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Portfolio Insurance Trading Rules

Richard Bookstaber
Joseph A. Langsam

Product development in portfolio insurance and dynamic hedging is moving along two fronts: hedging technology and risk management design. Risk management design answers the question of what should be insured. The current generation of this product looks at the asset/liability position of the client, addressing the surplus and the liability exposure rather than simply protecting the equities in isolation. Hedging technology focuses on mathematical models of how the hedge is adjusted to meet the payoff objective. Since the hedging technology is generally both proprietary and analytically complex, it is difficult to fully understand or verify the claims that are made.

The purpose of this article is to define and analyze the properties of the various portfolio insurance models. Since these models are rooted in mathematics, their true properties can be clearly and unambiguously stated. The properties we will discuss may appear esoteric, but will be familiar to anyone who has worked with portfolio managers in this area.

I. DEFINITIONS: WHAT ALL THE NEW TERMS MEAN

Option-Replication Strategy

A strategy that gives the same payoff that can be obtained by holding a set of options. Generic portfolio insurance is an option-replication strategy. It gives the same payoff as buying a port option on the underlying portfolio.

Figure 1 gives an example of an option-replication payoff. The payoff provides a floor of \$100, less the cost of the portfolio insurance hedge. It gives the same payoff as a put option with an exercise price of \$100 and a year to expiration. The hedging cost is the cost of the option.

Richard Bookstaber is a Managing Director at Moore Capital Management.

Joseph A. Langsam is Managing Director at Morgan Stanley Dean Witter.

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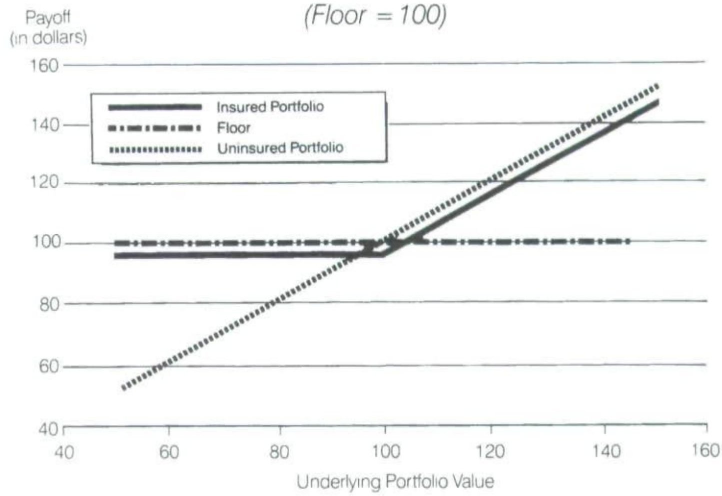

Figure 1
Portfolio insurance: payoff profile.

This is not the only payoff that is possible with options. Indeed, it is well established that any payoff that is a function of the underlying portfolio value can be obtained with a suitable set of options. And since any option can be replicated through a dynamic hedging strategy, there are no limits to the payoffs that are possible through option replication strategies.[1]

Path-Independent Strategy

A strategy whose payoff depends only on the value of the portfolio at the time of the payoff and on other parameters of the hedge. The payoff specifically does not depend on the particular path taken by the portfolio over the course of the hedging period.

Option strategies are path-independent strategies. The terminal payoff of the strategy depicted in Figure 1 is a function only of the underlying portfolio value at expiration, and is therefore path independent.

An example of a path-dependent strategy is the stoploss strategy shown in Figure 2. In this strategy, the portfolio is stopped out if it drops below the present value of the floor. The portfolio is then placed in a zero coupon bond maturing at expiration, thereby assuring the floor is met.

The payoff for the hedge cannot be stated solely in terms of the portfolio value at expiration. We also must know the path of the portfolio over the hedging period, since the ending value of the stoploss strategy will depend on whether the portfolio dropped below the stoploss level at any time over the period. In some cases a stoploss strategy will give a payoff equal to the underlying portfolio, since the portfolio value may have never dropped below the stoploss point. In the case shown

[1]Obviously, there will be a cost for these payoffs. The shape of the payoff profile as a function of the portfolio value can be determined by the appropriate option replication strategy, but the cost of that strategy will apply a constant term to the payoff function. Also, the strategies of option replication are generally stated only in terms of time horizon and portfolio value. They do not depend on the path the portfolio took to arrive at its ending value. The ability of option to meet payoff objectives is discussed in Ross (1976), and Breeden and Litzenberger (1978).

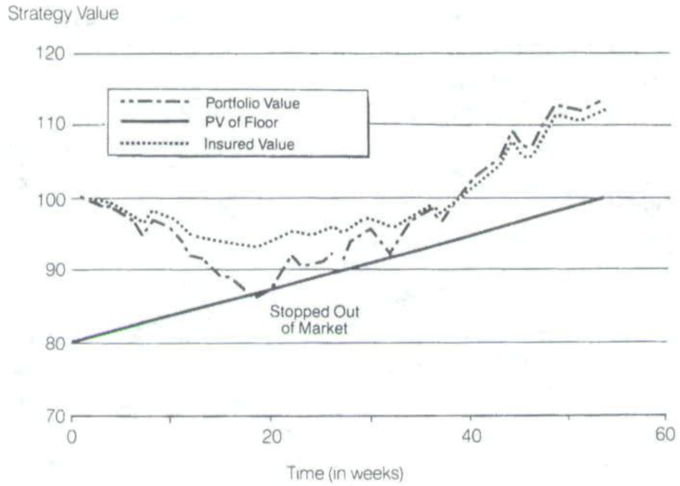

Figure 2
Payoff of alternative hedging strategies.

here, however, the portfolio will be stopped out, leading to an ending value of \$100, while the portfolio itself rises above \$112 by the end of the year. In contrast to this, the portfolio insurance program rises with the market to end the year above \$111.

