A Flexible Binomial Option Pricing Model

YISONG "SAM" TIAN*

This article develops a flexible binomial model with a "tilt" parameter that alters the shape and span of the binomial tree. A positive tilt parameter shifts the tree upward while a negative tilt parameter does exactly the opposite. This simple extension of the standard binomial model is shown to converge with any value of the tilt parameter. More importantly, the binomial tree can be recalibrated through the tilt parameter in order to position nodes relative to the strike price or barrier of an option. The rate of convergence is improved as a result. © 1999 John Wiley & Sons, Inc. Jrl Fut Mark 19: 817–843, 1999

INTRODUCTION

The binomial option pricing model of Cox, Ross, and Rubinstein (1979; CRR hereafter) is a well known and widely used model for valuing standard options. The model is straightforward to use and generally converges fairly fast to the continuous-time limit—the Black-Scholes model. It also allows for a clear understanding, especially for beginners, of arbitrage pricing, hedging, and replicating portfolios. Various extensions to the original CRR binomial model have been proposed, such as Boyle (1988), Nelson and Ramaswamy (1990), Hull and White (1990), Tian (1993),

The author would like to thank Phelim Boyle, Ken Vetzal, and two anonymous referees for helpful comments and suggestions. A special thank goes to editor Robert I. Webb for his guidance. An earlier version of the paper won the best paper award in options and futures at the 1996 annual meeting of the Northern Finance Association, sponsored by the *Chicago Board of Trade*.
*Correspondence author, Schulich School of Business, York University, Toronto, ON M3J 1P3, Canada.

■ *Yisong "Sam" Tian is an Associate Professor of Finance in the Schulich School of Business at York University in Toronto, Ontario.*

The Journal of Futures Markets, Vol. 19, No. 7, 817–843 (1999)

CCC 0270-7314/99/070817-27

and Leisen and Reimer (1995), to just mention a few. The motivation of these subsequent models is either improving the rate of convergence or generalizing the model for more complex stochastic processes or for pricing more complex derivatives. Both binomial and trinomial extensions have been developed, which are commonly known as tree-based models.

It is also well known that the convergence of tree-based models, binomial or trinomial, is not smooth. In other words, more time steps may not guarantee more accurate prices. Depending on the position of the strike price (for standard options) or barrier (for barrier options) relative to nodes in the tree, the approximation error may vary widely. For example, Heston and Zhou (1997) observe that the pricing error of the CRR binomial model for standard options fluctuates between the order of $O(1/N)$ and $O(1/\sqrt{N})$, as N, the number of time steps, increases. Boyle and Lau (1994) and Ritchken (1995) show that the pricing error of standard tree-based models for barrier options fluctuates widely in a sawtooth pattern. This is a serious problem because even significantly more time steps may not guarantee more accurate prices. Smooth convergence is thus desirable because more time steps do guarantee more accurate prices. More importantly, extrapolation methods can be used to enhance convergence in this case.

Comparing with trinomial (or higher dimensional multinomial) models, binomial models such as the CRR model lack flexibility in the sense that the jump size of the binomial tree is fixed for a given set of option parameters and time step (N). Trinomial trees, on the other hand, are more flexible because the jump size is only fixed up to an arbitrary parameter (called the dispersion parameter). Ritchken (1995), Cheuk and Vorst (1996), and Boyle and Tian (1997) utilize this flexibility of the trinomial model to value barrier options more accurately. The improvement in numerical accuracy is quite impressive.

This article develops a flexible binomial model with a "tilt" parameter that alters the shape and span of the binomial tree. The original CRR binomial model can be regarded as having zero tilt, because the asset price after an up (a down) move followed by a down (an up) move remains unchanged from before. In other words, the central nodes of the binomial tree form a horizontal line passing through the initial asset price.[1] In the flexible binomial model, the tree can tilt upwards or downwards if the tilt parameter is positive or negative, respectively. With a positive tilt parameter, the up (down) move is larger (smaller) than the corresponding up (down) move in a standard tree. Consequently, the central nodes form an

[1]The central nodes of a binomial tree are defined as the middle node in a time period. In other words, there are equal numbers of nodes above and below it. The initial asset price is a central node.

upward sloping line. The resulting tree span is thus shifted upward. The exact opposite is true for binomial trees with a negative tilt parameter.

This simple extension of the CRR model is shown to converge with any finite value of the tilt parameter. More importantly, the flexibility of the new model allows for appropriate adjustment to the binomial tree so that standard and barrier options can be valued more accurately. Specifically, smooth convergence is possible for standard European and American options, and extrapolation methods are used to enhance the rate of convergence. For barrier options, the binomial tree is adjusted so that the barrier is positioned optimally relative to layers of nodes in the tree which enhances numerical accuracy.

THE FLEXIBLE BINOMIAL MODEL

Consider the pricing of a derivative security written on an underlying asset for which the price follows a geometric Brownian motion given by:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz, \tag{1}$$

where μ and σ are the instantaneous drift and volatility rate, respectively. For the purpose of pricing derivative securities, the above stochastic process can be restated under the Equivalent Martingale Measure (EMM) (i.e., the risk-neutral probability measure), by replacing μ with r, the risk-free rate. This is different from nature's probability measure, which is based on the expected drift rate, μ.

Consider the pricing of an option maturing at time T. In an N-period binomial model, the time to maturity $[0, T]$ is partitioned into N equal subintervals of length $\Delta t = T/N$. Let $t_i = i\Delta t$ denote the partition points, for $i = 0, 1, 2, \ldots, N$. During each subinterval (also called a time step), $(t_i, t_{i+1}]$, the asset price is assumed to remain constant until the end of the subinterval. If the asset price is S at the beginning of the subinterval, it is assumed to jump either upward to uS with probability p or downward to dS with probability $1 - p$, where $0 < d < 1 < u$ and $0 \leq p \leq 1$. The number N is generally referred to as the number of time steps or periods. The binomial model is fully specified by the jump parameters u and d and the probability parameter p. Given these parameters, a unique binomial tree can be built using forward induction.

Label nodes in the tree as (i, j) where i indicates the time period (t_i) and j is the number of up moves from node $(0, 0)$ to node (i, j). The asset price at node (i, j) is given by:

$$S(i, j) = S_0 u^j d^{i-j} \tag{2}$$

where $0 \leq j \leq i$ and S_0 is the initial asset price at $t = 0$. Given this binomial tree of asset prices, options can be valued using a backward recursive procedure. The terminal payoff of the option is determined first, and option values in prior time periods are calculated as the expected future option value discounted at the risk-free rate.

The binomial parameters must be chosen such that option prices converge to the continuous-time limit. This requires that the discrete distribution of the asset price, represented by the binomial tree, converges to the continuous-time limit of a lognormal distribution. The risk-neutrality condition imposes a restriction on the probability, which also has the interpretation that the binomial drift term matches exactly its continuous-time counterpart:

$$pu + (1 - p)d = e^{r\Delta t}.$$

This simplifies to the familiar probability formula:

$$p = \frac{e^{r\Delta t} - d}{u - d}. \tag{3}$$

It can be shown that option prices calculated using the binomial model converge to the Black-Scholes price as long as the jumps are specified as follows:

$$u = e^{\sigma\sqrt{\Delta t} + O(\Delta t)}, \text{ and } d = e^{-\sigma\sqrt{\Delta t} + O(\Delta t)}, \tag{4}$$

where $O(\Delta t)$ denotes terms of order Δt or higher.

