Pricing and Hedging S&P 500 Index Options with Hermite Polynomial Approximation: Empirical Tests of Madan and Milne's Model

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The universal use of the Black and Scholes option pricing model to value a wide range of option contracts partly accounts for the almost systematic use of Gaussian distributions in finance. Empirical studies, however, suggest that there is an information content beyond the second moment of the distribution that must be taken into consideration.This article applies a Hermite polynomial-based model developed by Madan and Milne (1994) to an investigation of S&P 500 index option prices from the CBOE when the distribution of the underlying index is unknown. The model enables us to incorporate the non-normal skewness and kurtosis effects empirically observed in option-implied distributions of index returns. Out-of-sample tests confirm that the model outperforms Black and Scholes in terms of pricing and hedging. © 1999 John Wiley & Sons, Inc. Jrl Fut Mark 19: 735–758, 1999

INTRODUCTION

Much of the literature following Black and Scholes's seminal article (1973) assumes that the asset underlying an option follows a log-normal

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diffusion process. The probabilistic characteristics of assets largely differ from one market to another. Commodity, fixed income, and foreign exchange markets are undoubtedly farther from the Black and Scholes world than the equity market. Equity options and equity index options have thus been the major field of scrutiny in empirical investigations of the model's drawbacks. These empirical studies largely document that the benchmark Black and Scholes option-pricing formula exhibits strong pricing biases across both moneyness and maturity (that is, the "smile" effect and the "volatility term structure") and that the model especially underprices deep out-of-the-money calls and puts. These findings are in contradiction with the assumption of log-normal distributions and suggest that a more general model is required to correctly price options. As a result, the search for alternative models has mostly focused on finding the "right" distributional assumption. Following the example of the stochastic volatility models initiated by Hull and White (1987), all the models discussed in the recent literature yield return distributions with negative skewness and high kurtosis, features commonly observed on empirical return distributions. Option prices will produce smile effects on the Black and Scholes implied volatility but will be recovered with the "right" return process, leaving the right volatility process independent from the strike price considered.

However, investigating alternative pricing models, Heynen (1994) concludes that none of them is likely to generate implied volatility patterns close to the observed implied volatility biases. In the same manner, Taylor and Xu (1993) find that the magnitude of the theoretical smiles generated by stochastic volatility models is, on average, twice as less as the magnitude of the observed smiles. Not only do the stochastic volatility models fail to match empirical smiles, but they also require very disputable assumptions on the volatility risk premium, not to mention their often-difficult and computer-intensive implementation.

There is no denying that, from a theoretical viewpoint, it would be very satisfactory to exhibit a fully parametric model, whether belonging to the class of stochastic volatility models or any other candidate (for instance, subordinated processes, stable processes, hyperbolic processes, and so forth). This partly explains why academics have dedicated so huge a literature to this search. Nevertheless, as long as no "right" model is found, a semiparametric model, which relies on the canonical Black and Scholes model but incorporates the deviation from log-normality in a very simple but intuitive fashion, is likely to produce a better fit. Although such a model will give birth to much more ad hoc techniques of pricing and hedging, it can prove to be of considerable value to practitioners.

As already discussed, the empirical patterns of the implied volatility are clearly indicative of implicit stock return distributions that are negatively skewed with higher kurtosis than allowable in the universal lognormal distribution. This suggests that there is an information content beyond the second moment of the distribution that must be taken into consideration. In this article a method developed by Madan and Milne (1994) to extend the Black and Scholes formula and account for non-normal skewness and kurtosis is presented. This method fits the first four moments of a distribution to a pattern of empirically observed option prices. Using S&P 500 index options, we examine the internal consistency of the implied parameters of the model as well as its out-of-sample pricing and hedging efficiency when compared to the Black and Scholes formula. It is shown that, not only does the model provide a better out-of-sample fit, but it also significantly improves the Black and Scholes delta hedging, with virtually no further mathematical complexity.

MADAN AND MILNE' S HERMITE POLYNOMIAL APPROACH

In this section, the theoretical foundations of the method developed by Madan and Milne (1994) and extended in Abken, Madan, and Ramamurtie (1996) are presented and discussed. They propose a method to value a large set of contingent claims when the underlying asset at maturity follows an unknown distribution under the statistical measure and the risk neutral measure. The method relies on the identification and the pricing of a set of "basis" claims.

The statistical probability space is classically denoted by $(\Omega, \mathcal{F}, P)$ for time $t \in [0, T]$ with the increasing right continuous filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ of a Brownian motion $(W_t)_{0 \leq t \leq T}$ initialized at 0. It is assumed that there exists a constant interest rate r paid by a money market account. Madan and Milne's key insight is to observe that although no pricing or hedging can be done without a perfect knowledge of the risk neutral measure Q (or the statistical measure P), it is always possible to use an approach developed by Elliot (1993), in which there is a change of measure from Q to a reference measure R which is assumed to be Gaussian. In other words, the unknown risk neutral density of the underlying asset at the maturity T, denoted by $Q(z)$, can be written as a product of a change of measure density and a reference measure density

$$Q(z) = \lambda(z)g(z) \tag{1}$$

where $\lambda(z)$ is the risk neutral change of measure density and $g(z)$ the reference measure density chosen to be a standard Gaussian

$$g(z) = \frac{1}{\sqrt{2\pi}} e^{-1/2z^2} \tag{2}$$

In this context any European call price can be written

$$C_t = e^{-r(T-t)} \int_{-\infty}^{+\infty} f(x)\lambda(x)g(x)dx \tag{3}$$

with $f(x) = (x - K)_+$.

Then they observe that if $f(\mathrm{x})$ and $\lambda(x)$ are two real functions taking real arguments [as it is the case for the European call in eq. (3)], the integral

$$\langle f, \lambda \rangle = \int_{-\infty}^{+\infty} f(x)\lambda(x)g(x)dx \tag{4}$$

defines a scalar product.