Path independence is a desirable property. A strategy that is not path independent gives an uncertain payoff, and therefore violates the very premise of portfolio insurance: giving a known payoff.[2]

Time Invariant

A strategy that does not depend on a fixed time horizon, or on the time remaining in the program.

The conventional protective put option strategy is not time invariant. Since it has a defined expiration date, and a payoff that is set according to that date, the hedging strategy will depend on the amount of time remaining to expiration. The closer the program comes to the expiration date, the more exacting the hedge will become in moving into the reserve asset if the portfolio is below the floor, or into the underlying portfolio if the portfolio is above the floor.

Linear Trading Rule

A rule for adjusting the exposure to the risky portfolio that takes the form

$$\text{Exposure} = \text{Target Exposure} + \text{Constant} \times \text{Market Measure}.$$

A linear trading rule proposed by some for dynamically hedging bonds takes the form

$$\text{Portfolio Duration} = \text{Index Duration} + b\,(\text{Market Yield} - \text{Expected Long-term Yield}).$$

[2]Because of this, path independence is generally an important condition for utility maximization.

Here the portfolio exposure and the target exposure are both measured in terms of interest rate sensitivity (duration) rather than in terms of market value, and the market measure is based on the difference between the market interest rate and a subjective view of long-term interest rates.[3]

Another popular linear trading rule is called Constant Proportion Portfolio Insurance.[4] A simplified approach to portfolio insurance, it says that the amount of the portfolio held in the risky portfolio is proportional to the surplus of the portfolio (that is, the value of the portfolio in excess of the specified floor). This linear trading rule takes the form

$$\text{Exposure} = \text{Constant} \times \text{Surplus}.$$

This strategy looks at the portfolio value in excess of the stated floor (the floor is assumed to grow over time at the rate of interest) and moves the amount of the portfolio placed in the risky asset up or down by an amount equal to a factor times this excess.[5]

In practice, the Constant Proportionality portfolio insurance strategy is a heuristic—one of a number of arbitrary filter rules or trading rules for meeting the floor. It does not contain a foundation for analytical study. Because of this, questions such as why one constant is preferrable to another, or what distributional properties will be obtained by changes in the constant, are difficult to answer. As a result, the Constant Proportionality strategy is difficult to use in applications where the objective is to alter the return distribution in a well-defined manner, or where the cost of the hedge needs to be specified.

Process-Free Strategy

A strategy that does not depend on the stochastic process driving the portfolio price.

A process-free strategy does not use the statistical characteristics of the market in developing its trading rule. The linear trading rules discussed above are all process-free strategies.

II. PROPERTIES OF ALTERNATIVE PORTFOLIO INSURANCE METHODS

Proposition 1

All Path-Independent Strategies are Option-Replication Strategies

What this proposition says is, "Show me any payoff generated by a path-independent strategy, and I can give you that payoff using an option-replication strategy." It follows immediately from the facts that a) any payoff can be created with the appropriate mix of options, and b) option strategies are path independent. Given options of all possible exercise prices and all times to expiration as building blocks,

[3]This strategy is similar to the well known adaptive expectations models done historically in economics. It will give better-than-market returns if interest rates are mean reverting around the specified long-term yield.

[4]This strategy is discussed in detail in A. Perold (1986).

[5]We will discuss here the cases for the constant being greater than one. Although in theory it is possible to have a constant less than one, it leads to a concave payoff function. Such a curvature is not consistent with the usual portfolio insurance objectives.

it is possible to create any payoff as a function of the underlying portfolio. And we can create any option through the option-replication techniques of dynamic hedging.

This proposition dispels the prospect of *new and improved* or *optimized* portfolio insurance methods that give path-independent payoffs for less than option-replication strategies. Since the same payoff could be generated using option replication, the ability to give the same payoff with an alternative technology would imply tremendous arbitrage profit opportunities. Anyone using this technology would be better off leaving the portfolio insurance business and trading in the option markets.

1a. All time-invariant, path-independent strategies are option-replication strategies. Since all path-independent strategies are option-replication strategies, it follows that time-invariant, path-independent strategies, as a subset of all path-independent strategies, will also be option-replication strategies.

One of the most popular portfolio insurance strategies is the *perpetual* strategy. This strategy does not have a defined ending period and gives a floor that grows over time; it is therefore a time-invariant strategy. One of the advantages of this strategy over conventional option replication is that it overcomes the sometimes artificial specification of a terminal date for the hedging period. If there is no true ending date for the hedge, and one is imposed as a part of the portfolio insurance design, the portfolio will face unnecessary resetting costs. On the other hand, if the portfolio does have a terminal hedging date, the use of a perpetual strategy will impose an unnecessary cost.

This corollary to Proposition 1 shows that perpetual strategies, insofar as they are path independent, are no more than option-replication strategies, albeit more complex than the conventional protective put option.[6]

Proposition 2

A Dynamic Hedging Strategy That has the Potential for Requiring the Investor to be Fully Invested, or Moving the Investor to be Fully Cashed Out, is a Path-Dependent Strategy

To understand why this is the case, note that if the portfolio rises far enough in value to put the investor completely into the risky portfolio, there is no further impact on the hedging strategy for later movements in the portfolio above that level; while if the portfolio is only partially invested when the price path occurs, later movements in the portfolio will lead to changes in the portfolio position.

2a. Perpetual strategies that begin fully invested are path dependent. One common portfolio strategy begins fully invested, and only begins hedging after the portfolio starts to decline. According to this proposition, such a strategy is path dependent. If the portfolio continually moves up after the program is initiated, there will be no hedging cost and the hedged portfolio will give the same payoff as the unhedged portfolio. On the other hand, if the portfolio first drops and later rises to the same ending value, the hedge will come into play and the hedging strategy will impose a cost.[7] This means the payoff is uncertain—a less-than-desirable feature in an

[6]Indeed, as is shown in the Appendix, such strategies can be closely approximated by a strategy of rolling options.