To see this, it is sufficient to verify that the first two moments of the binomial distribution converge to their continuous-time counterparts and the conditions of the central limit theorem are satisfied. Define $\hat{\mu}$ and $\hat{\sigma}^2$ as the mean and variance of $S(t + \Delta t)/S(t)$ implied by the binomial tree, conditional on $S(t)$:

$$\hat{\mu} = pu + (1 - p)d,$$

$$\hat{\sigma}^2 = p(u - \hat{\mu})^2 + (1 - p)(d - \hat{\mu})^2.$$

Using Taylor's expansion, it is trivial to show that:

$$u = 1 + \sigma\sqrt{\Delta t} + O(\Delta t),$$

$$d = 1 - \sigma\sqrt{\Delta t} + O(\Delta t),$$

$$p = \frac{1}{2} + O(\sqrt{\Delta t}).$$

A little algebra simplifies the expression for the mean and variance to.[2]

$$\hat{\mu} = e^{r\Delta t},$$
$$\hat{\sigma}^2 = \sigma^2 \Delta t + O(\Delta t^{3/2}).$$

The first equation above shows that the binomial mean of $S(t + \Delta t)/S(t)$ matches exactly its continuous-time counterpart, $e^{r\Delta t}$. The binomial variance of $S(t + \Delta t)/S(t)$ does not exactly match its continuous-time counterpart, $e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1)$. However, it is straightforward to show that:

$$\hat{\sigma}^2 - e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1) = O(\Delta t^{3/2}).$$

Consequently, the volatility term is matched in the limit as $\Delta t \to 0$.

Given the properly matched mean and variance, the binomial distribution converges to the lognormal distribution defined by eq. (1) (with μ replaced with r). To ensure option prices calculated from the binomial model converge to the Black-Scholes price, the following condition of the central limit theorem needs to be verified:

$$\frac{p|u - \hat{\mu}|^3 + (1 - p)|d - \hat{\mu}|^3}{\hat{\sigma}^3 \sqrt{N}} \to 0, \tag{5}$$

as $\Delta t \to 0$. Using Taylor's expansion, it is straightforward to show that:

$$\frac{p|u - \hat{\mu}|^3 + (1 - p)|d - \hat{\mu}|^3}{\hat{\sigma}^3 \sqrt{N}} = O(\sqrt{\Delta t}).$$

Convergence of the binomial model to its continuous-time limit is thus ensured.[3]

To construct the so-called flexible binomial model (the FB model hereafter), the following specification is proposed:[4]

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t} + \lambda\sigma^2 \Delta t}, \\ d &= e^{-\sigma\sqrt{\Delta t} + \lambda\sigma^2 \Delta t}, \end{aligned} \tag{6}$$

where λ is an arbitrary constant, called the "tilt parameter," that can be

[2]The first equation follows directly from the risk-neutrality condition and the definition of $\hat{\mu}$. It is stated here with the second equation for completeness.

[3]This argument is similar to the one used in Cox, Ross, and Rubinstein (1979).

[4]Leisen (1998) independently proposed a binomial model that has some similar features as the flexible binomial model developed here. See his SMO model on p. 17. However, his choice of u and d is rather ad hoc. A systematic way of determining u and d is proposed here through the choice λ.

positive, zero or negative. Since the FB model is specified only up to an arbitrary constant, it has an extra degree of freedom over the standard binomial model. This degree of freedom may allow us to modify the model in certain ways in order to accomplish a particular objective. Several interesting use of the arbitrary constant is presented in subsequent sections. In addition, λ does not have to be constant. As long as it is bounded (i.e. $\lambda = O(1)$), $\lambda\sigma^2\Delta t = O(\Delta t)$ holds. This ensures that eq. (6) conforms to eq. (4), and the convergence of the FB binomial model is thus ensured.[5] This property will become quite useful in subsequent sections.

It is important to double check if nonnegative probabilities are obtained with the redefined jumps u and d in eq. (6). This nonnegativity condition is equivalent to the no arbitrage condition. From eq. (3), it is clear that nonnegative probabilities are ensured if:

$$d \leq e^{r\Delta t} \leq u.$$

It is trivial to show that this is equivalent to the following condition:

$$\left|\lambda - \frac{r}{\sigma^2}\right| \leq \frac{1}{\sigma\sqrt{\Delta t}}. \tag{7}$$

It is clear that any bounded value of λ satisfies this condition for a sufficiently small Δt (or sufficiently large N). This reenforces convergence, which is already ensured by the central limit theorem.[6]

The CRR binomial model is a special case of the FB binomial model with $\lambda = 0$. In other words, the CRR binomial model defines the following jumps and probability:

$$\begin{aligned} u_0 &= e^{\sigma\sqrt{\Delta t}}, \\ d_0 &= e^{-\sigma\sqrt{\Delta t}}, \\ p_0 &= \frac{e^{r\Delta t} - d_0}{u_0 - d_0}. \end{aligned} \tag{8}$$

In that sense, the FB model is a generalization of the original CRR model.

The arbitrary parameter λ can be regarded as the degree of tilt in the binomial tree. When $\lambda = 0$, the CRR binomial model is realized and the center of the tree forms a horizontal line. In other words, an up move

[5]In other words, convergence is ensured if there exists a finite, positive number, M, such that $|\lambda| \leq M$.

[6]Negative probability may occur when N is small. This is the case even for the CRR model ($\lambda = 0$), which is clear from eq. (7).

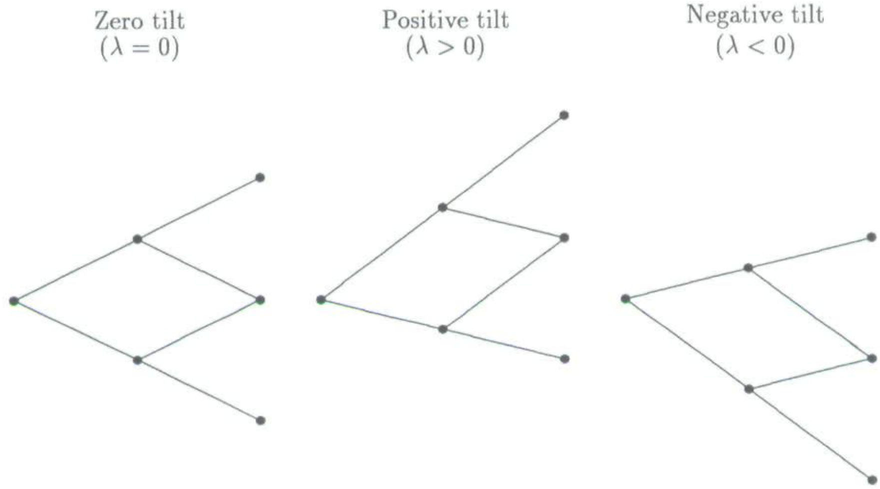

FIGURE 1
The flexible binomial tree for different λ values.

and a down move bring the asset price to exactly the same level it started from. This is obvious as $u_0 d_0 = 1$ by definition. The tree has zero tilt in this case. If $\lambda > 0$, the binomial tree tilts upwards, meaning that an up move and a down move raise the level of the asset price. When $\lambda < 0$, the exact opposite is true. Figure 1 illustrates the effect of the "tilt" parameter on binomial trees.

Table I presents some numerical results from applying the FB binomial model to call options. The asset price is 100, the strike price is 100, the asset price volatility is 0.2, the maturity of the option is six months, and the riskfree rate is 0.06. Three λ values are used in the table, 1, 0, and -1, representing positive, zero and negative tilt. The number of time steps used ranges from 50 to 5,000. The Black-Scholes price of the option is also provided in the table for comparison purposes. It is clear that binomial prices converge to the Black-Scholes price for all three λ values, including the special case of the CRR binomial model ($\lambda = 0$). It appears that the pricing error of the binomial model is less than a penny with 150 time steps or more for all three values of the tilt parameter. This confirms the earlier analytical analysis of the convergence properties of the FB binomial model.