Using the differential equation

$$H_k(x) = (-1)^k \frac{\partial^k g(x)}{\partial x^k} \frac{1}{g(x)} \tag{5}$$

to generate Hermite polynomials that are standardized to unity as follows

$$h_k(x) = \frac{H_k(x)}{\sqrt{k!}} \tag{6}$$

an orthonormal basis for the Gaussian reference measure is obtained. This means that any contingent claim payoff represented by a real function $f(x)$ can be represented in a unique fashion as a linear combination of elements of the basis—that is, there exist real constants $\alpha_0, \alpha_1, \ldots$ such that

$$f(x) = \sum_{k=0}^{\infty} \alpha_k h_k(x) \tag{7}$$

where

$$\alpha_k = \int_{-\infty}^{+\infty} f(x)h_k(x)g(x)dx \tag{8}$$

Observing that because of orthogonality

$$\langle f(x), h_j(x)\rangle = \sum_{k=0}^{\infty} \alpha_k \langle h_k(x), h_j(x)\rangle = \alpha_j \tag{9}$$

the coefficient α_k represents the covariance of the kth basis element claim $h_k(x)$ with the derivative payoff $f(x)$. It can be interpreted as the quantity of "basis" claim $h_k(x)$ to hold in a portfolio to replicate the contingent claim payoff $f(x)$.

In this context the price of the contingent claim $f(x)$ can be expressed as

$$P(f(x)) = \sum_{k=0}^{\infty} \alpha_k \pi_k \tag{10}$$

where π_k is the implicit price of Hermite polynomial claim $h_k(x)$.

Given a particular contingent claim payoff $f(x)$, the coefficients α_k are well defined by eq. (8) and thus the prices π_k can be inferred from the observed market prices of the contingent claim.

If it is now assumed that the price of the underlying asset S_t under the reference measure R is given by

$$S_T = S_t e^{\mu(T-t)+\sigma\sqrt{T-t}z_t - 1/2\sigma^2(T-t)} \tag{11}$$

with $z_t \approx N(0, 1)$, then the coefficients α_k for a European call option (for example, $f(x) = (x - K)_+$) can be obtained (see Madan & Milne, 1994) by

$$\alpha_k(S_t, K, \mu, \sigma, T - t) = \left.\frac{\partial^k G(u, S_t, K, \mu, \sigma, T - t)}{\partial u^k}\right|_{u=0} \frac{1}{\sqrt{k!}} \tag{12}$$

where $G(u, S_t, K, \mu, \sigma, T - t)$ is the call generating function defined by

$$G(u, S_t, K, \mu, \sigma, T - t) = S_t e^{\mu(T-t)+\sigma\sqrt{T-t}u}\, N(d_1(u)) - KN(d_2(u)) \tag{13}$$

with

$$d_1(u) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(\mu + \frac{\sigma^2}{2}\right)(T - t) + \sigma\sqrt{T - t}u}{\sigma\sqrt{T - t}}$$

and

$$d_2(u) = d_1(u) - \sigma\sqrt{T - t}$$

The option price in terms of the Hermite polynomial prices of elementary claims is

$$C_t(S_t,\ K,\ T\ -\ t,\ \mu,\ \sigma)\ =\ \sum_{k=0}^{\infty}\ \pi_k\alpha_k(S_t,\ K,\ \mu\ \sigma,\ T\ -\ t) \qquad (14)$$

At this stage, given a set of option prices depending on various exercise prices for a particular maturity, and under the assumption of a truncated sum, the parameters π_k can be inferred from the market prices. Suppose that a Hermite polynomial approximation is used up to order L, then the option price by Hermite polynomial approximation of a European call takes the form

$$C_t(S_t,\ K,\ T\ -\ t,\ \mu,\ \sigma)\ =\ \sum_{k=0}^{L}\ \pi_k\alpha_k(S_t,\ K,\ \mu,\ \sigma,\ T\ -\ t) \qquad (15)$$

Because μ and σ are unknown parameters, if the sum is truncated at lag L, at least $L\ +\ 3$ exercise prices will be needed to infer the parameters.

DATA SOURCES

The data used in this study consist of tick-by-tick prices for the European-style S&P 500 index options traded on the Chicago Board of Options Exchange from 1 January 1991 through 31 December 1991. The data base includes the time of the quote, the expiration date (options expire on the third Friday of the expiration month), the exercise price, call or put, the option price, and volume. Similar information for the underlying S&P 500 index are also available on a tick-by-tick basis. On both the index and the option markets, transactions that do not give rise to price changes are also recorded.

Daily bid and ask discounts from United States Treasury-bills are hand-collected from *the Wall Street Journal*. The risk-free rate of interest for each option is computed daily as the continuously compounded interest rate based on the midpoint of the bid and ask discounts from the T-bills for which their maturities straddle the expiration date of the option.The European S&P 500 index options must be adjusted for discrete dividends. Daily dividends paid on the index are culled from the *S&P 500 Information Bulletin*. On a given day , for an option contract expiring at maturity date , the present value of the future dividends, denoted by , is computed by

$$PVD_t\ =\ \sum_{i=1}^{T-t}\ D(t\ +\ i)e^{-iR(t,i)}$$

where $R(t,\ i)$ is the i-period yield-to-maturity calculated using the yields on the two T-bills observed at time t whose maturities straddle the dividend payoff $D(t + i)$. Then, for each option price, the tick-by-tick S&P 500 index database is used to track the last index price quoted before the option transaction occurs. The use of this particular index price for the theoretical valuation of the option strongly mitigates the bias associated with non synchronous trading in the option and the underlying asset as documented in Harvey and Whaley (1991).[1] The present value of the future dividends is then subtracted from this index price, to obtain the dividend-free S&P 500 index level corresponding to this particular option.[2]

To reduce the likelihood of errors, data screening procedures are used (see, for instance, procedures used by Harvey & Whaley, 1991, 1992). Because implied volatilities of short-term options are very sensitive to small errors in the call price and may convey liquidity-related biases, only options that have at least 10 days to maturity are considered. Moreover, to mitigate the impact of price discreteness on option valuation, price quotes lower than \$3/8 are not included. Finally, quotations violating the lower arbitrage boundary (Merton, 1973) for European calls with dividends

$$\max(0, S_t - K, S_t - PVD_t - Ke^{-(T-t)R(t,T)}) \leq C_{OBS,t}$$

are excluded from the sample to ensure that implied volatilities can be calculated. Eventually 39,492 original calls fulfill the above described criteria.