[7]Since this perpetual strategy is costless if the market goes up, it must cost more than the option replicating strategy in other market environments. In particular, it will be more costly in downward markets.

insurance strategy. The payoff cannot be known without knowing the path the portfolio took to get to its ending value.

2b. Linear trading rules are path dependent in practice. Linear trading rules require the position in the portfolio to grow without bound as the portfolio value moves away from the floor.[8] In practice, of course, there is a limit to how much investment can be made in the portfolio. The pension fund has a limited amount of assets, and a limited authority to lever on those assets. As we know from Proposition 2, the potential for becoming fully invested implies path dependence.

2c. The constant proportionality portfolio insurance strategy is path dependent in practice. Since, as we noted in Section II, the Constant Proportionality portfolio insurance strategy is a linear trading rule, this proposition follows immediately from Proposition 2b. This strategy is path dependent if the portfolio is not permitted to lever without constraint, since at some point the portfolio may become fully invested in the market.[9] Because all pension funds put some restriction on the amount of leverage that is acceptable, this means that in practice the Constnt Proportionality portfolio insurance strategy will be path dependent.

The argument made by the proponents of these strategies to defend them against the criticism of path dependency is that their cost can be determined by the number of times they *trigger*—that is, the number of times a hedge adjustment must be made. On this basis, the argument goes, the strategy is not path-dependent. It is true that if an adjustment is made whenever the portfolio value changes by a predetermined amount (it remains to be shown why such a rule for hedge adjustments is ideal), it is easy to calculate the cost of each adjustment. But the number of triggers can only be determined by knowing the path the portfolio takes.[10]

To see this, look at Figure 3. Based on triggers set every three points, the lower path has 31 triggers over the course of the program. The upper path has only 15 triggers, and therefore imposes less than half the hedging cost. Although both paths have the same volatility, and both lead to the same ending value, one may be off by a factor of two in pricing the hedge if a path-dependent strategy is used without knowledge of the actual path that occurred. By contrast, the hedging cost will be identical when a path-independent, option-replication strategy is used.

Proposition 3

A Process-Free Strategy Is Dominated by the Correct, Process-Specific Strategy

The name process free suggests a hedging strategy unencumbered by the need to specify the process driving the market. But this freedom comes at a price, and that price is potentially inefficient hedging.

[8]Here we restrict ourselves to those rules that lead to convex payoff functions. Payoff functions that are consistent with the objectives of portfolio insurance are convex.

[9]This path dependency is noted by Perold.

[10]Some of those unfamiliar with the mathematical underpinnings of stochastic processes argue that the number of triggers essentially is the same sort of information as volatility. This is incorrect. As we have just demonstrated, the number of triggers can only be determined after knowing the path, even for two paths of the same volatility. Volatility, however, is a statistical parameter that is independent of the particular path that is followed. In fact, its very mathematical foundation is founded on the notion of path independence.

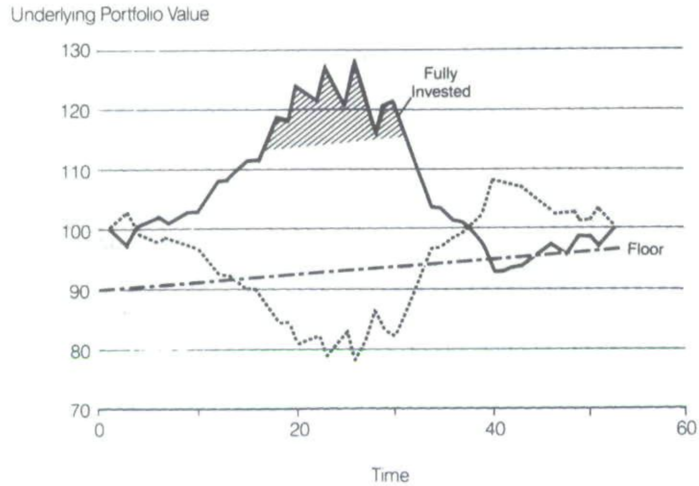

Figure 3
The path dependence of linear trading rules.

The problem with hedges based on process-free strategies is that they do not take advantage of all the information available in the market. In particular, they ignore information about the price behavior of the portfolio. Are prices more volatile intraday than over longer time periods? Do prices tend to move about smoothly, in intervals of fractions of a point at a time, or do they jump erratically several points at a time? What statistical distribution do returns follow? We have information that bears on these questions.[11] Using that information cannot help but add value to the hedging model and the decision of hedging readjustment.

III. WHAT WE CAN GAIN BY USING OPTIONS TO CREATE OPTIONS

Since the objective of a path-independent dynamic hedging strategy is to replicate an option payoff, it seems natural to use the options available in the marketplace in the dynamic hedge. There are useful applications for market options. There are also misuses and false claims attributed to their application.

Proposition 4

The Conventional Strategy of Rolling Options Is Path Dependent and Leads to a Higher Expected Cost for the Hedge

One way to assure a floor of $100 in one year is to roll listed options with an exercise price of $100. That is, buy an option with three months to expiration, upon its expiration buy another three-month option, and so forth. As Table I shows, the cost of this strategy depends on the path of the market over the course of the

[11]For example, research spanning over a quarter of a century has shown that equity return are approximately log-normally distributed, but are slightly more "fat-tailed" than a log-normal distribution. The returns more closely approximate a log-normal distribution over longer time periods. Returns are serially correlated over short time periods. For bonds, we know that prices tend to decrease in volatility as they approach expiration, and short-term interest rates have greater volatility than longer-term interest rates.

