A Note on Pricing Asian Derivatives with Continuous Geometric Averaging

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A general expression is derived for the price of a European-style Asian contingent claim in which the terminal value depends on both the underlying asset price and the continuous geometric average of the price of the underlying asset over the life of the claim. Specific formulas are derived for Asian call, put, and binary options, as well as for the average strike binary options. © 1999 John Wiley & Sons, Inc. Jrl Fut Mark 19: 845–858, 1999

INTRODUCTION

Assume that trading is done in a perfect continuous security market with no transaction costs. Consider a security (for instance, a stock) for which the share price is governed by the Ito stochastic differential equation

$$dS = (\mu - D)Sdt + \sigma SdX \tag{1.1}$$

where μ and σ are positive constants, $\{X(t), t \geq 0\}$ is a standard Brownian motion, and D represents a constant rate of continuous dividend payment. Assume all risk-free investments earn the same fixed constant rate of continuously compounded return $r > 0$. Define, for each $t \geq 0$,

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$$I(t) = \int_0^t \log S(u)du. \tag{1.2}$$

Note that if $t > 0$, $\exp(I(t)/t)$ is the continuous geometric average share price over the time period $[0, t]$.

A derivative security based on the asset governed by eq. (1.1) in which the payoff function depends on some type of average behavior of the asset over its life is called an Asian derivative security. According to Zhang (1997), Bankers Trust was the first financial institution to offer Asian type derivatives in the form of options and offered them through their Tokyo office (hence the name). Because average asset prices exhibit less volatility than the prices themselves, Asian options are very useful when dealing with assets that are vulnerable to price manipulations, such as currencies and commodities including oil and metals. However, as a rule, this attenuation of volatility results in lower values for Asian options as compared to their vanilla (for instance, those priced using Black-Scholes) counterparts. Considered exotic, Asian options are mostly sold in the over-the-counter or off-exchange markets (for example, in interbank transactions) and therefore there is not much public data available on the details of specific contracts and prices.

In this note, an Asian derivative security with life extending over $[0, T]$ and terminal payoff function f that depends jointly on $I(T)$ and $S(T)$ is considered. It is assumed that the payoff may only be made at expiration time T (hence, this is a "European-style" Asian derivative). A general pricing expression is developed that gives the value of such a derivative at time t, $0 \leq t \leq T$.

Asian derivatives are discussed in Hull (1993), Karatzas (1997), Rogers et al. (1994), Wilmott et al. (1993), and Wilmott et al. (1997), for example. Though much progress has been made in studying distributions of arithmetic average processes of geometric Brownian motion (Dufresne 1998), for example), closed-form pricing formulae for Asian derivatives that use arithmetic averaging are unavailable. For these, numerical methods aimed at solving the PDEs satisfied by the price of the derivative security must be used. These are treated nicely in general in Wilmott et al. (1993), Wilmott et al. (1997), and Wilmott (1998) and specifically for Asian options in Rogers et al. (1994). Alternatively, approximation methods, such as those developed in Turnbull et al. (1991), are available. In the case treated here, namely continuous geometric averaging, explicit pricing formulae are available for some specific claims but are not given in many of the popular texts (for example, Duffie, 1996; Duffie, 1988; Hull, 1993; Karatzas, 1997; and Merton, 1994, to name a few). A notable exception is Kolb (1997) who gives an explicit formulae for the geometric-average-based call that includes both the discrete and continuously sampled cases (however, no derivation is given). Also, a brief non analytical discussion of the continuous geometric averaging case is given in Hull (1993), but the prices that would follow from that treatment would only be valid for time $t = 0$, and not at any other intermediate time. That treatment may be based on Kemna et al. (1990), where an exact formula is derived for the geometric-average–based Asian call option for the special case $t = 0$ and then used in a Monte Carlo-based attack on arithmetic-average-based Asian calls. Finally, Wilmott et al. (1993) and Wilmott (1997) offer a "recipe" for writing down the explicit formulae for the geometric-average-based call and put based on solving the corresponding PDE (with boundary conditions) satisfied by the value function, but the resulting formulae are incorrect, and the expressions given in Wilmott (1998) for the Asian call and put are also incorrect.