Following a classical approach (see, for instance, Sheikh, 1991, or Heynen, 1994), the remaining records are then assigned to 9 cells corresponding to three groups for the time to maturity and three groups for

[1]Harvey and Whaley (91) explain that the timing difference that arises when closing prices on S&P 100 index options (3:15 PM) and daily closing index prices (3:00 PM) are used to calibrate an option pricing model, may induce negative first-order serial correlation in the implied volatility changes. In order to demonstrate the magnitude of the spurious behavior that may be induced by this problem, they use the S&P 100 index closing price and the last transaction prices recorded before 3:00 PM on the option market.I follow the same methodology: for each option transaction that occurs during the trading day, I tracked back the last S&P 500 index transaction price that occurred before the option was traded. The average timing difference between the option transaction and the underlying transaction is of a few seconds. We can argue that the S&P 500 index value is thus the corresponding index level at the moment when the option price is recorded. There is no non synchronous price issue here.

[2]Harvey and Whaley (1992) investigate the importance of taking into consideration the futures dividends in option pricing. They also stress on the seasonal patterns of S&P 100 index dividends that are highly predictable over a period of time corresponding to an option's life. Thus using ex-post observed dividends or estimates of these futures dividends to compute the present value of the dividends paid until the option expiration will significantly affect the results. It is common practice in empirical studies to take the present value of ex-post dividends (see, for instance, Dumas, Fleming, & Whaley, 1996; Day & Lewis, 1992).

TABLE I

Historical S&P 500 Index Options Statistics

	Moneyness S/K	Days to Expiry: Short Term <60	Medium Term 60–120	Long Term >120	Subtotal
Out-of-the money	<0.98	2.95	7.74	7.93	
		(2.64)	(2.13)	(2.71)	
		{4204}	{3781}	{2281}	{10266}
At-the-money	0.98–1.02	682	14.10	20.86	
		(3.35)	(3.11)	(3.38)	
		{9598}	{6942}	{3527}	{20067}
In-the-money	>1.02	17.76	23.48	29.41	
		(4.47)	(3.51)	(4.54)	
		{2167}	{3576}	{3416}	{9159}
Subtotal		{15969}	{14299}	{9224}	{39492}

Notes: For each moneyness-maturity category, Table I reports the average call price, the standard deviation (in parentheses) and the total number of observations (in braces) for the sample period 1 January 1991 through 31 December 1991. The 39,492 calls described in the table are the data resulting from the data screening procedures described in this article.

the moneyness. The call option is said to be at-the-money when the ratio S/K belongs to the interval [0.98, 1.02]; out-of-the-money when $S/K <$ 0.98; in-the-money when $S/K >$ 1.02. Moreover, the options can be classified according to the time to expiry with the following division: (i) short-term options $\tau < 60$ days, (ii) medium-term options $60 \leq \tau \leq 120$, and (iii) long-term options $\tau > 120$ days.

Table I gives some standard sample properties of the S&P 500 call prices for the different groups of moneyness and time to expiry. The 39,492 calls described in this table are the results of the data screening procedures presented in this section and will be used throughout this study.

ESTIMATION PROCEDURES AND IN-SAMPLE PERFORMANCE

The first set of estimation procedures assesses the comparative in-sample performance of the Black and Scholes option pricing model and the Madan and Milne Hermite polynomial model. In order not to restrict the analysis to daily prices picked up at an arbitrary point in the trading day (that is, the closing prices or the options prices observed at a fixed time before the closing time), all the option transaction prices of the day are used in the following estimation procedures as well as in the in-sample and out-of-sample performance tests introduced in the article.

The Black and Scholes Option Pricing Model

Because of the European nature of S&P 500 index options, implied volatilities can be calculated inverting the dividend-adjusted Black and Scholes formula

$$C_t(K, \tau) = (S_t - PVD_t)N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{16}$$

with

$$d_1 = \left\{\ln[(S_t - PVD_t)/K] + \frac{1}{2}(r + \sigma^2)\tau\right\}\Big/\sigma\sqrt{\tau}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

where

S_t is the value of the index at time t

PVD_t is the present value at time t of all dividends paid over the remaining life of the option

$N(\cdot)$ is the cumulative normal density function

r is the risk-free rate of interest

$\tau = T - t$ is the time to expiration

The Black and Scholes dividend-adjusted formula specifies six inputs: (i) an index price, (ii) a strike price, (iii) a risk-free interest rate, (iv) an option maturity, (v) the present value of the dividends, and (vi) a return standard deviation. The first five inputs are directly observable market data or can easily be estimated on the market. Because the return standard deviation is not directly observable, a return standard deviation implied by option prices can be estimated.

Let C_{BS} $(BSIV)$ denote the theoretical Black and Scholes price of an option with particular time to expiration τ and strike price indexed by K as defined by eq. (16), given an estimate for the volatility of the return on the index over the time to expiration $BSIV$. The market-observed call price for this option is denoted by C_{OBS}. The daily Black and Scholes implied volatility $BSIV$ for each expiration and moneyness class is chosen to minimize the sum of the squared differences

$$\min_{BSIV} \sum_{i=1}^{N} [C^i_{OBS} - C^i_{BS}(BSIV)]^2 \tag{17}$$

TABLE II

Black and Scholes Implied Volatility

		Black and Scholes Implied Volatility		
Sample Period	*Moneyness S/K*	*Short Term <60*	*Medium Term 60–120*	*Long Term >120*
1 January 1991	<0.98	0.1635	0.1603	0.1535
31 December 1991		(0.0275)	(0.0223)	(0.0265)
	0.98–1.02	0.1645	0.1674	0.1591
		(0.0258)	(0.0245)	(0.0219)
	>1.02	0.1916	0.1715	0.1601
		(0.0359)	(0.0331)	(0.0311)
	All Strikes	0.1640	0.1629	0.1583
		(0.0245)	(0.0243)	(0.0281)
1 January 1991	<0.98	0.1758	0.1723	0.1650
30 June 1991		(0.0296)	(0.0278)	(0.0277)
	0.98–1.02	0.1733	0.1766	0.1677
		(0.0294)	(0.0289)	(0.0292)
	>1.02	0.1988	0.1783	0.1679
		(0.0386)	(0.0327)	(0.0339)
	All Strikes	0.1749	0.1732	0.1655
		(0.0279)	(0.0268)	(0.0271)
1 July 1991	<0.98	0.1477	0.1452	0.1403
31 December 1991		(0.0126)	(0.0134)	(0.0117)
	0.98–1.02	0.1545	0.1595	0.1515
		(0.0156)	(0.0138)	(0.0143)
	>1.02	0.1781	0.1821	0.1697
		(0.0254)	(0.0222)	(0.0237)
	All Strikes	0.1487	0.1473	0.1507
		(0.0187)	(0.0199)	(0.0291)