Table I-A
ROLLING OPTIONS AND PORTFOLIO INSURANCE

	Time					Total Premium Cost	Insured Portfolio Payoff	
	Today	3 Months	6 Months	9 Months	1 Year		Rolled	Not Rolled
Insured Floor	100.00	100.00	100.00	100.00	100.00			
			Stable Market					
Market Value	100.00	100.00	100.00	100.00	100.00		83.49	89.88
Option Price	4.13	4.13	4.13	4.13				
Option Premium	4.13	4.13	4.13	4.13		16.51		
			Increasing Market					
Market Value	100.00	105.00	110.00	115.00	120.00		109.31	109.88
Option Price	4.13	7.68	12.05	16.83				
Option Premium	4.13	2.68	2.05	1.83		10.69		
			Decreasing Market					
Market Value	100.00	95.00	90.00	85.00	80.00		93.51	89.88
Option Price	4.13	1.74	0.53	0.10				
Option Premium	4.13	1.74	0.53	0.10		6.49		
			One-year Option					
Option Price	10.12							
Option Premium	10.12					10.12		

year. If the market is stable and the value of the portfolio is at $100 at each rolling point, the option will cost far more than the one-year option ($16.51 versus $10.21 in this example). This is because the price of an option does not grow linearly with time. An option with six months to expiration costs less than twice what a three-month option costs; an option with a year to expiration costs far less than four times that of one with three months to expiration. If the market moves away from the exercise price, the relative cost of the rolling strategy diminishes. If the market moves enough, the price will be lower for rolling options than it will be for the one-year strategy. In expected terms, however, the rolling strategy will cost more, and will also bring an unnecessary source of uncertain cost to the portfolio insurance program.[12]

Of course, there is no need to have an exercise price equal to the end-of-year

[12]Certain, more sophisticated rolling strategies, such as those that replicate options rolled continuously, do not share this problem. Such strategies can not be executed in the market with listed options, however.

Table I-B
ROLLING OPTIONS AND PORTFOLIO INSURANCE

	Time					Total Premium Cost	Insured Portfolio Payoff	
	Today	3 Months	6 Months	9 Months	1 Year		Rolled	Not Rolled
Insured Floor	93.24	94.89	96.56	98.25	100.00			
			Stable Market					
Market Value	100.00	100.00	100.00	100.00	100.00		84.25	89.88
Option Price	8.87	7.53	6.27	5.13				
Option Premium[a]	2.11	4.07	4.50	5.07		15.75		
			Increasing Market					
Market Value	100.00	105.00	110.00	115.00	120.00		107.55	109.88
Option Price	8.87	11.97	15.19	18.47				
Option Premium[a]	2.11	3.51	3.42	3.41		12.45		
			Decreasing Market					
Market Value	100.00	95.00	90.00	85.00	80.00		87.79	89.88
Option Price	8.87	3.93	1.11	0.16				
Option Premium[a]	2.11	5.47	2.78	1.85		12.21		
			One-year Option					
Option Price	10.12							
Option Premium	10.12					10.12		

[a]Includes additional funds required to purchase the next option at the higher exercise price.

floor of $100. The payoff to the option need only be enough to accrue to the floor by the end of the year if invested in the reserve asset. A more sensible rolling strategy might be that depicted by the second part of Table I. Here the exercise prices are $93.24, 94.89, 96.56 and 98.25 for the first, second, third and fourth options, respectively. Each of these exercise prices is equal to the value of $100 discounted at 7% over the time period. While this strategy diminishes the cost and uncertainty of the strategy, it still leads to uncertain cost and a higher expected cost.

The reason for the higher expected cost of rolling options is not difficult to see. Rolling options provides the same floor at the end of the hedging period as does holding a single option with a time to expiration equal to the hedging period, but it provides other features as well. In particular, rolling options provides reset opportunities along the way, opportunities to change the strategy and to obtain a defined payoff. These opportunities can be valuable, but they also make the strategy more costly.

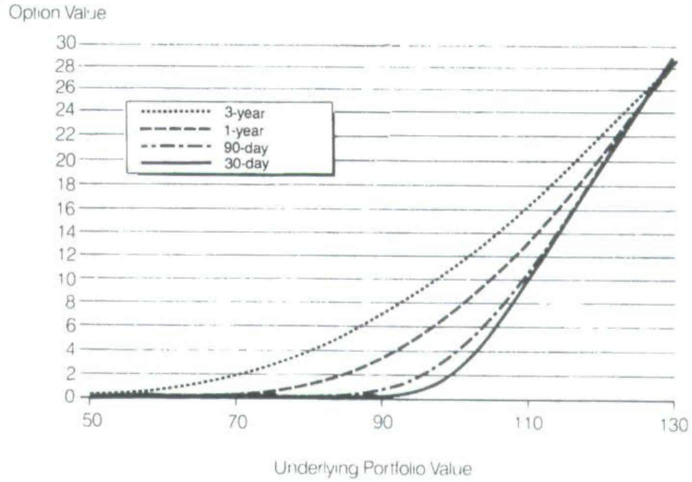

Figure 4
The curvature of a call option.

While rolling options has major drawbacks, options do play a vital role as an instrument for dynamic hedging. At any point in time, a dynamic hedge is determined by the model-determined hedge ratio. A hedge ratio of .80 means the hedge portfolio should have 80% of the exposure to the market that the underlying portfolio has. If a portfolio of $100 million is being dynamically hedged and the hedge ratio is .8, $20 million of the portfolio must be sold off or shorted out. One way of shorting out that exposure is by selling $20 million of futures. Since the short futures position will move one-for-one in the direction opposite of the portfolio being hedged, the net result will be deserved reduction in exposure.

