.

In an application to wheat storage hedging in Michigan, there is strong evidence for rejecting the constant conditional covariance model in favor of a GARCH model of wheat price volatility. It is also found that the GARCH model provides superior hedging performance than either the constant hedge ratio model or the moving sample variances and covariances model. However, the GARCH model performs only marginally better than a simple constant hedge ratio estimate. This suggests that assuming constant optimal hedge ratios, and using linear regression approaches to optimal hedge ratio estimation, may be an adequate approximation in this application. Whether similar conclusions will emerge from other applications of these methods to optimal hedge ratio estimation remains a question for future research.

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