Notes: The daily Black and Scholes implied volatility, *BSIV*, is chosen to minimize the sum of the squared differences in eq. (17) using all the option prices on a given day for a moneyness-maturity category. Table II reports the average *BSIV* for each category with the standard deviation in parentheses. Results are given for the sample period 1 January 1991 to 31 December 1991 as well as the two subperiods 1 January 1991 to 30 June 1991 and 1 July 1991 to 31 December 1991. Expiration classes are reported in columns and moneyness categories appear in lines. The last line of each period, denoted by "all strikes," represents the average of the daily *BSIV* obtained when minimization (17) is launched daily with all option prices grouped by maturity category, but regardless of the moneyness. *S/K* still represents the moneyness.

where N represents the number of option prices on a particular day for a particular group of time to expiration and moneyness. Initial values $BSIV_0$ for the minimization program are computed using an adapted version of the Bharadia, Christofides, and Salkin (1996) quadratic method for the calculation of implied volatility.

Table II reports the average of the daily Black and Scholes implied volatility $BSIV$ for the entire sample period obtained by running the minimization procedure in eq. (17) for each class of moneyness and time to maturity, along with the results of this minimization procedure performed by class of maturity, regardless of the moneyness (denominated by the heading "all strikes" in Table II). The minimization is performed on the whole one-year sample as well as the two subperiods 1 January 1991 to 30 June 1991 and 1 July 1991 to 31 December 1991.

Note that in Table II the implied volatility of calls exhibits the traditional dependence on strike price and maturity. These features confirm that the assumed constant parameter of the Black and Scholes model would yield theoretical prices that depart significantly from their observed counterparts. Moreover, even when volatility is computed according to the moneyness and maturity category, the parameter varies significantly from one subperiod to the other, revealing the instability of the volatility, which is very bad news for forecasting and hedging.

The Madan And Milne Hermite Polynomial Model

In the second set of estimation procedures, Madan and Milne's option pricing model introduced earlier in this article is used. The Hermite polynomial equation in (15) is truncated at order four to take into account skewness and kurtosis departure from normality in the distribution of returns. In order to ensure that the truncated expansion yields a risk neutral density for $Q(z)$ in eq. (1) we need to impose $\pi_0 = e^{-r(T-t)}$. In addition, constraining the first two moments of the return process under the transformed measure to equal the true moments under the reference measure, imposes two further restrictions on the implicit prices of risk π_k: $\pi_1 = 0$ and $\pi_2 = 0$ (see Appendix A or Abken, Madan, & Ramamurtie, 1996, for a detailed discussion of this issue).

These restrictions are very intuitive when one is reminded that the implicit prices of risk π_k are directly related to the moment of the same order. The function $\alpha_0(S_t, K, \mu, \sigma, T - t)$ gives a Black and Scholes price for the European call at maturity that needs to be discounted by $\pi_0 = e^{-r(T-t)}$. Because it is required that the change of measure does not affect the first two statistical moments of the distribution, the prices of risk π_1 and π_2 are set to zero. To conclude, the following two prices of risk, π_3 and π_4, related respectively to the moments of order three and four, are used to take into account the fact that the skewness and kurtosis of the empirical distribution of returns are different from their values obtained with the lognormal distribution assumed in the reference measure. Option prices will then be used to capture the departure from the skewness and kurtosis of a Gaussian distribution in the option-implied distribution of the S&P 500 index.

The option pricing formula in eq. (15) can then be expressed as follows:

$$C_{MM}(\mu, MMIV, \pi_3, \pi_4) = C(S_t - PVD_t, K, T - t, \mu, MMIV)$$
$$= e^{-r(T-t)}\alpha_0(S_t - PVD_t, K, T - t, \mu, MMIV)$$
$$+ \pi_3\alpha_3(S_t - PVD_t, K, T - T, \mu, MMIV)$$
$$+ \pi_4\alpha_4(S_t - PVD_t, K, T - T, \mu, MMIV) \quad (18)$$

where *MMIV* stands for the Madan and Milne implied volatility.

Appendix B gives the theoretical form of the coefficients $\alpha_k(S_t - PVD_t, K, T - t, \mu, \sigma)$ computed with eqs. (12) and (13).

On a given day, regardless of the moneyness, the option prices are divided according to the time to expiry. For each class of maturity, the parameters μ and *MMIV* are estimated along with the implicit prices of risk π_3 and π_4 with the minimization procedure

$$\min_{\mu, MMIV, \pi_3, \pi_4} \sum_{i=1}^{N} [C^i_{OBS} - C^i_{MM}(\mu, MMIV, \pi_3, \pi_4)]^2 \quad (19)$$

The average daily values of the parameters obtained with this procedure for each class of maturity are reported in Table III.

Note that the standard errors of the structural parameters of the Madan and Milne model estimated by minimizing the sum of the squared pricing errors between the market price and the model-determined price for each option are, on average, ten times less than the corresponding standard errors obtained with the Black and Scholes model and reported in Table II.

In-Sample Performance

The Hermite polynomial approximation model of Madan and Milne presented in the previous section along with the minimization program in eq. (19) allow to estimate the implied parameters of the model for each option maturity category. As a result, in order to assess and compare the in-sample pricing fit of this model with the Black and Scholes model, all the call options available on a given day are grouped into maturity category. These daily prices are used as input to compute either the Black and Scholes implied volatility or the Madan and Milne implied parameters. The in-sample performance of each model is then measured by the average of the daily in-sample sum of the squared errors and the average of the absolute deviation of the theoretical prices from the observed prices. Results are reported in Table IV and Table V, respectively, for the Black and Scholes model and the Madan and Milne model.