Suppose there is an option in the market that moves in price by $.50 for each dollar move in the portfolio.[13] As an alternative to rolling $20 million of futures, one could write $40 million of the option to hedge out the $20 million of portfolio exposure. The net result for the dynamic hedge will be the same.

What is the difference between using this option—or, for that matter, any option—rather than the futures contract in the hedge? When is one better than the other? By construction, we know they are identical when the hedge is started. Their comparative performance lies in how well they meet the hedging needs as time progresses, and as the price of the underlying portfolio changes.

Figure 4 shows the payoff curve of a dynamic hedge for various times remaining to expiration. The payoff curves steepen gradually around the floor as time to expiration approaches. For portfolio values below the floor, the payoff becomes flat and the payoff profile reaches a slope of 45°, moving one-for-one with the market as the market rises. The curvature of the payoff increases as the time to expiration approaches, becoming regular at expiration. This curvature is called its gamma.

[13]Such an option has a delta of .5. The delta is the change in dollar value of an option with a one dollar change in the underlying security. The hedge ratio of the hedged portfolio is the delta of the underlying portfolio (which is, by definition, equal to unity) plus the delta of the protective put. Since the price of put options moves in the opposite direction of the underlying portfolio, they have a negative delta. The hedge ratio is thus less than one.

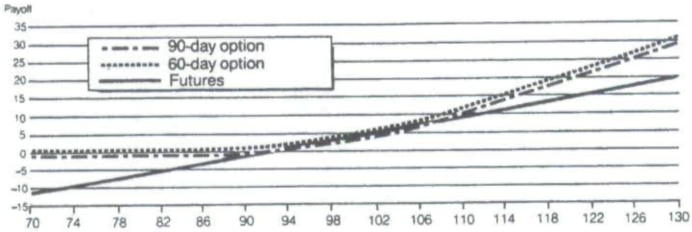

Figure 5-a
Hedging a 90-day with a 60-day option.

The greater the curvature, the greater the gamma. A straight line, having no curvature, has a gamma of zero.

The greater the gamma of the profile, the more demanding a future-based hedge will be. Small changes in the portfolio value will lead to frequent hedge adjustment, since the straight-line futures hedges will need to be drawn frequently to fit the step curvature. In an option hedge which itself has a high gamma, options can be used to better match the curvature of the hedge portfolio payoff profile. Figure 5-a illustrates the use of both futures and short-maturity options. The objective in this figure is to replicate the payoff of a protective put strategy with three months remaining. With such a short period remaining to expiration, the gamma is high, and a market option with two months to expiration better matches the curvature than does the linear futures hedge. Figure 5-b illustrates the other possibility, that the curvature of the dynamic hedging strategy is better matched by the futures. In this case, the dynamic hedge has three years to expiration, and the gamma of the market option is too severe to make it an attractive hedging vehicle.

As these two cases illustrate, it is the curvature of the hedge that dictates the desirability of using an option-based hedge. Because the gamma is a decreasing function of time, the comparatively short-maturity market options will be less useful in hedging, the longer the time to expiration of the hedge. Furthermore, since the gamma is lower the future the portfolio value moves from the exercise price (as is evident in Figure 4), options will be less useful the further the portfolio value moves from the floor. Table 2 lists the gamma of options with various times to expiration for a range of portfolio values. The decision to use a market option for hedging rather than the futures of the underlying asset itself depends on the following rule.

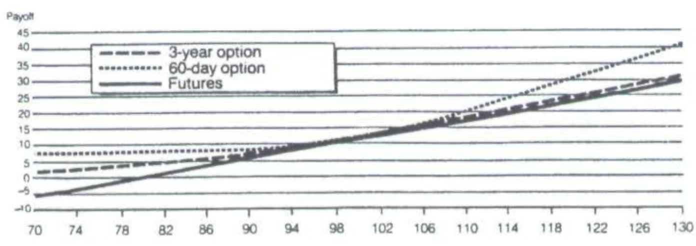

Figure 5-b
Hedging a 3-year with a 60-day option.

Table 2
OPTION GAMMA

The Curvature of Options as a Function of Time to Expiration

Portfolio Value	Three-Year Option	One-Year Option	60-Day Option	30-Day Option
95	0.0101	0.0195	0.0430	0.0501
96	0.0100	0.0195	0.0456	0.0571
97	0.0099	0.0193	0.0475	0.0629
98	0.0098	0.0192	0.0486	0.0670
99	0.0096	0.0190	0.0490	0.0692
100	0.0095	0.0187	0.0487	0.0692
101	0.0093	0.0184	0.0476	0.0672
102	0.0092	0.0181	0.0459	0.0633
103	0.0090	0.0177	0.0436	0.0580
104	0.0089	0.0173	0.0408	0.0516
105	0.0087	0.0169	0.0377	0.0448

[a]Assumptions: Exercise Price = \$100; Volatility = .20

Proposition 5

An Option Will be a More Efficient Hedge than a Futures if the Gamma of the Hedge Program is Greater than One-Half the Gamma of the Hedging Instrument

Since the gamma of a futures is zero, if the gamma of the hedge is less than half that of the option being contemplated as the hedging vehicle, the curvature of futures will better match the hedge.

Unless an option in the market has the same exercise price and time to expiration as the portfolio insurance program, it will not match both the required hedge ratio and gamma. An option or futures position can be adjusted to give the righthedge ratio—to have a slope equal to the slope of the hedged portfolio—but it will still deviate in curvature. The hedge will therefore need to be adjusted as the market changes. If more than one hedging instrument is used (an option and a futures, for example) both the hedge ratio and the curvature of the hedge position can be matched. We now have two instruments, and can choose the proportion of each in order to solve for two objectives: hedge ratio and curvature. Such a strategy is called a delta/gamma neutral hedge; the delta refers to the hedge ratio, and the gamma refers to the curvatures. Such hedges are used in the large-scale exposure management of option trading operations, and with some judgment can be applied successfully in dynamic hedging applications.[14]

[14]The judgment is needed because the options are also affected by the passage of time, yet another dimension that must be considered in dynamic hedging. Also, as the complexity of the hedging structure increases, the potential for misspecification errors in the hedge (due to errors in volatility estimates, mispricing or model design) increases. The relationship of delta, gamma and theta—the impact of time on option prices—is discussed in Bookstaber and Putcha (1985).