ENHANCED CONVERGENCE FOR STANDARD OPTIONS

Because the FB binomial model converges for any value of the tilt parameter (λ), it might be possible to select a particular value that enhances

TABLE I

Call Prices with Different Tilt Parameter Values

Time steps (N)	$\lambda = 1$	$\lambda = 0$	$\lambda = -1$
50	7.1829	7.1276	7.1785
100	7.1648	7.1417	7.1624
150	7.1553	7.1464	7.1537
200	7.1491	7.1488	7.1480
300	7.1576	7.1512	7.1570
400	7.1591	7.1524	7.1585
500	7.1585	7.1531	7.1580
600	7.1572	7.1535	7.1568
700	7.1557	7.1539	7.1554
800	7.1542	7.1541	7.1539
900	7.1557	7.1543	7.1554
1000	7.1565	7.1545	7.1563
5000	7.1556	7.1556	7.1556
Black-Scholes price:		7.1559	

Notes: The asset price is 100, the strike price is 100, the asset price volatility is 0.2, the call option has six months remaining to maturity, the risk-free rate is 0.06, and the number of time steps ranges from 50 to 5,000. Three values for the tilt parameter (λ) are considered, covering positive, zero, and negative tilt, respectively.

the rate of convergence of binomial prices to their continuous-time limit. In this section, it is shown how the value of λ should be selected in order to enhance the rate of convergence for European calls and American puts. The basic idea and results apply to other types of standard options as well.

It is well known that option prices calculated using the CRR binomial model converge to the Black-Scholes price in a wavy, erratic fashion. Heston and Zhou (1997) show that the pricing error of an N-step CRR binomial model for standard options fluctuates between the rate of $O(1/N)$ and $O(1/\sqrt{N})$. This means that convergence is not smooth for the CRR model, and the pricing error does not decline monotonically as N increases. In other words, there are refinements for which the accuracy of the approximation is quite good while the pricing error can be quite large for other refinements. Figure 2 illustrates such convergence behavior for an in-the-money call option with the following parameters: The asset price is 100, the strike price is 95, the volatility is 0.2, the maturity of the option is six months, and the risk-free rate is 0.06. The line labeled "CRR" is a plot of the pricing error of the CRR model against the number of time steps. The pricing error is defined as the difference between the CRR price and the Black-Scholes price. As the number of time steps increases from 10 to 100, it is clear from Figure 2 that convergence is not smooth. Furthermore, the pricing error oscillates between positive

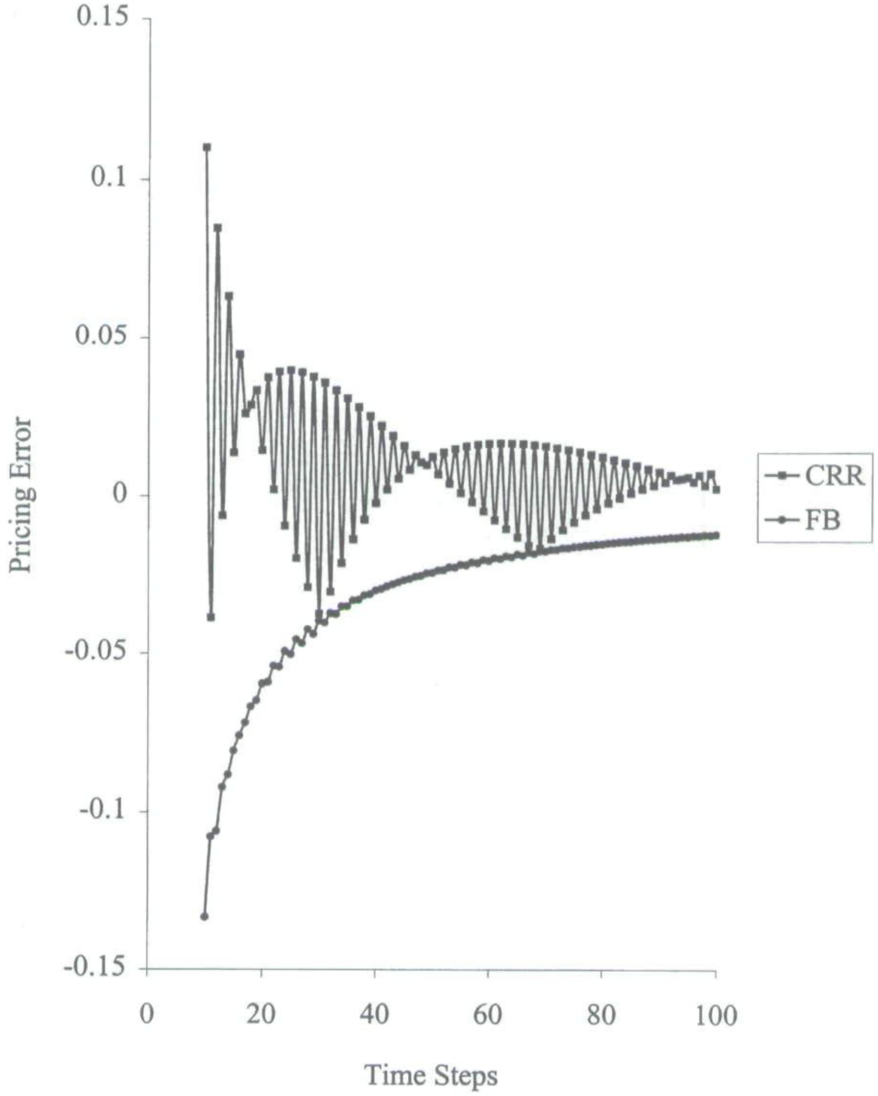

FIGURE 2

Smoothness of convergence. The asset price is 100, the strike price is 95, the asset price volatility is 0.2, the call option has six months remaining to maturity, the riskfree rate is 0.06, and the number of time steps ranges from 10 to 100. In the figure, CRR and FB denote prices from the CRR and flexible binomial model, respectively.

and negative values. Such convergence behavior is quite common, not just limited to the particular case shown here.

Smooth convergence is important since extrapolation techniques can be used to greatly enhance the rate of convergence. With the CRR binomial model, extrapolation techniques cannot be used because convergence is not smooth. With the FB binomial model, however, there is a simple way to achieve smooth convergence. This is done by selecting a tilt parameter such that a node in the tree coincides exactly with the strike price at the maturity of the option.[7]

To see how this is done, consider an N-period FB binomial model. The terminal asset price after N periods and a total of j up moves is given by:

$$S(N, j) = S_0 u^j d^{N-j}, \ 0 \leq j \leq N.$$

Let the initial value of λ be:

$$\lambda_0 = 0,$$

which represents the CRR model. The node closest to the strike price X, (N, j_0), can be determined by solving the following equation:

$$S_0 u_0^{\eta} d_0^{N-\eta} = X. \tag{9}$$

This leads to a solution for η:

$$\eta = \frac{\log(X/S_0) - N \log (d_0)}{\log(u_0/d_0)}. \tag{10}$$

Of course, the right-hand side of eq. (10) is usually not an integer. The closest node to X is found by defining j_0 as:

$$j_0 = \left[\frac{\log(X/S_0) - N \log (d_0)}{\log(u_0/d_0)}\right], \tag{11}$$

where $[\cdot]$ denotes the closest integer to its argument. To ensure that node (N, j_0) coincides exactly with the strike price X, a new λ is selected such that:

$$S_0 u^{j_0} d^{N-j_0} = X. \tag{12}$$

Solving for λ, the following equation is derived:

$$\lambda = \frac{\log(X/S_0) - (2j_0 - N)\sigma\sqrt{\Delta t}}{N\sigma^2 \Delta t}. \tag{13}$$

With this particular choice of λ, the strike price is always located on node

[7]The idea of placing the strike price at an optimal position relative to nodes in the tree is not new. See Leisen and Reimer (1995).