TABLE III

Madan and Milne Implied Parameters

		Madan and Milne Implied Parameters			
Sample Period	*Days to Expiry*	μ	*MMIV*	π_3	π_4
1 January 1991	<60	0.0562	0.1296	0.1425	0.1673
31 December 1991		(0.0019)	(0.0045)	(0.0067)	(0.0072)
	60–120	0.0378	0.1326	0.5446	0.2247
		(0.0039)	(0.0034)	(0.0127)	(0.0029)
	>120	0.0564	0.1512	0.1137	0.2154
		(0.0033)	(0.0021)	(0.0016)	(0.0063)
1 January 1991	<60	0.0538	0.1263	0.1327	0.1633
30 June 1991		(0.0021)	(0.0072)	(0.0051)	(0.0049)
	60–120	0.0321	0.1314	0.3956	0.2228
		(0.0037)	(0.0027)	(0.0056)	(0.0037)
	>120	0.0537	0.1497	0.1193	0.2146
		(0.0019)	(0.0031)	(0.0014)	(0.0057)
1 July 1991	<60	0.0596	0.1342	0.1562	0.1729
31 December 1991		(0.0016)	(0.0065)	(0.0084)	(0.0095)
	60–120	0.0464	0.1345	0.7703	0.2276
		(0.0042)	(0.0042)	(0.0189)	(0.0006)
	>120	0.0592	0.1528	0.1078	0.2163
		(0.0044)	(0.0027)	(0.0018)	(0.0069)

Notes: Each day in the sample period, the Madan and Milne implied parameters (including *MMIV*, the Madan and Milne implied volatility) are estimated by minimizing the sum of the squared pricing errors between the market prices and the model-determined prices for each option according to eq. (19) with a Hermite polynomial approximation of order four. Parameters π_3 and π_4 account for the non-normal skewness and kurtosis in the S&P 500 Index returns. Table III gives the average of the daily estimated parameters for each maturity category, regardless of the moneyness, and for the period 1 January 1991 through 31 December 1991 as well as for the two sub-periods 1 January 1991 to 30 June 1991 and 1 July 1991 to 31 December 1991. Numbers in parentheses are the standard deviations.

It is important to notice that even if the Black and Scholes model enables to compute an implied volatility for all option strikes and maturities all together, the computation of this parameter according to the maturity category does not represent a restriction of the test. Indeed, if the volatility were constant as it is predicted by the Black and Scholes framework, then the same implied volatility would be found for each class of maturity. It is known that volatility varies with strike price and maturity. Computing an implied volatility by maturity category is already an improvement of the Black and Scholes model. In this context, it is sufficient to exhibit the superiority of the Madan and Milne model over the Black and Scholes model, with implied parameters computed by maturity categories, to conclude that the Madan and Milne model outperforms the Black and Scholes model.

TABLE IV

In-Sample Performance of the Black and Scholes Model

Sample Period	*Days to Expiry*	*Number of observations*	*Daily Average BSIV*	*Average Observed Call Price*	*Average of Daily In-Sample Sum of the Squared Differences*	*Average of Daily Absolute Deviation of Theoretical Price from Observed Price*
1 January 1991	<60	15969	0.1640	7.29	11.84	0.2828
31 December 1991			(0.0245)	(3.36)	(4.45)	(0.0595)
	60–120	14299	0.1629	14.77	11.31	0.2873
			(0.0243)	(3.01)	(3.38)	(0.0637)
	>120	9224	0.1583	20.83	12.01	0.4160
			(0.0281)	(3.63)	(4.26)	(0.0906)
1 January 1991	<60	9323	0.1749	7.18	11.65	0.5213
30 June 1991			(0.0279)	(3.12)	(4.62)	(0.0568)
	60–120	8614	0.1732	14.68	11.17	0.4759
			(0.0268)	(2.95)	(3.25)	(0.0614)
	>120	4728	0.1655	20.78	11.88	0.8262
			(0.0271)	(3.81)	(4.19)	(0.0965)
1 July 1991	<60	6646	0.1467	7.44	12.11	0.6069
31 December 1991			(0.0187)	(3.41)	(4.20)	(0.0631)
	60–120	5685	0.1473	14.91	11.53	0.9373
			(0.0199)	(3.28)	(3.57)	(0.0670)
	>120	4496	0.1507	20.88	12.15	0.8375
			(0.0291)	(3.92)	(4.33)	(0.0839)

Notes: Table IV assesses the in-sample pricing fit of the Black and Scholes model. The daily average implied volatility is reported for each maturity category with its standard error and the number of observations averaged. The third column of this table gives the average observed call price. The average of the daily in-sample sum of the squared errors and the average of the absolute deviation of the theoretical price from the observed price are provided in the last two columns as measures of the in-sample performance of the model. Numbers in parentheses are the standard errors.

First, the dependence on maturity of the estimated implied volatility differs across the two models. On average the Black and Scholes volatility is less sensitive to the maturity than the Madan and Milne volatility. For instance, during the period 1 January 1991 through 31 December 1991, the Black and Scholes volatility ranges from 0.1640 to 0.1583 when maturity increases—that is, an absolute variation of 3.60% of the initial value. Over the same period the Madan and Milne volatility increases from 0.1296 to 0.1512—that is, an increase by 16.67% of the initial value. Although this seems somewhat surprising at first sight, it is important to remember that the Madan and Milne parameters (including *MMIV*) are time-dependent by construction with the version of the model implemented in this study. As a result, the time dependence of the vola-