Proposition 6

Option-Based Hedges Cannot Hedge Away the Volatility Risk of Longer-Term Portfolio Insurance Programs

One argument for using options in dynamic hedging is that they eliminate the impact of uncertain market volatility on the hedging cost. The price of an option in the market is set according to expected volatility over the life of the option. Because the option cost is set at the time of purchase, changes in the realized volatility will not affect it. Changes in volatility will affect a dynamic hedge, however. It is the realized volatility over the course of the portfolio insurance program that determines its cost. If the volatility differs from the expected, the cost of the hedge will differ as well.

While an option contract locks in the volatility implied at the time of purchase, the strategy is subject to shifts in the market's assessment of volatility as the option is rolled. The effect of changes in volatility will be realized every time the option is rolled over and a new option, priced according to the then-existing expectations of volatility, is purchased. For example, if volatility rises, the price of the options purchased later in the hedging program will be higher. Since on average the market expectation of volatility will equal the volatility later realized in the market, the market options will not be able to hedge out the volatility risk for hedges that extend beyond the time to expiration of the market option itself.

IV. CONCLUSION

If portfolio insurance is simply defined as assuring a portfolio will not drop below a specified value, any number of trading rules will do. A stoploss strategy that completely moves into cash once the portfolio drops to the present value of the floor, a constant proportionality rule that moves linearly toward cash as the cushion between the portfolio value and the present value of the floor drops, a strategy of rolling listed options with an exercise price equal to the present value of the floor, the replication of a put option on the portfolio—all will provide the requisite protection.

How can we differentiate between portfolio insurance methods? What are the characteristics we should look for in a portfolio insurance model, and which of the above methods will best meet our needs?

In this article we have focused on several important characteristics. One is path independence. The impact of a strategy that is not path-independent on a portfolio's return characteristics cannot be clearly specified until the actual course of the market is observed. We have shown that only option-replicating strategies provide path independence.

A second characteristic is the time horizon of the hedging program. Many programs can profitably employ time-invariant, or perpetual, strategies, which have no specified ending date. Until now, all proposed perpetual strategies have been path-dependent; the only way to eliminate the need for resetting a program has been to accept path dependence. In this report we have shown that one can create a path-independent perpetual strategy through option replication.

A third characteristic that is often suggested as a criterion for portfolio insurance programs is simplicity of model design. Certainly when two models give the same result, the simpler model should be chosen. But as we have shown, many simplified models sacrifice analytical integrity. Why should a portfolio insurance strategy be chosen on the basis of simplicity when a more complex—but immediately computable—method provides superior characteristics?

A fourth characteristic is the certainty of payoff. Dynamic hedges by nature impose a cost, and this cost is a function of a number of uncertainties in the market including volatility, interest rates and model misspecification. We have shown how market options can be used to reduce this uncertainty by providing an adjustment for the curvature, or gamma, of the hedge. We have also shown how the option markets are frequently misused through rolling strategies that actually impose greater costs on the portfolio insurance program.

As portfolio insurance moves from straightforward equity applications to more difficult fixed income portfolios and applications in asset/liability and surplus management, the issues addressed in this report will bear increasingly greater weight. As floor is added upon floor, the plumb of the foundation becomes more critical. As the complexity of dynamic hedging increases, the analytical foundation will likewise become more important, and many heuristic approaches applied today will no longer be adequate.

Appendix

The basic set up, which we will subsequently modify, is taken from Harrison and Pliska (1981) and is given as follows:

The triplet (Ω, F, P) is a probability space; I is an index set, where I is either $[0, T]$ or $\{0, 1, 2, \ldots T\}$; and S is a $k + 1$ dimensional strictly positive stochastic process. The process S generates a filtration $\mathbf{F} = \{\mathrm{F}_t : t \in I\}$, where F_t is the smallest σ algebra such that each S_i with $i \leqq t$ is measurable. F_0 is the σ algebra generated by the null sets of P so S_0 is constant almost everywhere, F equals F_T, and $\mathbf{F}$ is assumed right continuous. The index set I represents time, the process S with coordinate functions at time t given by $S_t^0, \ldots S_t^k$ models the price of a fixed set of commodities, and the filtration $\mathbf{F}$ is a model for information flow which states that information at time t is only attainable through observations on past and present prices.

A trading strategy ϕ is a $k + 1$ dimensional locally bounded process whose components are predictable for the filtration $\mathbf{F}$. In addition, for the case of the continuous index $[0, T]$, further restrictions are placed upon the set of admissible trading strategies to rule out suicide strategies and to insure suitable integrability properties. For a strategy ϕ we define a value process $V(\phi)$ and a gains process $G(\phi)$ as follows:

$$V_t(\phi) = \phi_t \cdot S_t \tag{1}$$

$$G_t(\phi) = \int_0^t \phi_u \cdot dS_u \tag{2}$$

In the case of the discrete index set, $G_t(\phi)$ reduces to $\Sigma_{i=1}^t \phi_i(S_i - S_{i-1})$. A trading strategy ϕ is said to be self financing if for each t in I, $V_t(\phi) = V_0(\phi) + G_t(\phi)$.