TABLE II

Prices and Error Ratios for European Call Options

	The flexible binomial model			*The CRR binomial model*		
Time steps N	*Price*	*Error*	*Error ratio*	*Price*	*Error*	*Error ratio*
			European call			
25	10.139765	−0.050294	—	10.229789	0.039731	—
50	10.165893	−0.024166	2.081184	10.202537	0.012478	3.183973
100	10.178175	−0.011883	2.033633	10.192395	0.002337	5.340583
200	10.184097	−0.005962	1.993260	10.195410	0.005352	0.436570
400	10.187085	−0.002974	2.004896	10.192466	0.002408	2.222578
800	10.188570	−0.001489	1.997437	10.189847	−0.000211	−11.413150
1600	10.189314	−0.000745	1.998967	10.190394	0.000336	−0.627982
3200	10.189686	−0.000372	2.000770	10.190234	0.000175	1.916679
6400	10.189873	−0.000186	2.003083	10.190230	0.000172	1.019975
Black-Scholes	10.190058			10.190058		
			American put			
25	2.493905	—	—	2.553684	—	—
50	2.507587	−0.013681	—	2.530140	0.023544	—
100	2.513679	−0.006092	2.245737	2.524845	0.005294	4.446938
200	2.516809	−0.003130	1.946452	2.525224	−0.000379	−13.9664
400	2.518496	−0.001687	1.855366	2.522484	0.002740	−0.138346
800	2.519292	−0.000796	2.119450	2.520055	0.002429	1.128062
1600	2.519672	−0.000380	2.093843	2.520397	−0.000342	−7.102269
3200	2.519856	−0.000184	2.064632	2.520230	0.000168	−2.039447
6400	2.519947	−0.000091	2.013085	2.520205	0.000025	6.773305

Notes: The asset price is 100, the strike price is 95, the asset price volatility is 0.2, the call option has six months remaining to maturity, the risk-free rate is 0.06, and the number of time steps starts at 25 and doubles each time subsequently. The left side of the table reports prices, pricing errors, and error ratios from the flexible binomial model. The corresponding results for the CRR binomial model are reported on the right side of table.

(N, j_0) at the maturity of the option. It is trivial to show that this λ value can be restated as:[8]

$$\lambda = \frac{2(\eta - j_0)\sqrt{\Delta t}}{\sigma T}. \tag{14}$$

As $|\eta - j_0| < 0.5$ by definition, $\lambda \to 0$ as $\Delta t \to 0$. Nonnegative probabilities are ensured as eq. (7) is clearly satisfied for sufficiently small Δt. Convergence is thus ensured even though λ depends on Δt.

Table II reports prices of European call and American put options, calculated using the FB binomial model with λ value determined by eq. (13). The asset price is 100, the strike price is 95, the asset price volatility is 0.2, the maturity of the option is six months, and the riskfree rate is

[8]Eq. (14) can be derived by dividing eq. (12) to eq. (9) and rearranging terms.

0.06. Prices calculated using the CRR binomial model are also reported in Table II for comparison. For the European call, the accuracy of the binomial model is assessed by comparison with the Black-Scholes price. For the American put, however, no closed-form solution exists and numerical accuracy is assessed by the convergence of the price series. It is clear from Table II that prices calculated using either binomial model converge fairly quickly, with the CRR model converging slightly faster. For the call option, accuracy within a penny of the Black-Scholes price is achieved with less than 200 time steps in both cases.

One important difference between the two models is the smoothness of convergence. Figure 2 illustrates the convergence behavior of both binomial models for a call option with a strike price of 95. The FB model converges virtually monotonically to the closed-form solution, showing smaller pricing error as more time steps are used. This is clearly not the case for the CRR model. Figure 2 also reveals that the CRR model converges at a faster rate than the FB model. However, this turns out to be far less important because an extrapolation technique can be used to greatly enhance the rate of convergence for the FB model.

Formally, the smoothness of convergence is measured by the error ratio, to be defined below. Consider a sequence of time steps, with each one twice the size of the proceeding one. In Table II, the sequence starts at 25 and ends with 6,400. Let $e(N)$ be the pricing error of the N-step binomial model—that is:

$$e(N) = C(N) - C_{\mathrm{BS}}, \tag{15}$$

where $C(N)$ is the N-step binomial price and C_{BS} is the Black-Scholes price. For European options, the pricing error of the binomial model is directly measurable as the Black-Scholes model provides a closed-form solution for the option price. The error ratio is defined as:

$$\rho(N) = \frac{e(N)}{e(2N)}. \tag{16}$$

It measures the improvement in accuracy as the number of time steps doubles. If accuracy does improve as N increases, the error ratio should be greater than 1 (in absolute value). Combining eqs. (15) and (16), the following relationship obtains:

$$\frac{\rho(N)C(2N) - C(N)}{\rho(N) - 1} = C_{BS}.$$

This means that the Black-Scholes price can be calculated precisely from two binomial prices if the error ratio is known. Of course, this is generally not the case. However, when convergence is smooth (the pricing error declines monotonically as the number of time steps increases), the error ratio converges to a constant. In that case, an extrapolation method is available to enhance the accuracy of the binomial model:

$$\hat{C}(2N) = \frac{\rho C(2N) - C(N)}{\rho - 1}, \tag{17}$$

where ρ is the limit of the error ratio series. It is straightforward to show that the above extrapolation formula can be rewritten as:

$$\hat{C}(2N) = C_{BS} + \frac{\rho - \rho(N)}{\rho - 1}\, e(2N), \tag{18}$$

which establishes the error contained in the extrapolated price. If the error ratio, $\rho(N)$, converges to ρ, the extrapolation error would be much smaller than the pricing error, $e(2N)$, from the binomial model. This is apparent from eq. (18). As a result, the extrapolation technique can improve the rate of convergence if convergence is smooth (that is, the error ratio converges).

The top panel in Table II reports the error ratio for a European call. It is clear that the error ratio is roughly constant and converges to 2 very quickly for the FB binomial model but not for the CRR binomial model. Additional numerical experimentation shows that the same conclusion applies to all European options priced using the FB model and the CRR model. The rate of convergence of the FB model can be enhanced by the extrapolation method defined in eq. (17). This is not possible for the CRR model.

To see how much the extrapolation technique improves convergence, consider the call prices reported in Table II with 100 time steps. The pricing error of the FB model is -0.011883, more than five times as large as the pricing error of 0.002337 in the CRR model. However, because the error ratio of the FB model converges to 2, the extrapolation technique defined in eq. (17) can be used to improve convergence. As the error ratio is 2.033633 at 100 time steps, extrapolation (using prices with 100 and 50 time steps) reduces the pricing error of the FB model to:

$$\frac{2.033633 - 2}{2 - 1} \times (-0.011883) = -0.000400,$$

which is nearly six times smaller than the error of the CRR model.

For American options, the pricing error of a binomial model is not observable because a closed-form solution of the option price is not available. As a result, the error ratio cannot be calculated directly. Instead, the error reduction is defined and calculated as follows, when the number of time steps doubles:

$$\varepsilon(N) = C(N) - C(2N). \tag{19}$$

Define a new ratio called the error reduction ratio:

$$\rho'(2N) = \frac{\varepsilon(N)}{\varepsilon(2N)} = \frac{C(N) - C(2N)}{C(2N) - C(4N)}. \tag{20}$$

Unlike the error ratio, the error reduction ratio is straightforward to calculate. It is trivial to show that the error reduction ratio is linked to the error ratio as follows:

$$\rho'(2N) = \frac{\rho(N) - 1}{\rho(2N) - 1}\,\rho(2N). \tag{21}$$

This implies that if the error ratio converges to a constant, the error reduction ratio also converges to the same constant. Therefore, if the error reduction ratio converges to a constant, the extrapolation technique in eq. (17) also improves the rate of convergence. This is especially useful for options with no closed-form solution such as American options. The bottom panel of Table II reports the error reduction ratio for an American put. It appears that the ratio converges to 2 in the FB binomial model but not the CRR binomial model.

Table III reports the extrapolated prices using the flexible binomial model (EFB hereafter) for European calls and American puts. Prices from the CRR binomial model and the Kamrad and Ritchken (1991) trinomial model (TRIN hereafter) are also reported in the table for comparison. From the EFB prices reported in Table III, it is clear that penny accuracy is achieved with only 50 time steps (extrapolated from the binomial prices with 25 and 50 time steps). In fact, the EFB price is accurate to the fourth decimal place with only 400 time steps used. In comparison, both the CRR and TRIN prices converge much slower than the EFB prices.