TABLE V

In-Sample Performance of the Madan and Milne Model

Sample Period	*Days to Expiry*	*Number of observations*	*Daily Average MMIV*	*Average Observed Call Price*	*Average of Daily In-Sample Sum of the Squared Differences*	*Average of Daily Absolute Deviation of Theoretical Price from Observed Price*
1 January 1991	<60	15969	0.1296	7.29	3.27	0.0582
31 December 1991			(0.0045)	(3.36)	(0.94)	(0.0106)
	60–120	14299	0.1326	14.77	3.46	0.0636
			(0.0034)	(3.01)	(0.83)	(0.0137)
	>120	9224	0.1512	20.83	3.18	0.0881
			(0.0021)	(3.63)	(0.88)	(0.0155)
1 January 1991	<60	9323	0.1263	7.18	3.16	0.0984
30 June 1991			(0.0022)	(3.12)	(0.88)	(0.0108)
	60–120	8614	0.1314	14.68	3.21	0.1025
			(0.0027)	(2.95)	(0.77)	(0.0141)
	>120	4728	0.1497	20.78	3.08	0.1690
			(0.0013)	(3.81)	(0.97)	(0.0149)
1 July 1991	<60	6646	0.1342	7.44	3.43	0.1425
31 December 1991			(0.0065)	(3.41)	(1.02)	(0.0103)
	60–120	5685	0.1345	14.91	3.84	0.1672
			(0.0042)	(3.28)	(0.92)	(0.0131)
	>120	4496	0.1528	20.88	3.30	0.1842
			(0.0027)	(3.92)	(0.78)	(0.0161)

On a given day, observed prices are grouped by maturity and used as input to compute the Madan and Milne implied parameters. The in-sample fit of the model is measured by the average of the daily in-sample sum of the squared errors described in eq. (19) and by the average of the daily absolute deviation of the theoretical price from the observed price. All statistics are given together with their associated standard deviations in parentheses.

tility does not reveal a model inconsistency, as is the case with the Black and Scholes model.

Second, as one would expect, the presence of more parameters in the Madan and Milne model enhances the model's fit as illustrated by the average of the daily in-sample sum of the squared errors and the average of the daily absolute deviation of the theoretical prices from their observed counterparts. Regardless of the maturity category, both statistics are on average four times smaller with the Madan and Milne model than with the Black and Scholes model.

OUT-OF-SAMPLE PERFORMANCE

It is shown in the previous section that the Hermite polynomial option pricing model outperforms the Black and Scholes model for in-sample tests. Nevertheless, this better fit is not very surprising when one keeps in mind that the Black and Scholes model relies on the estimation of a single parameter, namely the *BSIV*, to fit the market prices. On the contrary, the truncated approximation up to order four yields an estimation of the Madan and Milne model based upon four parameters. It could be argued that the better fit is simply the consequence of a larger number of parameters—that is, more degrees of freedom in the fitting procedure. Although an increasing number of parameters often leads to an increasing better in-sample fit, out-of-sample pricing in the presence of more parameters can actually cause overfitting. Out-of-sample tests are required to confirm the superiority of the Madan and Milne model for option pricing.

To assess the out-of-sample forecasting power of each model, the following test is performed. All option prices within the same maturity class are used to estimate the models' parameters with the minimization programs (17) or (19) in a current-day sample. Then, for each sample day, prior-day parameter estimates are used to compute theoretical option prices. These theoretical prices obtained either with Black and Scholes or Madan and Milne previous day parameters are then compared to the current day market-observed prices. To this end, the model determined price is subtracted from its observed counterpart, and both the average absolute and the average percentage pricing errors are computed with their associated standard errors. The average of the daily out-of-sample sum of the squared errors is also reported. Table VI and Table VII report the results of this study respectively for the Black and Scholes model and the Madan and Milne model.

First, the three pricing error measures presented in this study rank the Madan and Milne model first. Incorporating new parameters to account for non-Gaussian skewness and kurtosis divide on average by 3.66 the daily out-of-sample sum of the squared error and by 4.43 the absolute deviation of the theoretical price from the market-observed price. Pricing improvements with the Madan and Milne model are particularly striking. For instance, a typical out-of-the-money call with moneyness less than 0.98 and with less than 60 days to expiration has an average price of \$2.95 from Table I. When the Black and Scholes model is applied to value this call, the resulting absolute error is, on average, \$0.5341 as shown in Table VI, but when the Madan and Milne model is applied, the average error goes down to \$0.1182 (that is, the error decreases from 18% to 4% of the real value).

Third, regardless of the pricing model, the absolute pricing error increases from short term to medium and to long term options. However, the worst value obtained with Madan and Milne for long term options

TABLE VI

Out-of-Sample Performance of the Black and Scholes Model

Same Period	*Days to Expiry*	*Number of observations*	*Average of the Daily Percentage Pricing Error*	*Average of Daily Out-of-Sample Sum of the Squared Differences*	*Average of Daily Absolute Deviation of Theoretical Price from Observed Price*
1 January 1991	<60	15887	−55.21%	27.32	0.5341
31 December 1991			(4.13)	(4.01)	(0.0743)
	60–120	14221	−31.17%	26.24	0.5411
			(4.26)	(3.32)	(0.0714)
	>120	9157	−21.08%	24.48	0.7533
			(3.97)	(3.89)	(0.0911)
1 January 1991	<60	9241	−48.67%	27.04	0.9127
30 June 1991			(4.10)	(3.93)	(0.0735)
	60–120	8536	−39.12%	25.99	0.8964
			(4.21)	(3.28)	(0.0709)
	>120	4661	−24.34%	24.27	1.4435
			(4.01)	(3.78)	(0.0898)
1 July 1991	<60	6646	−64.31%	27.71	1.2871
31 December 1991			(4.18)	(4.12)	(0.0754)
	60–120	5685	−19.25%	26.62	1.3652
			(4.33)	(3.38)	(0.0721)
	>120	4496	−17.70%	24.71	1.5733
			(3.93)	(4.00)	(0.0924)

Notes: Black and Scholes implied volatility (*BSIV*) estimated prices, for each maturity category, from prior-day option price observations are used to compute current-day theoretical Black and Scholes option prices. The assessment of the out-of-sample performance of the model is presented through the following three estimators: the average of the daily absolute deviation of the theoretical prices computed with prior-day parameters from the current-day observed prices, the average of the daily percentage pricing errors and the average of the daily out-of-sample sum of the squared errors.

during the second sub-period, \$0.3502, is significantly better than the best pricing fit performed by the Black and Scholes model for short term options (\$0.5341).

At last, note that all the pricing error average measures exhibit a small standard deviation with Madan and Milne. Not only does the model provide a better out-of-sample fit with respect to Black and Scholes, but it also insures a smaller deviation of the pricing error from its average value.