In this note a general formula is derived, using the risk-neutral valuation approach, for the price of an Asian derivative with continuous geometric averaging and payoff function f depending jointly on the underlying price at expiration and the continuous geometric average of the underlying price. The prototypical claim for this setting is the average-strike call or put, with payoff function (the notation $x^+ = (|x| + x)/2$ signifies the positive part of the real number x)

$$f(S, I) = (S - \exp(I/T))^+ \tag{1.3}$$

for the call, and

$$f(S, I) = (\exp(I/T) - S)^+ \tag{1.4}$$

for the put. There is no known closed form for these cases despite the use of the more tractable geometric average, but the general expression given herein makes these easily attacked via numerical integration. The general formula is then used to derive some explicit pricing formulas for specific claims with payoffs given as follows:

$$f(S, I) = (\exp(I/T) - K)^+ \tag{1.5}$$

$$f(S, I) = (K - \exp(I/T))^+ \tag{1.6}$$

$$f(S, I) = B1_{\{\exp(I/T)>K\}} \tag{1.7}$$

$$f(S, I) = B1_{\{\exp(I/T)<K\}} \tag{1.8}$$

$$f(S, I) = B1_{\{S>\exp(I/T)\}} \tag{1.9}$$

$$f(S, I) = B1_{\{S<\exp(I/T)\}}, \tag{1.10}$$

where $K > 0$ is a constant exercise price and $B > 0$ is a constant payoff. The payoffs (1.5) and (1.6) correspond to the call and put, respectively, (1.7) and (1.8) correspond to the binary call and put, respectively, and (1.9) and (1.10) correspond to the average-strike binary call and put, respectively. Finally, to correct the analyses set out in Wilmott et al. (1993) and Wilmott et al. (1997), the PDE approach to deriving the pricing formulae is illustrated for the case of the call.

Throughout, the notation 1_A stands for the indicator function that takes value 1 when the event A occurs, and 0 otherwise; $E(\cdot)$ and $E^*(\cdot)$ will denote mathematical expectation operators.

THE GENERAL PRICING FORMULA

Using Ito calculus, the share price satisfying (1.1) is given by

$$S(u) = S(t) \exp\left((\mu - D - \sigma^2/2)(u - t) + \sigma(X(u) - X(t))\right) \tag{2.1}$$

for $u \geq t \geq 0$. Using this in (1.2) it follows that for expiration time T, $T \geq t \geq 0$,

$$I(T) = I(t) + (T - t) \log S(t) + \frac{(\mu - D - \sigma^2/2)(T - t)^2}{2} + \sigma \int_t^T (X(u) - X(t))du. \tag{2.2}$$

Using (2.1), it also follows that

$$\ln S(T) = \ln S(t) + (\mu - D - \sigma^2/2)(T - t) + \sigma(X(T) - X(t)). \tag{2.3}$$

Next, consider the joint distribution of $(\ln S(T), I(T))$, conditioned on $S(t) = S$ and $I(t) = I$. The stochastic term in (2.2) has mean 0 and variance

$$\sigma^2 \int_t^T \int_t^T (\min(u, v) - t)dudv = \sigma^2 \frac{(T - t)^3}{3},$$

and the stochastic term in (2.3) has mean 0 and variance $\sigma^2(T - t)$. The covariance of these terms is

$$\sigma^2 E\left((X(T) - X(t)) \int_t^T (X(u) - X(t))du\right)$$

$$= \sigma^2 \int_t^T E(X(T) - X(t))(X(u) - X(t))du$$

$$= \sigma^2 \int_t^T (u - t)du = \sigma^2 \frac{(T - t)^2}{2}.$$

Thus, the random vector

$$\begin{pmatrix} \sigma(X(T) - X(t)) \\ \sigma \int_t^T (X(u) - X(t))du \end{pmatrix}$$

has a bivariate normal distribution with mean vector $(0,0)'$ and variance-covariance matrix given by

$$\sigma^2 \begin{pmatrix} (T - t) & \frac{(T - t)^2}{2} \\ \frac{(T - t)^2}{2} & \frac{(T - t)^3}{3} \end{pmatrix}.$$