Figure 3 plots the pricing errors of the European call with a 95 strike price. The EFB pricing errors are compared with the CRR and TRIN pricing errors. The difference is quite remarkable. The EFB pricing error is barely noticeable after about 30 time steps. For similar accuracy, 100 or more time steps are required for the CRR and TRIN models.[9]

[9]Similar results obtain for other strike prices and types of options. Additional figures are omitted here for brevity.

TABLE III

Enhanced Convergence for European Calls and American Puts

	European call			American put		
N	*CRR*	*TRIN*	*EFB*	*CRR*	*TRIN*	*EFB*
Strike price = 95						
20	10.2046	10.2094	10.2041	2.5462	2.5347	2.5487
50	10.2025	10.1887	10.1920	2.5301	2.5193	2.5213
100	10.1924	10.1878	10.1905	2.5248	2.5195	2.5198
150	10.1925	10.1931	10.1900	2.5223	2.5230	2.5201
200	10.1954	10.1909	10.1900	2.5252	2.5209	2.5199
300	10.1897	10.1902	10.1901	2.5201	2.5203	2.5201
400	10.1925	10.1898	10.1901	2.5225	2.5202	2.5202
500	10.1886	10.1910	10.1901	2.5193	2.5210	2.5201
1000	10.1907	10.1905	10.1901	2.5208	2.5205	2.5200
Black-Scholes[a]	10.1901	10.1901	10.1901	2.5200	2.5200	2.5200
Strike price = 100						
20	7.0854	7.1352	7.1549	4.4612	4.4826	4.4910
50	7.1276	7.1476	7.1663	4.4803	4.4895	4.4978
100	7.1417	7.1517	7.1559	4.4867	4.4912	7.4932
150	7.1464	7.1531	7.1575	4.4888	4.4918	4.4934
200	7.1488	7.1538	7.1559	4.4898	4.4921	4.4928
300	7.1512	7.1545	7.1559	4.4908	4.4923	4.4929
400	7.1524	7.1548	7.1559	4.4913	4.4925	4.4929
500	7.1531	7.1551	7.1559	4.4916	4.4925	4.4928
1000	7.1545	7.1555	7.1559	4.4922	4.4927	4.4928
Black-Scholes[a]	7.1559	7.1559	7.1559	4.4927	4.4927	4.4927
Strike price = 105						
20	4.8127	4.8066	4.8421	7.2605	7.2579	7.2729
50	4.7987	4.7826	4.7893	7.2608	7.2451	7.2502
100	4.7989	4.7934	4.7890	7.2544	7.2503	7.2487
150	4.7892	4.7958	4.7927	7.2503	7.2518	7.2498
200	4.7982	4.7901	4.7940	7.2544	7.2486	7.2502
300	4.7875	4.7921	4.7922	7.2477	7.2495	7.2496
400	4.7953	4.7935	4.7922	7.2520	7.2503	7.2496
500	4.7919	4.7931	4.7925	7.2497	7.2502	7.2496
1000	4.7935	4.7922	4.7925	7.2505	7.2495	7.2496
Black-Scholes[a]	4.7923	4.7923	4.7923	7.2495	7.2495	7.2495

Notes: The asset price is 100, the strike prices are 95, 100, and 105, the asset price volatility is 0.2, the put option has six months remaining to maturity, the risk-free rate is 0.06, and the number of time steps ranges from 20 to 1,000. In the table, CRR and TRIN are prices calculated from the CRR binomial model and the Kamrad and Ritchken (1995) trinomial model. EFB is the price from the flexible binomial model, extrapolated from prices with *N* and *N*/2 time steps.

[a]For American puts, closed-form solution is not available and the flexible binomial price with 10,000 time steps is used instead.

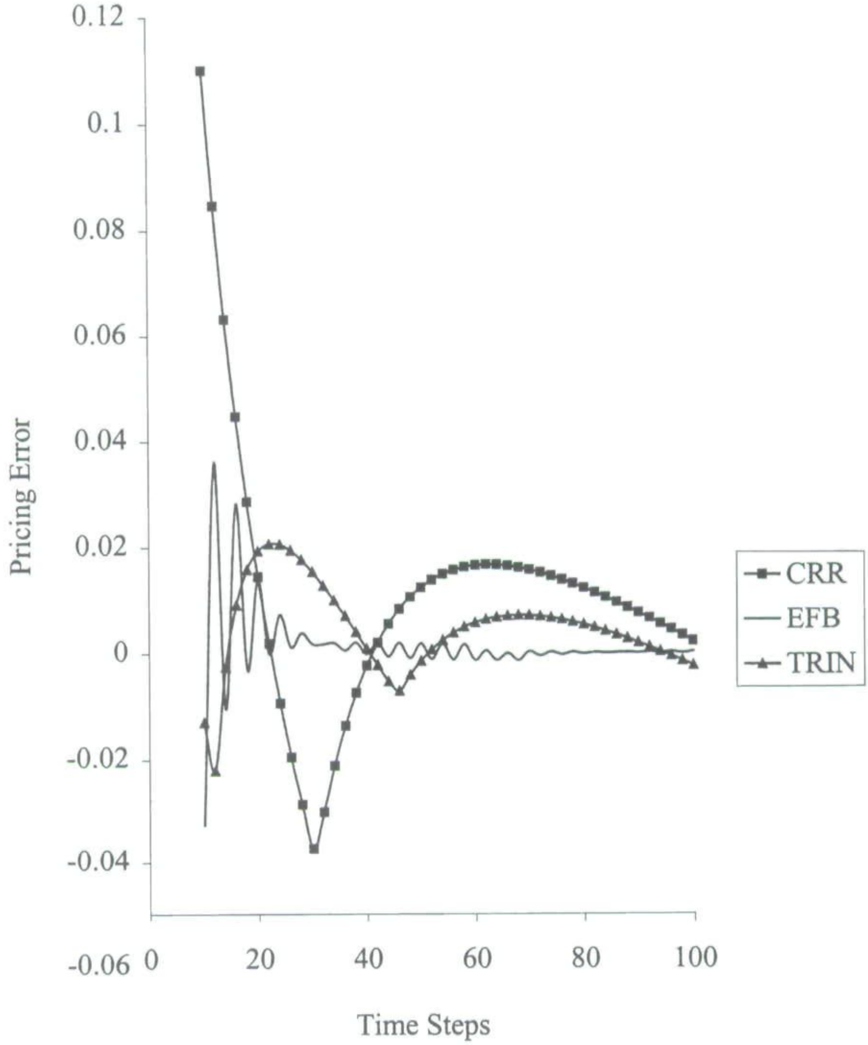

FIGURE 3

Enhanced convergence using the extrapolation technique. The asset price is 100, the strike price is 95, the asset price volatility is 0.2, the call option has six months remaining to maturity, the riskfree rate is 0.06, and the number of time steps ranges from 10 to 100. In the figure, CRR, TRIN and EFB denote prices from the CRR binomial model, Kamrad and Ritchken (1995) trinomial model, and the extrapolated flexible binomial model, respectively.

PRICING BARRIER OPTIONS

In over-the-counter derivatives markets, barrier option has become one of the most actively traded exotic options. In fact, many traders no longer regard barrier options as exotic options. One reason for its popularity is cost effectiveness as compared with ordinary options. Barrier options can be much less expensive than ordinary options, and, at the same time, equally effective in hedging activities if the barrier is chosen properly.

In this section, the flexible binomial model is applied to the valuation of barrier options. It is well known that standard tree-based models should be modified before they are used to value barrier options. The tree must be reshaped such that the barrier is located at a certain position relative to layers of nodes in the tree.[10] With the flexible binomial model, the tilt parameter provides the necessary freedom for such adjustment. This is not possible with the CRR binomial model. Because the barrier can be continuous or discrete, the two types of barrier options are examined separately.

Continuous Barrier Options

Continuous barrier options are options with a continuously observed barrier. If the barrier is crossed at any time during the life of the option, the barrier condition applies (either knocking out or knocking in the option). Closed-form solution exists for most types of continuous barrier options, with the first one going as far back as Merton (1973).