Many stochastic processes, such as Brownian motion and discrete processes with independent increments have the following reflection principle:

If ω_1 and ω_2 are in Ω, with $S_t(\omega_1) = S_t(\omega_2)$ then there is a $\omega_{1,2}$ in Ω with $S_i(\omega_1) = S_i(\omega_1)$ if $i \leqq t$, and $S_i(\omega_{1,2}) = S_i(\omega_2)$ if $i \geqq t$.

For a model defined by (Ω, F, P), I, S, a time t, and a value $\bar{S} = S_t(\omega)$ for some ω in Ω, we can define a submodel by $(\Omega^{t,\bar{S}}, \mathrm{F}^{t,\bar{S}}, P^{t,\bar{S}})$, I_t, ${}^tS$. Here $\Omega^{t,\bar{S}} = \{\omega \in \Omega;\ S_t(\omega) = \bar{S}\}$, $\mathrm{F}^{t,\bar{S}} = \Omega^{t,\bar{S}} \cap \mathrm{F}$, $P^{t,\bar{S}} = P\,|_{S_t=S}$, $I = \{u \text{ in } I\text{: } u \geqq t\}$, and ${}^tS = S$ restricted to $\Omega^{t,\bar{S}}$. If ϕ is a trading strategy for the larger model and the model has the reflection principle, we can attempt to define for each ω_0 in $\Omega^{t,\bar{S}}$, the trading strategy $\omega_0\phi$ defined as follows: For each ω_1 in $\Omega^{t,\bar{S}}$, $\omega_0\phi(\omega_1) = \phi(\omega_{0,1})$. We will restrict our attention to those models with the reflection principle and with the property that for each ω_0 in $\Omega^{t,\bar{S}}$, and each trading strategy ϕ, $\omega_0\phi$ is a trading strategy for the smaller model.

For a given option ψ with exercise at time T, let ψ_t be the value at time t of the option. For example, if ψ_T is the value for a European call option, expiring at time T, with strike price E, and with the call on the commodity whose price is given by S^1, then $\psi_T = \max(S_T - E, 0)$. An option replication strategy replicating the option ψ is a self financing trading strategy ϕ with the property that for each t in I, $V_t(\phi) = \psi_t$. A self financing trading strategy ϕ is path independent at time t if there is a function f such that $V_t(\phi) = f(S_t)$.

Central to modern option theory is the no arbitrage axiom, which in our model is initially represented as follows:

If ϕ and ψ are self financing trading strategies with $V_T(\phi) = V_T(\psi)$ and $V_0(\phi) \geqq V_0(\psi)$, then $V_0(\phi) = V_0(\psi)$.

Theorem 1

If ϕ is a self financing trading strategy path independent for time T, then ϕ is path independent for times t less than T.

Proof

We need to show that if ω_1 is in Ω, then for every ω_0 in $\Omega^{t,S_t(\omega_1)}$, $(V_t\phi)\omega_1 = (V_t\phi)\omega_2$. Let $\bar{S} = S_t(\omega_1)$ and consider the model given by $(\Omega^{t,\bar{S}}, \mathrm{F}^{t,\bar{S}}, P^{t,\bar{S}})$, I_t and ${}^tS$. Let ω_2 be in $\Omega^{t,\bar{S}}$; by assumption $\omega_1\phi$ and $\omega_2\phi$ are both trading strategies. Since ϕ is self financing, $V_t\phi - V_0\phi = \int_0^t \phi_u \cdot dS_u$ and so for $\gamma \geqq t$, $V_\gamma\phi = \int_0^\gamma \phi_u \cdot dS_u + V_0\phi = \int_t^\gamma \phi_u \cdot dS_u + V_t\phi + [V_0\phi - V_t\phi + \int_0^t \phi_u \cdot dS_u] = \int_t^\gamma \phi_u \cdot dS_u + V_t\phi$. Therefore, both $\omega_1\phi$ and $\omega_2\phi$ are self financing in the restricted models. For some f, $V_T(\phi) = f(S_T)$, so $V_T(\omega_1\phi) = V_T(\omega_2\phi)$. Both $V_t(\omega_1\phi)$ and $V_t(\omega_2\phi)$ are constant almost everywhere $(P^{t,\bar{S}})$, so by the no arbitrage assumption $V_t(\omega_1\phi) = V_t(\omega_2\phi)$ and $(V_t\phi)\omega_1 = \phi_t(\omega_1) \cdot S_t(\omega_1) = V_t(\omega_1\phi) = V_t(\omega_2\phi) = \phi_t(\omega_2) \cdot S_t(\omega_2) = (V_t\phi)\omega_2$.

Theorem 2

A strategy that at time 0 consists of buying and holding a portfolio of European options is path independent and self financing up to the time of the first expiration.

Proof

A buy and hold strategy is self financing provided none of the securities throws off a cash flow. Each European option is path independent at its expiration and hence for each time up until its expiration. A portfolio of strategies, each path independent to time t, is itself path independent to time t.

Lemma 2

Let f_x: $\mathbf{R} \rightarrow \mathbf{R}$ be defined by $f_x(t) = \max(t - x, 0)$ and let $1(\cdot)$ define the constant function where $1(t) = 1$. Then the linear span of $\{1(\cdot), f_x\text{: } x \in [a, b]\}$ is the set of piecewise linear real valued functions on $[a, b]$.

Proof

A continuous function f on the interval $[a, b]$ is piecewise linear if there exists a partition $a = p_0 < p_1 < p_2 \cdots < p_n = b$ such that for $i = 1, 2, \ldots n$, f is linear on $[p_{i-1}, p_i]$.