HEDGING STRATEGIES

Eq. (7) gives a unique representation of a particular claim $f(x)$ in terms of the elements of the basis $h_k(x)$. While estimating the parameters of a

TABLE VII

Out-of-Sample Performance of the Madan and Milne Model

Same Period	*Days to Expiry*	*Number of observations*	*Average of the Daily Percentage Pricing Error*	*Average of Daily Out-of-Sample Sum of the Squared Differences*	*Average of Daily Absolute Deviation of Theoretical Price from Observed Price*
1 January 1991	<60	15887	−19.47%	6.94	0.1182
31 December 1991			(1.11)	(0.92)	(0.0145)
	60–120	14221	−9.02%	7.21	0.1212
			(1.15)	(0.78)	(0.0146)
	>120	9157	1.37%	7.18	0.1737
			(1.03)	(0.94)	(0.0217)
1 January 1991	<60	9241	−17.85%	6.82	0.2010
30 June 1991			(1.08)	(0.91)	(0.0138)
	60–120	8536	−8.56%	7.11	0.1934
			(1.12)	(0.74)	(0.0153)
	>120	4661	2.44%	7.23	0.3445
			(0.97)	(1.01)	(0.0226)
1 July 1991	<60	6646	−21.72%	7.11	0.2869
31 December 1991			(1.15)	(0.94)	(0.0154)
	60–120	5685	−9.71%	7.36	0.3223
			(1.20)	(0.83)	(0.0135)
	>120	4496	0.26%	7.13	0.3502
			(1.09)	(0.87)	(0.0207)

Notes: Table VII sums up the out-of-sample pricing performance of the Madan and Milne model. The estimation procedure presented in Section IV is used to back out the Madan and Milne implied parameters on a day-by-day basis. Current day's model-based option prices are then computed using the parameters implied by the previous-day calls whose maturities lie in the same category. Next, the model-determined prices with prior day parameters are subtracted from their observed current-day counterparts and the average of the daily percentage pricing errors, and the average of the daily out-of-sample sum of the squared errors and the average of the daily absolute deviation of the theoretical prices from the observed prices are given with their associated standard errors to measure the pricing fit of the model.

truncated Hermite polynomial model, a countable basis is constructed that can be effectively used to ease pricing and hedging strategies. The previous two sections document the pricing performance of the Hermite polynomial approximation truncated at order four. This representation can be used to build a static hedging strategy. One has to purchase α_k units of the basis element $h_k(x)$.

However, in order to compare the hedging performance of this model with the Black and Scholes pricing model, a classical dynamic hedging strategy is constructed. Indeed, to explore the Madan and Milne model's implications for hedging strategies using options, formulas for option delta are derived to build a traditional hedge based upon a single instrument, namely the S&P 500 index contracts.

Let $n_s(t)$ denote the number of S&P 500 index contracts to be purchased to hedge a short position in a call option with strike price K and τ periods to expiry ($n_s(t)$ is given by the option delta). The residual cash position $n_0(t)$ required to construct a replicating portfolio is $n_0(t) = C(t, \tau) - n_s(t)S_t$. Although the theory assumes a continuous rebalancing of the portfolio, in practice, the hedging strategy is made through discrete rebalancing. It is assumed in the following tests that the replicating portfolio is rebalanced every working day.

At time t the self-financed portfolio consists in a short position on the option and a long position in $n_s(t)$ S&P 500 contracts. The residual, $n_0(t)$, is invested in an instantaneously maturing risk-free bond. The hedging error one day ahead can be expressed as follows:

$$H(t + 1) = n_s(t)S_{t+1} + n_0(t)e^{1*r(t)} - C(t + 1, \tau - 1) \quad (20)$$

The new self-financed portfolio is then computed with the quantities $n_0(t+1)$ and $n_s(t+1)$, and is used to assess the hedging error on day τ + 2. The calculations are then repeated for each option in the sample for every day until maturity.

Finally, two estimates of the hedging performance are constructed. First, the average absolute hedging error

$$AAHE = \frac{1}{\tau - t} \sum_{i=1}^{\tau - t} |H(t+i)| \quad (21)$$

then, the average hedging error

$$AHE = \frac{1}{\tau - t} \sum_{i=1}^{\tau - t} H(t + i) \quad (22)$$

In summary, the hedging results presented in Tables VIII and IX are obtained through the following procedure. First, the model's parameters are estimated on day t and used together with the current index level and the short term interest rate to compute the hedge. On day $t + 1$, the hedging error resulting from prior day hedge is computed according to eq. (20). The new hedging ratios are also computed to rebalance the portfolio. These steps are repeated for each option on every trading day. Then the average absolute hedging error and the average hedging error are reported in Tables VIII and IX.

Both the average absolute hedging error and the average hedging error highlight the superiority of the Madan and Milne model for hedging purposes. The average absolute pricing error is, on average, more than

TABLE VIII

Hedging Performance of the Black and Scholes Model

Sample Period	*Days to Expiry*	*Average Absolute Hedging Errors*	*Average Hedging Errors*
1 January 1991	<60	0.4417	−0.0966
31 December 1991		(0.102)	(0.0151)
	60–120	0.4608	−0.0753
		(0.107)	(0.0144)
	>120	0.5408	−0.0214
		(0.105)	(0.0139)
1 January 1991	<60	0.4423	−0.0987
30 June 1991		(0.097)	(0.0148)
	60–120	0.4612	−0.0768
		(0.098)	(0.0142)
	>120	0.5422	−0.0208
		(0.101)	(0.0137)
1 July 1991	<60	0.4408	−0.0933
31 December 1991		(0.1094)	(0.0156)
	60–120	0.4602	−0.0730
		(0.1196)	(0.0147)
	>120	0.5394	−0.0220
		(0.1088)	(0.0141)

Notes: For each maturity category, the Black and Scholes implied volatility *BSIV* is estimated on each sample day t. The risk-free rate and the index level are also collected to compute the hedge ratios $n_s(t)$ and $n_0(t)$. On day $t + 1$, the hedging error on the prior day hedge is computed with eq. (20) as well as the new hedge ratios to rebalance the portfolio. This procedure is repeated for each option on every trading day. The hedging performance is measured through the average absolute hedging error and the average hedging error described in eqs. (21) and (22).

five times smaller with the Hermite polynomial model. Note that the average hedging error has the same sign for each category whatever the model considered. Nevertheless, the magnitude of the error in dollars is far less important with the Madan and Milne model. For instance, the hedging error for medium term options is of $\$-0.0753$ with the Black and Scholes delta hedging and of \$0.0016 with Madan and Milne.