When using a tree-based model, binomial or trinomial, to value a continuous barrier option, it is well known that a layer of nodes must be placed exactly on the barrier. Otherwise, erratic and wavy convergence behavior is observed and large pricing error may persist even if a large number of time steps is used. Consequently, the standard tree model needs to be modified so that a layer of nodes are always on the barrier. Ritchken (1995) shows how this can be accomplished using a trinomial tree, by changing the distance between layers of nodes through a "dispersion parameter." With the CRR binomial model, such a dispersion parameter is not available and the adjustment needed for pricing barrier options is thus not possible. However, with the flexible binomial model, the tilt parameter can be used to alter the shape of the tree, in order to put a layer of nodes on the barrier.

To see how this is done, consider a down-and-out option with the barrier observed continuously. Let X and B_d be the strike price and barrier of the option, respectively. At any time during the life of the option, the option becomes void if the asset price reaches or drops below B_d. To avoid trivial cases, it is assumed that $B_d < S_0$. This is to prevent the option from being knocked out immediately.

[10]See Boyle and Lau (1994), Derman, Emanuel, Ergener, and Bardhan (1995), Ritchken (1995), and Cheuk and Vorst (1996) for further details.

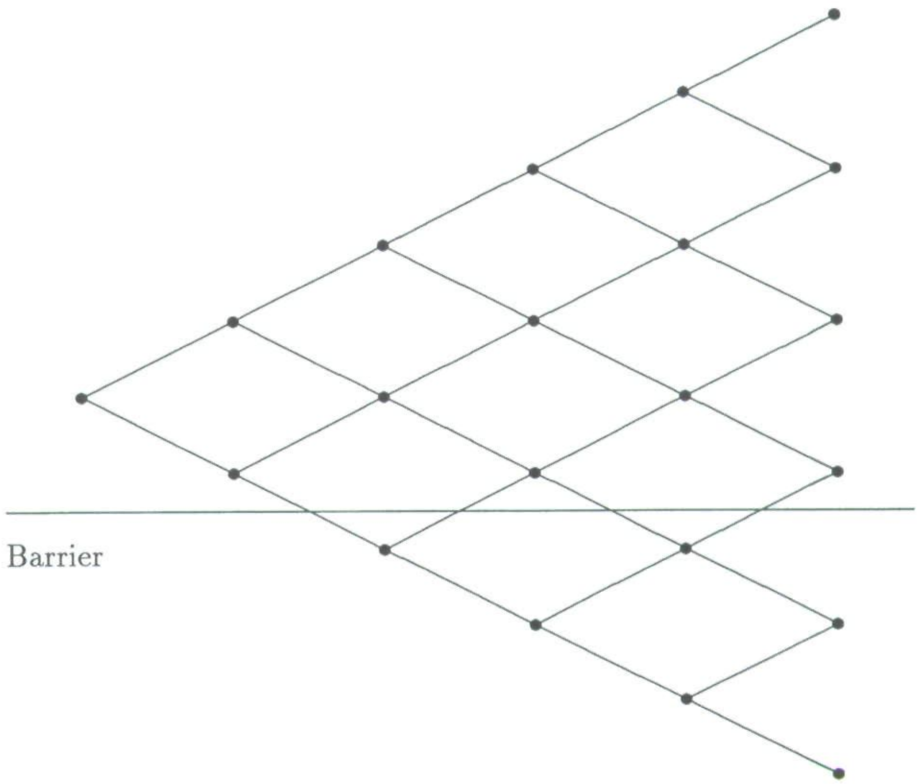

FIGURE 4
The CRR binomial tree with a barrier.

If the CRR model is used to value barrier options, a binomial tree that resembles the one shown in Figure 4 typically obtains. The tree spans "symmetrically" above and below the initial price of the underlying asset. In other words, the tree has zero tilt in the sense that an up move and a down move result in an asset price that is exactly the same as the previous asset price. As a result, the tree can be described as horizontal layers of nodes linked together through a branching process. The flat barrier typically lies between two adjacent horizontal layers of nodes, as shown in Figure 4.

In order to value continuous barrier options accurately, a layer of nodes need to lie exactly on the barrier. Using a similar technique as in the previous section, the tilt parameter is determined to be:

$$\lambda = \frac{\log(B_d/S_0) + N_0\sigma\sqrt{\Delta t}}{N_0\sigma^2\Delta t}, \tag{22}$$

where

$$N_0 = \left\lceil \frac{\log(B_d/S_0)}{\log d_0} \right\rfloor. \tag{23}$$

This modifies the binomial tree such that the barrier is reached exactly after N_0 consecutive down moves from the initial asset price, S_0.

To proceed further, it is necessary to ensure that the binomial model still converges with this new tilt parameter. This requires λ in eq. (22) to be bounded. A little algebra shows that the tilt parameter can be rewritten as:

$$\lambda = \frac{\varepsilon}{\varepsilon\sigma\sqrt{\Delta t} - \log(B_d/S_0)}, \tag{24}$$

where $\varepsilon = N_0 - \log(B_d/S_0)/\log d_0$. As $|\varepsilon| \leq 0.5$ by definition and $\log(B_d/S_0) > 0$ is a constant, λ is bounded for sufficiently small Δt. The non-negative probability condition in eq. (7) is thus satisfied, and convergence of the binomial model is ensured.

However, the binomial tree is now "tilted" (upward if $\lambda > 0$ or downward if $\lambda < 0$) with the tilt parameter specified by eq. (22). There are no longer any layers of nodes in the tree that are horizontal, which is not helpful for pricing options with a barrier that is horizontal. This is clear because:

$$ud = e^{2\lambda\sigma^2\Delta t}.$$

Horizontal layers of nodes are possible only if $\lambda = 0$, which ensures that an up move and a down move will not change the level of asset price. With a tilted tree, the chosen λ value only ensures that the barrier goes through a node at time t_{N_0}, but not at other times.

To resolve this problem, a two-stage flexible binomial tree (a TSFB tree hereafter) is proposed, with the asset price movement governed by different parameters in two different segments of the tree. The first stage of the tree is designed to have its lowest node locate exactly on the barrier. This requires a flexible binomial tree with a properly chosen λ value as shown above. The second stage of the tree is designed to resemble the CRR tree so that horizontal layers of nodes are present in the tree. Care must be taken to ensure that the binomial tree still recombines after the change of parameters.

Specifically, the first stage of the tree consists of all nodes during the first N_0 time steps, where the asset price movement is governed by the parameters $(u, d, p, 1 - p)$ determined by the tilt parameter defined in eq. (22). This ensures that the tree is exactly on the barrier after N_0

consecutive down moves from S_0. The second stage of the tree consists of all nodes after the first N_0 time steps, where the asset price movement is governed by the parameters $(u_0, d_0, p_0, 1 - p_0)$ calculated with a zero tilt parameter (as in the CRR binomial tree).

The tree clearly recombines properly during the first stage. Does it continue to recombine when the parameters change during the second stage? The answer is yes. To see this, consider two adjacent nodes at time t_{N_0}, (N_0, j) and $(N_0, j + 1)$ for $0 \le j \le N_0 - 1$. The tree would continue to recombine if an up move from (N_0, j) leads to the same price as a down move from $(N_0, j + 1)$, i.e.:

$$S(N_0, j)u_0 = S(N_0, j + 1)d_0.$$

This is equivalent to:

$$S_0 u^j d^{N_0 - j} u_0 = S_0 u^{j+1} d^{N_0 - j - 1} d_0,$$

or simply:

$$\frac{u}{d} = \frac{u_0}{d_0}.$$

This is obviously true for any λ from the definition of the four parameters in eqs. (6) and (8). Consequently, the binomial tree recombines throughout both stages. An example of such a binomial tree is illustrated in Figure 5.