It follows that, conditioned on $S(t) = S$ and $I(t) = I$,

$$\ln S(T) =^d \ln S + (\mu - D - \sigma^2/2)(T - t) + \sigma N_1 \sqrt{T - t} \quad (2.4)$$

$$I(T) =^d I + (T - t) \log S + \frac{(\mu - D - \sigma^2/2)(T - t)^2}{2}$$

$$+ N_2\sigma \sqrt{\frac{(T - t)^3}{3}}, \quad (2.5)$$

where $=^d$ signifies "equal in distribution" and where $(N_1, N_2)'$ has a bivariate normal distribution with mean vector $(0,0)'$ and variance-covariance matrix

$$\begin{pmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{pmatrix}.$$

A convenient way to represent this in terms of independent standard normal deviates is to let

$$\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} \rho Z_1 + Z_2 \sqrt{1 - \rho^2} \\ Z_1 \end{pmatrix} \tag{2.6}$$

where Z_1, Z_2 are independent and identically distributed standard normal random variables, and

$$\rho = \frac{\sqrt{3}}{2}. \tag{2.7}$$

Now suppose a terminal payoff function f is given that maps

$$\{(s, i): s \geq 0, \ -\infty < i < \infty\}$$

into the nonnegative real numbers. Assume that f is Borel measurable and that for any real a, b and $c \geq 0$, $d \geq 0$ the expectation $Ef(\exp(a + bN_1), c + dN_2)$ exists and is finite. If an Asian derivative security pays $f(S(T), I(T))$ at expiration time T, then its value at time t, $0 \leq t \leq T$ is given by $\exp(-r(T - t))E^* f(S(T), I(T))$ where E^* is expectation, conditioned on $S(t) = S$ and $I(t) = I$, taken under a probability measure with respect to which the r-discounted value of a portfolio consisting purely of the underlying asset is a martingale (see Baxter et al. (1996), for example, or Ross (1997) for an elementary treatment of this fact).

To find this probability measure, consider a portfolio initially consisting of 1 share of the underlying purchased for $S(t_0)$ at time t_0. The value of the portfolio at time $t > t_0$ is, accounting for dividend payments by reinvesting them in shares for the portfolio, $\exp(D(t - t_0))\, S(t)$. This is true because a dividend payment of $DS(u)du$ is made during each time interval $(u, u + du]$, so in units of shares, the dividend is Ddu. This means that the number of shares in the portfolio, N, evolves as $dN = N\, Ddu$ with $N(t_0) = 1$ so that the number of shares at time t is $\exp(D(t - t_0))$. At time t the discounted value of the portfolio is therefore $\exp(-(r - D)(t - t_0))S(t)$. To be a martingale, it must be true that

$$\begin{aligned} &E(\exp(-(r - D)(t - t_0))S(t) \mid S(u), u \leq t_0) \\ &= S(t_0)e^{-(r-D)(t-t_0)}e^{(\mu-D)(t-t_0)} = S(t_0), \end{aligned}$$

which is true for all $t \geq t_0 \geq 0$ if $\mu = r$ in (2.1) and we continue to treat X as a standard Brownian motion. Under this probability structure, (2.5) and (2.4) hold with μ replaced by r. Summarizing this leads to the following theorem.

Theorem 2.1. Let f be a Borel measurable function from

$$\{(s,\ i)\colon s \geq 0,\ -\infty < i < \infty\}$$

into the nonnegative real numbers satisfying

$$\int_{-\infty}^{\infty} f\,(e^{a+b(\rho z_1 + z_2\sqrt{1-\rho^2})},\ c + dz_1)\ \exp\left(-\frac{z_1^2 + z_2^2}{2}\right) dz_1 dz_2 < \infty$$

with $\rho = \sqrt{3}/2$ for all real a, b and all real $c \geq 0$, $d \geq 0$. Define

$$U_1(z_1,\ z_2;\ S,\ \tau) = Se^{(r-D-\sigma^2/2)\tau + \sigma(\rho z_1 + z_2\sqrt{1-\rho^2})\sqrt{\tau}}$$

and

$$U_2(z_1;\ S,\ I,\ \tau) = I + \tau \log S + \frac{(r - D - \sigma^2/2)\tau^2}{2} + \sigma z_1 \sqrt{\frac{\tau^3}{3}}.$$