Let g be a piecewise linear real valued function defined on $[a, b]$ and let $a = p_0 < p_1 \cdots < p_n = b$ be a partition of $[a, b]$ such that for $i = 0, 1, 2, \ldots n - 1$, $g \left|_{[p_i, p_{i+1}]}\right.$ is linear. For $i = 1, 2, \ldots n$, let x_i be any element of (p_{i-1}, p_i) and let $c_i = g'(x_i)$. Let $\hat{g} = g(a) \cdot 1(\cdot) + c_1 f_a + \Sigma_{i=1}^{n-1} (c_{i+1} - c_i) f_{p_i}$, then $\hat{g}(a) = g(a) \cdot 1 + c_1 \cdot 0 + \Sigma_{i=1}^{n-1} (c_{i+1} - c_i) \cdot 0 = g(a)$ and for $i = 1, 2, \ldots n$, $\hat{g}'(x_i) = c_1 + \Sigma_{j=1}^{i-1} (c_{j+1} - c_j) = c_i$, thus $\hat{g} = g$. Since $1(\cdot)$ and f_x are piecewise linear, the proof is completed.

It is well known that the closure of the piecewise linear functions in $[a, b]$ in the sup norm topology of $[a, b]$ is $[a, b]$ (the space of continuous real valued functions on $[a, b]$).

Theorem 3

Let ϕ be a self financing path independent strategy at time T where there is a continuous function f: $\mathbf{R} \rightarrow \mathbf{R}$ and index i such that $V_T(\phi) = f(S_T^i)$. Let ϵ and b be positive numbers, then there exists a strategy ψ consisting of buying and holding a portfolio of zero coupon bonds and European call options all expiring at time T whose value at time T is within ϵ of $V_T\phi$ provided S_t^i is in $[0, b]$.

Proof

The strategy ψ is not a trading strategy in our restricted sense, since it does not necessarily involve trading in the $k + 1$ commodities whose prices at time t are given by S_t. Nevertheless, a buy and hold strategy of zero coupon bonds and options is self financing and path independent up to maturity, so we let ψ_t represent its value at time t. The value of the zero coupon bond maturing at time T is constant at time T and independent of prices at time T. The value of the European call option on the security whose prices are given by S^i, with the strike price E and expiration at time T is given by $f_E(S_T^i)$. Theorem 3 is proven by a simple application of the previous lemma.

To prove the next theorem we need to slightly strengthen the no arbitrage axiom to include the following:

If α^j and β are the values of self financing strategies (not necessarily trading strategies), with $\alpha_T^j > \beta_T$ and $(\alpha_T^j - \beta_T) \underset{j}{\rightarrow} 0$, then $\alpha_0^j > \beta_0$ and $\alpha_0^j - \beta_0 \underset{j}{\rightarrow} 0$. Convergence is almost everywhere.

Theorem 4

Let ϕ be a self financing path independent strategy at time T. Let T_1 and T_2 be times in I, $(T_1 < T_2)$, and let f_1 and f_2 be continuous functions such that $V_{T_1}(\phi) = f_1(S_{T_1}^i)$ and $V_{T_2}(\phi) = f_2(S_{T_2}^j)$ for some indices i and j. Let ϵ, δ, a, and b be positive numbers such that for $S_{T_1}^i$ in $[a, b]$, $V_{T_1}(\phi) > \delta$. Then there exists a self financing strategy ψ consisting of, at time 0, buying a portfolio of zero coupon bonds and European options on commodity i expiring at T_1 and then at T_1 rebalancing and purchasing zero coupon bonds and European options on commodity j expiring at time T_2 such that for $(S_{T_1}^i, S_{T_2}^j)$ in $[a, b] \times [a, b]$, $|\psi_{T_1} - V_{T_1}(\phi)| + |\psi_{T_2} - V_{T_2}(\phi)| < \epsilon$.

Proof

For $\epsilon_1 > 0$, we can find portfolios ψ^1 and ψ^2, where ψ^1 is comprised of zero coupon bonds and options expiring at T_1 and ψ^2 is comprised of zero coupon bonds and options expiring at T_2 such that for $(S_{T_1}^i, S_{T_2}^j)$ in $[a, b] \times [a, b]$, $|\psi_{T_1} - V_{T_1}(\phi)| < \epsilon_1$ and $|\psi_{T_2} - V_{T_2}(\phi)| < \epsilon_1$. Without loss of generality we may assume that $\psi_{T_2} > V_{T_2}(\phi)$ and thus $\psi_{T_1}^2 \geqq V_{T_1}(\phi)$. At time 0, buy portfolio ψ^1, at time T_1, sell and purchase $\psi_{T_1}^1/\psi_{T_1}^2$ units of portfolio ψ^2. Now $|\psi_{T_1} - V_{T_1}(\phi)| < \epsilon_1$ and $|\psi_{T_1}^1/\psi_{T_1}^2 \cdot \psi_{T_2}^2 - V_{T_2}(\phi)| < \epsilon_1 + |\psi_{T_1}^1/\psi_{T_1}^2 - 1| \cdot |\psi_{T_2}^2|$, whenever $(S_{T_1}^i, S_{T_2}^j)$ is in $[a, b] \times [a, b]$. Now $V_{T_2}(\phi) = f_{T_2}(S_{T_2}^j)$, so for $S_{T_2}^j$ in $[a, b]$, $V_{T_2}(\phi)$ is bounded by some constant M and so $|\psi_{T_2}^2| \leqq M + \epsilon$. We can thus find ϵ_1 positive and small such that for $(S_{T_1}^i, S_{T_2}^j)$ in $[a, b] \times [a, b]$, $|\psi_{T_1}^1 - V_{T_1}(\phi)| < \epsilon/2$ and $|\psi_{T_1}^1/\psi_{T_1}^2 \cdot \psi_{T_2}^2 - V_{T_2}(\phi)| < \epsilon/2$.

Corollary

A path independent self financing strategy whose value at each time t is given by a continuous function f_t of the price S_t^i of a single traded commodity can be approximated arbitrarily closely by a rolling option strategy.

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