The TSFB tree has the desirable property that it reaches the barrier exactly after N_0 consecutive down moves from S_0 and horizontal layers of nodes prevail after that. In this tree, the asset price at node (i, j) is given by

$$S(i, j) = \begin{cases} S_0 u^j d^{i-j}, & \text{if } i \le N_0, \\ S_0 d^{N_0} u_0^j d_0^{i - N_0 - j}, & \text{if } i > N_0. \end{cases} \tag{25}$$

With the tree modified this way, standard recursive procedure can be used to value continuous barrier options.

Discrete Barrier Options

The previous subsection shows how the flexible binomial model can be used to value continuous barrier options. However, such barrier options are not practical and are rarely traded in the actual marketplace. Instead, discrete barrier options are commonly traded in over-the-counter mar-

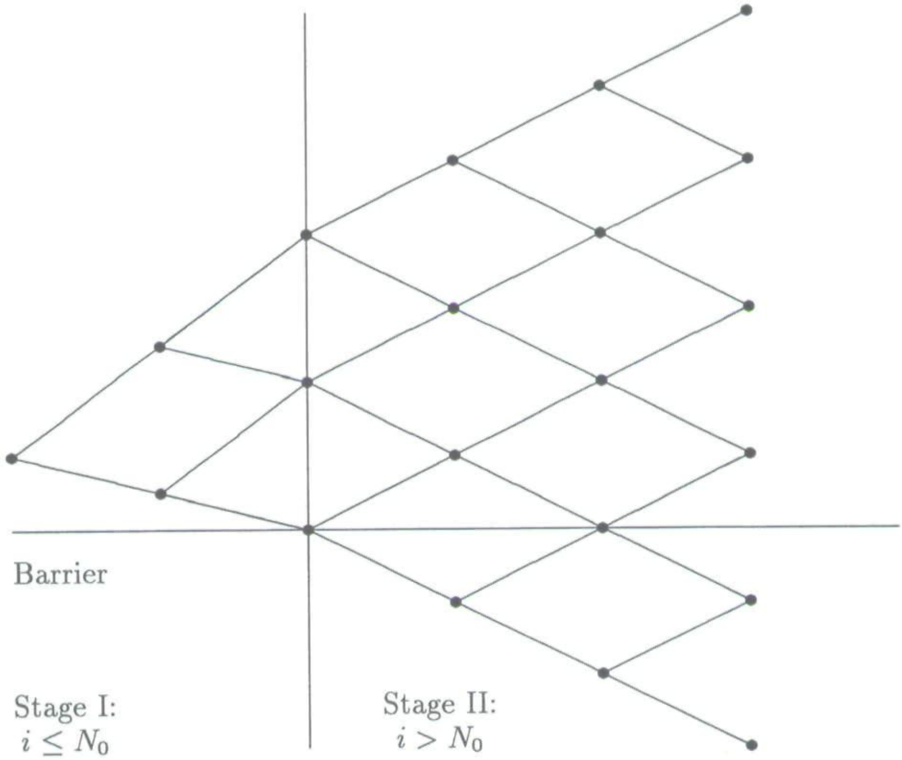

FIGURE 5
The two-stage flexible binomial tree adjusted to a continuous barrier.

kets. This is because asset prices in the real world are not available on a continuous basis. Exchanges do not open 24 hours a day and are closed on weekends and holidays. This means that most asset prices and indices are only available during specific hours of the day and certain days of the week. In addition, monitoring asset price continuously is much more costly even if it is available on a continuous basis. Consequently, it is more common to specify a barrier that is only in effect at discrete time intervals. For example, it is common to specify a daily knock-out barrier such that the option is cancelled if the closing price reaches or crosses the barrier on any trading day during the life of the option. This defers substantially from the corresponding continuous barrier option, since the option is still alive if the asset price crossed the barrier during the day but retreated back at the market close. This is an example of a barrier monitored only once a day (called daily fixing), or a daily barrier.

For options with barriers monitored only at discrete-time intervals, Cheuk and Vorst (1994, 1996) show that it is optimal to have the barrier lie exactly halfway between two nodes at any time the barrier is in effect. For the flexible binomial model, this is straightforward to do. Suppose the barrier is imposed K times during the life of the option, including the maturity of the option. At other times, the barrier is ignored. For technical simplicity, it is assumed that the K observation times are evenly spread over the option's life, with the first one at time T/K and the last one at the option maturity.[11]

Before making the necessary adjustments to the tree, it is important to note that the CRR binomial tree has a unique feature: The same asset price can be reached again only after an even number of periods, but not after an odd number of periods. This is because CRR binomial trees permit only up and down moves, not horizontal moves. The same price can be reached again only after an equal number of up and down moves. Consequently, it is necessary to have an even number of time steps between barrier observation points, in order to have similar tree structure at these points. This requires that the total number of time steps in the binomial tree be of the form $N = nK$, where $n \geq 2$ is an even number that specifies the number of time steps between two observation points. With this setup, the barrier observations points are $t_{i \cdot n}$ for $i = 1, 2, \ldots, K$.

To proceed further, consider a TSFB tree as illustrated in Figure 6. The first stage has a non-zero tilt parameter while the second stage has a zero tilt parameter. This two-stage tree is fully specified by N_0 and λ, the number of time steps and tilt parameter during the first stage, respectively. The lowest node in the first stage, $(N_0, 0)$, is put exactly on the barrier. This point is labelled Point A in Figure 6. With the tree structured this way, the barrier does not go through the first stage of the tree, but does go through the second stage where horizontal layers of nodes are present.

Next it is necessary to place the barrier exactly halfway between two nodes at all observation points. To do so, consider a sub-tree spanned by Points A, B, and C, as shown in Figure 6. The barrier lies exactly in the center of this sub-tree. At maturity, which is an observation point, the barrier must lie exactly halfway between two nodes. This requires that the central node of the sub-tree (Point P) in the previous period (t_{N-1}) lies exactly on the barrier. As a result, the number of time periods between Points A and P, $N - 1 - N_0$, must be an even number. Denote Point P as $(N - 1, j_0)$, where j_0 is the number of up moves from (0,0) to Point P. Since only down moves are involved from (0,0) to Point A, j_0 is also

[11]The technique described here can be modified slightly to deal with any type of observation points, evenly or unevenly spread over the life of the option.

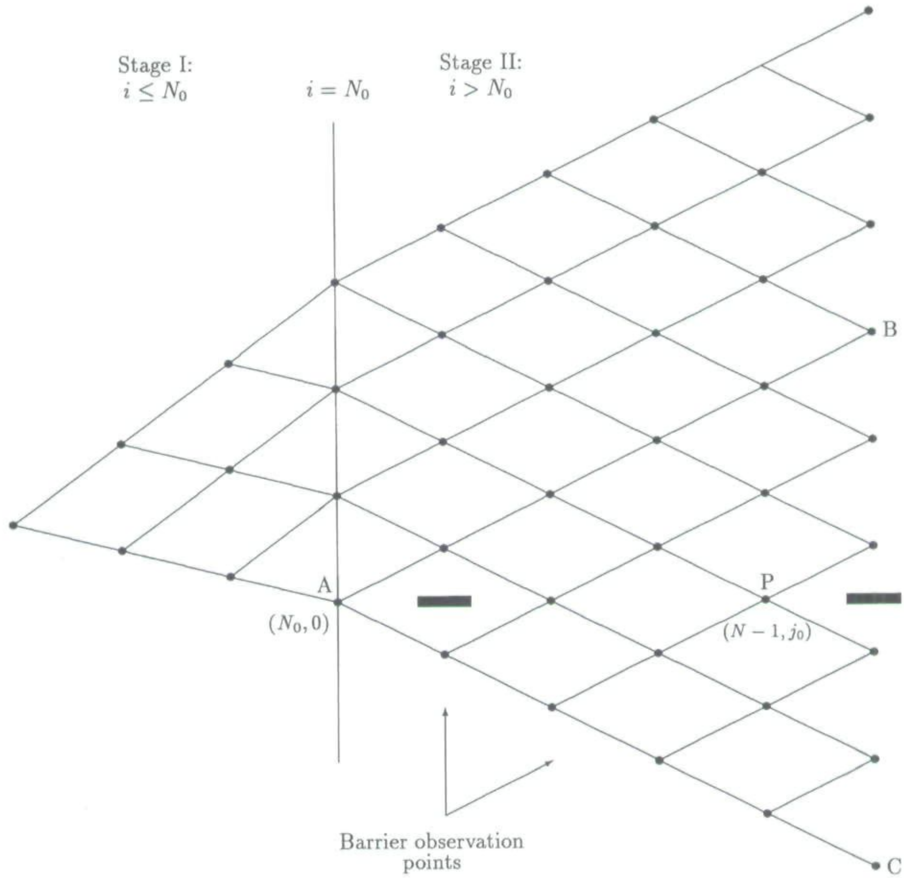

FIGURE 6
The two-stage flexible binomial tree adjusted to a discrete barrier.