Then the Asian contingent claim that pays $f(S(T), I(T))$ at time $T > 0$ has value

$$V(S,\ I,\ t) = e^{-r(T-t)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(U_1(z_1,\ z_2;\ S,\ T - t),\ U_2(z_1;\ S,\ I,\ T - t))\ \frac{\exp\left(-\dfrac{z_1^2 + z_2^2}{2}\right)}{2\pi}\ dz_1 dz_2$$

at time t when the share price is $S = S(t)$ and $I = I(t)$, $0 \leq t \leq T$. If in addition, $f(S, I) = g(I)$ for a suitable Borel function g, then the Asian contingent claim that pays $g(I(T))$ at time $T > 0$ has value

$$V(S,\ I,\ t) = e^{-r(T-t)} \int_{-\infty}^{\infty} g\left(I + (T - t)\log S + \frac{(r - D - \sigma^2/2)(T - t)^2}{2} + \sigma z \sqrt{\frac{(T - t)^3}{3}}\right) \frac{e^{-z^2/2}}{\sqrt{2\pi}}\ dz$$

at time t when the share price is $S = S(t)$ and $I = I(t)$, $0 \leq t \leq T$.

With the help of the following lemma, whose proof is a straight forward manipulation of probability integrals involving the standard normal, calls and puts can easily be priced:

Lemma 2.2. Let Z be a standard normal random variable and

$$\Phi(y) = \int_{-\infty}^{y} \frac{\exp(-z^2/2)}{\sqrt{2\pi}}\, dz$$

the standard normal cumulative distribution function. Then

$$\begin{aligned} &E(F\exp(\bar{\sigma}Z + \bar{\mu}) - k)^+ \\ &\quad = F\exp(\bar{\mu} + \bar{\sigma}^2/2)\Phi\left(\frac{\log(F/k) + \bar{\mu} + \bar{\sigma}^2}{\bar{\sigma}}\right) - k\Phi\left(\frac{\log(F/k) + \bar{\mu}}{\bar{\sigma}}\right) \end{aligned}$$

and

$$\begin{aligned} &E(F\exp(\bar{\sigma}Z + \bar{\mu}) - k)^- \\ &\quad = -F\exp(\bar{\mu} + \bar{\sigma}^2/2)\Phi\left(\frac{-\log(F/k) - \bar{\mu} - \bar{\sigma}^2}{\bar{\sigma}}\right) + k\Phi\left(\frac{-\log(F/k) - \bar{\mu}}{\bar{\sigma}}\right) \end{aligned}$$

for any constants F, k, $\bar{\mu}$, and $\bar{\sigma}$.

Taking $g(I) = (\exp(I/T) - K)^+$ in the theorem is tantamount to taking $k = K$ (the exercise price), $F = \exp(I/T)S^{(T-t)/T}$, $\bar{\mu} = (r - D - \sigma^2/2)(T - t)^2/2T$, and $\bar{\sigma} = (\sigma/T)\sqrt{(T - t)^3/3}$ in the lemma. This immediately gives the price for the Asian call with continuous geometric averaging:

$$\begin{aligned} V_{Call}(S, I, t) = e^{-r(T-t)} \Bigg[F\exp\Bigg(&\frac{(r - D - \sigma^2/2)(T - t)^2}{2T} \\ &+ \frac{\sigma^2(T - t)^3}{6T^2}\Bigg)\Phi(d_1) - K\Phi(d_2)\Bigg]; \end{aligned} \quad (2.8)$$

and putting $g(I) = (\exp(I/T) - K)^-$ immediately gives the price of the Asian put with continuous geometric averaging:

$$\begin{aligned} &V_{Put}(S, I, t) = e^{-r(T-t)} \Bigg[K\Phi(-d_2) - \\ &F\exp\left(\frac{(r - D - \sigma^2/2)(T - t)^2}{2T} + \frac{\sigma^2(T - t)^3}{6T^2}\right)\Phi(-d_1)\Bigg] \end{aligned} \quad (2.9)$$

where

$$d_1 = \frac{T\log(F/K) + (r - D - \sigma^2/2)(T - t)^2/2 + \sigma^2(T - t)^3/3T}{\sigma\sqrt{T - t)^3/3}} \quad (2.10)$$

and

$$d_2 = \frac{T \log(F/K) + (r - D - \sigma^2/2)(T - t)^2/2}{\sigma\sqrt{(T - t)^3/3}}. \quad (2.11)$$

Note that when $t = 0$ is substituted in (2.8) through (2.11), they reduce to the formulae given in Kemna et al. (1990) and Hull (1993). Therefore the Kemna et al. (1990) and Hull (1993) formulae correspond to the case in which there is T time left in the option life, whereas (2.8) through (2.11) give the values when there is $T - t$ time left in the option, for all $0 \leq t \leq T$. Also, the formula in Kolb (1997) agrees when it is taken to the continuous limit (that is, when continuous geometric averaging is used).