the number of up moves from Point A to Point P in the sub-tree ABC. In addition, Points A and P are both on the (horizontal) barrier, implying an equal number of up and down moves between these two points. Thus,

$$N - 1 - N_0 = j_0 + j_0,$$

or equivalently,

$$N_0 = N - 1 - 2j_0.$$

Now it is obvious that j_0 and λ must be chosen such that Point A is exactly on the barrier:

$$j_0 = \left[\frac{N - 1 - \dfrac{\log(B_d/S_0)}{\log d_0}}{2} \right]. \tag{26}$$

$$\lambda = \frac{\log(B_d/S_0) + N_0\sigma\sqrt{\Delta t}}{N_0\sigma^2\Delta t}. \tag{27}$$

Because other observation points are even number of periods away from maturity, the barrier is also exactly halfway between two nodes at these points.

Numerical Results

Some numerical examples may help illustrate how the above two-stage flexible binomial model works, using down-and-out call options. The asset price is 100, the strike price is 100, asset price volatility is 0.2, the option has six months remaining to maturity, the riskfree rate is 0.1, and the barrier is specified at 95. Four separate cases are considered, including a continuously observed barrier, a daily observed barrier, a weekly observed barrier, and a monthly observed barrier. For simplicity, it is assumed that a calendar year has exactly 52 weeks and 250 business days.

Table IV reports prices of down-and-out call options calculated using the TSFB binomial model outlined above and the trinomial model of Kamrad and Ritchken (1991). For the trinomial model, adjustments recommended by Ritchken (1995) and Cheuk and Vorst (1996) are adopted. For continuous barrier options, the number of time steps used ranges from 50 to 10,000. Accuracy of the approximation is evaluated by comparison with the closed-form solution.[12] For discrete barrier options, the number of time steps between barrier observation points ranges from 2 to 100. Closed-form solution is not available in this case, the analytical approximation provided by Broadie, Glasserman and Kou (1997) is calculated and reported in the table.[13] It appears that prices from both the binomial and trinomial model converge quickly, and the rate of convergence is fairly comparable between the two models. For the continuous

[12]Closed-form solution exists for virtually all continuous barrier options. For a comprehensive coverage, see Derman and Kani (1993), Rich (1994), Rubinstein and Reiner (1991), and Zhang (1997). For more complex barriers, see Heynen and Kat (1994a,b), Kunitomo and Ikeda (1992), and Zhang (1995).

[13]The approximation error using the Broadie, Glasserman, and Kou (1997) technique is in the order of $O(1/\sqrt{K})$, where K is the number of times the barrier is checked during the life of the option. For frequently checked barriers (say daily), the error is quite small. However, for infrequently checked barriers (say monthly), the error can be large.

TABLE IV

Down-and-Out Call Options

	Continuous barrier			Discrete barrier: Daily (K = 125)		Discrete barrier: Weekly (K = 26)		Discrete barrier: Monthly (K = 6)	
Time steps N[a]	*TSFB*	*TRIN*	*Time steps n[a]*	*TSFB*	*TRIN*	*TSFB*	*TRIN*	*TSFB*	*TRIN*
50	5.7309	5.7179	2	6.2597	6.1970	6.7873	6.7825	7.5020	n/a
100	5.7219	5.7177	4	6.2162	6.1812	6.7053	6.6042	7.4047	7.3158
200	5.7172	5.7176	6	6.2003	6.1743	6.6753	6.6343	7.3738	7.3326
300	5.7174	5.7171	8	6.1922	6.1739	6.6590	6.6236	7.3582	7.5184
400	5.7171	5.7169	10	6.1873	6.1732	6.6534	6.6278	7.3438	7.3136
500	5.7166	5.7168	20	6.1783	6.1713	6.6351	6.6219	7.3334	7.3117
600	5.7166	5.7166	30	6.1751	6.1700	6.6292	6.6192	7.3211	7.3138
700	5.7170	5.7166	40	6.1734	6.1700	6.6260	6.6190	7.3220	7.3117
800	5.7167	5.7166	50	6.1724	6.1696	6.6236	6.6188	7.3191	7.3107
900	5.7165	5.7165	60	6.1718	6.1695	6.6230	6.6185	7.3155	7.3111
1000	5.7168	5.7165	70	6.1714	6.1693	6.6221	6.6182	7.3163	7.3105
2000	5.7164	5.7164	80	6.1710	6.1692	6.6214	6.6180	7.3162	7.3107
5000	5.7164	5.7163	90	6.1708	6.1692	6.6210	6.6178	7.3149	7.3108
10000	5.7163	5.7163	100	6.1705	6.1691	6.6206	6.6177	7.3133	7.3104
Closed-form	5.7163		BGK[b]	6.1688		6.6205		7.2829	

Notes: The asset price is 100, the strike price is 100, the asset price volatility is 0.2, the down-and-out call has six months remaining to maturity, the risk-free rate is 0.1, and the barrier is specified at 95. For the continuous barrier case, the number of time steps ranges from 50 to 10,000. For the discrete barrier case, the number of time steps between barrier observations ranges from 2 to 100. TSFB and TRIN are prices calculated using the two-stage flexible binomial model and the Kamrad and Ritchken (1991) trinomial model.

[a] N is the total number of time steps used in the binomial model for the continuous barrier option whereas n is the number of time steps used between two observation points for the discrete barrier option.

[b] BGK is the analytical approximation to the discrete barrier call price developed by Broadie, Glasserman, and Kou (1997). The approximation error is in the order of $O(1/\sqrt{K})$ where K is the number of times the barrier is in effect during the life of the option.

down-and-out call, penny accuracy obtains with only 100 time steps. For the discrete down-and-out call, prices appear to be converged after 30 time steps between barrier observation points. Of course, without a closed-form solution, the exact accuracy is difficult to gauge. It is reassuring that the converged price appears to agree with the Broadie, Glasserman and Kou (1997) analytical approximation.

Unlike standard options, convergence is not smooth in the case of barrier options. This is because it is no longer possible to put a node on the strike price of the option. The one degree of freedom is used to put the barrier at an optimal position. As a result, extrapolation techniques cannot be used to further enhance the rate of convergence for barrier options. This applies to both binomial and trinomial models.

CONCLUSIONS

This article has developed a flexible binomial model with a tilt parameter that alters the shape and span of the binomial tree. A positive tilt parameter shifts the tree upward, whereas a negative tilt parameter does exactly the opposite. This new binomial model is a simple generalization of the original Cox, Ross, and Rubinstein (1979) binomial model, which is equivalent to the flexible binomial model with a zero tilt parameter. It is shown that the flexible binomial model converges with any value of the tilt parameter. More importantly, the flexibility of the new model allows for certain adjustments to the binomial tree so that standard options and barrier options are valued more accurately. Specifically, smooth convergence obtains for standard European and American options, and a simple extrapolation method is used to enhance the rate of convergence. For barrier options, the flexible binomial tree is recalibrated so that the barrier is positioned optimally in relation to layers of nodes in the tree, which enhances numerical accuracy.

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