Using eq. (2.5) directly, the binary call and put prices are obtained using $g(I) = B1_{\{\exp(I/T)>K\}}$ and $g(I) = B1_{\{\exp(I/T)<K\}}$, respectively as:

$$V_{CNCall}(S, I, t) = e^{-r(T-t)} B\Phi(d_2) \quad (2.12)$$

and

$$V_{CNPut}(S, I, t) = e^{-r(T-t)} B\Phi(-d_2) \quad (2.13)$$

with d_2 given by (2.11).

The average-strike call or put, with payoff (1.3) or (1.4), respectively, is easily handled via an application of numerical integration along with the theorem. On the other hand, the average-strike binary options have explicit formulae. Using (1.9) and (1.10) in the theorem readily yields

$$V_{CNCall}^{AV\ STRIKE}(S, I, t) = e^{-r(T-t)} B\Phi\left(\frac{a - c}{\sqrt{b^2 + d^2 - 2\rho bd}}\right) \quad (2.14)$$

for the average-strike binary call, and

$$V_{CNPut}^{AV\ STRIKE}(S,I,t) = e^{-r(T-t)} B\Phi\left(\frac{c - a}{\sqrt{b^2 + d^2 - 2\rho bd}}\right) \quad (2.15)$$

for the average-strike binary put. In eqs. (2.14) and (2.15), ρ is given in (2.7), and

$$a = (r - D - \sigma^2/2)(T - t)(1 - (T - t)/2)$$

$$b = \sigma\sqrt{T - t}$$

$$c = \frac{I}{T} - \frac{t}{T}\ln S$$

$$d = \sigma\sqrt{\frac{(T - t)^3}{3}}.$$

PDE APPROACH

By using a continuously hedged portfolio of the underlying and the derivative security, it is essentially shown in Wilmott et al. (1993) and Wilmott et al. (1997) that the value W of an Asian derivative based on the continuous geometric mean, satisfies

$$\frac{\partial W}{\partial t} + \log S\,\frac{\partial W}{\partial I} + \frac{\sigma^2}{2}\,S^2\,\frac{\partial^2 W}{\partial S^2} + (r - D)S\,\frac{\partial W}{\partial S} - rW = 0.$$

(Here a slight adjustment to the portfolio to account for the dividend payment has been made.) Assuming that the final condition

$$W(S,\, I,\, \mathrm{T}) = f(I)$$

depends only on I, the substitutions

$$y = \frac{I + (T - t)\log S}{T}$$

$$W(S,\, I,\, t) = V(y,\, t)$$

transforms the PDE to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\left(\frac{T - t}{T}\right)^2 \frac{\partial^2 V}{\partial y^2} + (r - D - \sigma^2/2)\left(\frac{T - t}{T}\right)\frac{\partial V}{\partial y} - rV = 0.$$

Finally, substituting

$$F = \exp(y)$$

yields the PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\left(\frac{T - t}{T}\right)^2 F^2\,\frac{\partial^2 V}{\partial F^2} \tag{3.1}$$

$$+ \left[(r - D - \sigma^2/2)\left(\frac{T - t}{T}\right) + \frac{\sigma^2}{2}\left(\frac{T - t}{T}\right)^2\right] F\,\frac{\partial V}{\partial F} - rV = 0.$$

Eq. (3.1) is nearly a Black-Scholes PDE with time-varying volatility, time-varying interest rate, and time-varying rate of continuous dividend payment. In fact, if the coefficient of V were $r(T - t)/T$, then this would be a Black-Scholes PDE. Nevertheless, the technique used to solve the Black-Scholes PDE with time-varying coefficients presented in Wilmott et al. (1993) and Wilmott et al. (1997) still works for (3.1). In particular, by making the further substitutions

$$\bar{F} = Fe^{\alpha(t)}$$
$$\bar{V} = Ve^{\beta(t)}$$

and

$$\bar{t} = \gamma(t)$$

where the functions α, β, and γ are given by

$$\begin{aligned}\alpha(t) &= \int_t^T \left[(r - D - \sigma^2/2)\left(\frac{T - u}{T}\right) + \frac{\sigma^2}{2}\left(\frac{T - u}{T}\right)^2\right] du \\ &= \left(r - D - \frac{\sigma^2}{2}\right)\frac{(T - t)^2}{2T} + \frac{\sigma^2(T - t)^3}{6T^2}\end{aligned}$$

$$\beta(t) = r(T - t)$$

and

$$\begin{aligned}\gamma(t) &= \int_t^T \sigma^2\left(\frac{T - u}{T}\right)^2 du \\ &= \frac{\sigma^2(T - t)^3}{3T^2},\end{aligned}$$

eq. (3.1) transforms to

$$\frac{\partial\bar{V}}{\partial\bar{t}} = \frac{\bar{F}^2}{2}\frac{\partial^2\bar{V}}{\partial\bar{F}^2}. \tag{3.2}$$

This latter equation may be solved for a particular solution, say $\bar{V}$, independently of the parameters, and in terms of the original variables, the solution of eq. (3.1) satisfying the final condition

$$V(F, T) = \bar{V}(F, 0)$$

is then

$$V(F,\, t) \;=\; e^{-\beta(t)}\bar{V}(Fe^{\alpha(t)},\, \gamma(t)). \tag{3.3}$$

Repeating the analysis of eq. (3.1) for the classical Black-Scholes equation with constant parameters, namely

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} F^2 \frac{\partial^2 V}{\partial F^2} + (r - D)F \frac{\partial V}{\partial F} - rV = 0, \tag{3.4}$$

would entail taking $\alpha(t) = \int_t^T (r - D)du = (r - D)(T - t)$, $\beta(t) = \int_t^T r du = r(T - t)$, and $\gamma(t) = \int_t^T \sigma^2 du = \sigma^2(T - t)$ en route to eq. (3.2). Of course, the solution to (3.4) satisfying $V(F,T) = (F - K)^+$ (i.e. the call) is given by the well-known Black-Scholes formula,

$$V_{BS}(F,\, t) \;=\; e^{-D(T-t)}F\Phi\left(\frac{\log(F/K) + (r - D + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}\right)$$

$$-Ke^{-r(T-t)}\Phi\left(\frac{\log(F/K) + (r - D - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}\right),$$

so that the particular solution to (3.2) is found by solving eq. (3.3):

$$\bar{V}(\bar{F},\, \bar{t}) \;=\; e^{r(T-t)}V_{BS}(e^{-(r-D)(T-t)}\bar{F},\, T - \bar{t}/\sigma^2) \tag{3.5}$$

$$= \bar{F}\Phi\left(\frac{\log(\bar{F}/K) + \bar{t}/2}{\sqrt{\bar{t}}}\right) - K\Phi\left(\frac{\log(\bar{F}/K) - \bar{t}/2}{\sqrt{\bar{t}}}\right).$$

Using (3.5) for $\bar{V}$ in eq. (3.3), and substituting for α, β, and γ as given preceding (3.2), yields the same formula for the Asian call as given in the risk-neutral analysis above. It is noted once again that the recipe outlined in Wilmott et al. (1993), pp. 196–197 and Wilmott et al. (1997), p. 232, will not lead to the correct formulae for the Asian derivative with continuous geometric averaging.

CONCLUSIONS

A general formula was developed for pricing Asian derivatives that use the continuous geometric mean of the underlying share price under the conditions that the terminal payoff depends on the underlying asset price at termination and its geometric mean at termination and that a constant and continuous rate of dividend payment from the underlying applies. This in turn was used to exhibit explicit pricing formulae for the call, put, and binary call and binary put for an Asian option that uses the continuous geometric mean. In addition, explicit formulae were derived for the average-strike binary Asian options. The formulae for the put and call are apparently not widely known or published, though other authors have alluded to the fact that they are tractable due to the use of the geometric mean as opposed to the arithmetic mean. Given the increasing interest in the mathematics of derivative pricing, it seems timely to present an exposition of these now.

Financially speaking, it seems of little importance whether the arithmetic or geometric mean is used in designing Asian derivatives. Indeed, the object is to base the claim value on some measure of central tendency of the share price over a time period, and for this purpose it seems just as effective to use the geometric mean, especially since explicit pricing formulae are available.

Finally, the explicit formulae given here are also important because they provide the ingredients necessary to implement variance reduction techniques, such as the method of control variates (see Ross, 1997a, for example), in computing Monte Carlo approximations for derivative securities involving the less tractable arithmetic average of the underlying price.

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