CONCLUSION

This article tests an approach of Madan and Milne (1994) for pricing contingent claims through a Hermite polynomial model. The model exhibits four parameters: a drift μ, a volatility *MMIV* and two coefficients π_3 and π_4 directly related to the skewness and kurtosis of the unknown distribution of the S&P 500 index prices. Significant skewness and excess kurtosis was found in the option-implied risk-neutral density of the S&P 500 index prices.

TABLE IX

Hedging Performance of the Madan and Milne Model

Sample Period	*Days to Expiry*	*Average Absolute Hedging Errors*	*Average Hedging Errors*
1 January 1991	<60	0.0836	−0.0134
31 December 1991		(0.0052)	(0.0042)
	60–120	0.0813	0.0016
		(0.0047)	(0.0035)
	>120	0.0821	0.0089
		(0.0045)	(0.0033)
1 January 1991	<60	0.0832	−0.0104
30 June 1991		(0.0048)	(0.0039)
	60–120	0.0807	0.0005
		(0.0049)	(0.0031)
	>120	0.0818	0.0072
		(0.0044)	(0.0030)
1 July 1991	<60	0.0842	−0.0181
31 December 1991		(0.0058)	(0.0045)
	60–120	0.0822	0.0033
		(0.0044)	(0.0041)
	>120	0.0824	0.0106
		(0.0046)	(0.0036)

Notes: The hedging performance of the Madan and Milne model is tested in Table IX with the average absolute hedging error and the average hedging error described in eqs. (21) and (22). The Madan and Milne implied parameters estimated on day t are used to compute the hedge ratios $n_s(t)$ and $n_0(t)$. On day $t + 1$, the hedging error on the prior day hedge is computed with eq. (20) as well as the new hedging ratios. This procedure is repeated for each option on every trading day.

In-sample and out-of-sample tests conclude that skewness and kurtosis adjustments added to the Black and Scholes formula with the Madan and Milne model, significantly improve the accuracy and consistency of option pricing. Further out-of-sample tests highlight the superiority of the model in terms of hedging.

In addition to better pricing and hedging performances, the model yields an explicit form for the risk-neutral density of the S&P 500 index returns (see Appendix A) in terms of the four implied parameters. This density can be used to price more complex contingent claims, taking into account the empirical deviations from lognormality in the S&P 500 index distribution.

APPENDIX A

The risk neutral density given in eq. (1) with respect to the standard normal measure can be expressed with the Hermite polynomial expansion

$$Q(z) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} e^{r(T-t)} \pi_k h_k(z) e^{-z^2/2}$$

For the empirical work, however, the Hermite expansion of the equivalent martingale measure $\lambda(z)$ is truncated at the fourth order. To ensure that the truncated expansion

$$Q(z) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{4} e^{r(T-t)} \pi_k h_k(z) e^{-z^2/2}$$

still defines a density (for example, $\int_{-\infty}^{+\infty} Q(z)dz = 1$), the restriction $\pi_0 = e^{-r(T-t)}$ needs to be imposed.

Following Abken, Madan, and Ramamurtie (1996), the two further restrictions $\pi_1 = 0$ and $\pi_2 = 0$ are added to ensure that the first two moments of the return process under the transformed measure are equal to the true moments under the reference measure.

Using the fact that the first five Hermite polynomial standardized to unit variance are

$$h_0(z) = 1,\ h_1(z) = z,\ h_2(z) = \frac{1}{\sqrt{2}}(z^2 - 1),$$

$$h_3(z) = \frac{1}{\sqrt{6}}(z^3 - 3z),\ h_4(z) = \frac{1}{\sqrt{24}}(z^4 - 6z^2 + z),$$

the resulting truncated risk neutral density with the constraint on the first two moments takes the form

$$Q(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\left[1 + \frac{3e^{r(T-t)}\pi_4}{\sqrt{24}} - \frac{3e^{r(T-t)}\pi_3}{\sqrt{6}} = \right.$$

$$\left. z - \frac{6e^{r(T-t)}\pi_4}{\sqrt{24}} z^2 + \frac{e^{r(T-t)}\pi_3}{\sqrt{6}} z^3 + \frac{e^{r(T-t)}\pi_4}{\sqrt{24}} z^4\right]$$

APPENDIX B

Using eqs. (12) and (13), Appendix B gives the result of the computation of the coefficients $\alpha_k(x, K, \mu, \sigma, T - t)$ up to order four. These coefficients are used in the minimization routine presented in eq. (18) to infer the parameters of the Madan and Milne model.

$$\alpha_0(x, K, \mu, \sigma, T - t) = xe^{\mu(T-t)}N(d_0(0)) - KN(d_2(0))$$

$$\alpha_1(x, K, \mu, \sigma, T - t) = \sigma\sqrt{T - t}\, xe^{\mu(T-t)}N(d_1(0))$$

$$\alpha_2(x, K, \mu, \sigma, T - t) = \frac{1}{\sqrt{2}}\left[\sigma^2(T - t)xe^{\mu(T-t)}N(d_1(0)) + \frac{\sigma\sqrt{T - t}xe^{\mu(T-t)}}{\sqrt{2\pi}}e^{-d_1^2(0)/2}\right.$$

$$\alpha_3(x, K, \mu, \sigma, T - t) = \frac{1}{\sqrt{6}}\left[\sigma^3(T - t)^{3/2}xe^{\mu(T-t)}N(d_1(0)) + \frac{2\sigma^2(T - t)xe^{\mu(T-t)}}{\sqrt{2\pi}}e^{-d_1^2(0)/2} - \frac{\sigma\sqrt{(T - t)}xe^{\mu(T-t)}}{\sqrt{2\pi}}d_1(0)e^{-d_1^2(0)/2}\right]$$

$$\alpha_4(x, K, \mu, \sigma, T - t) = \frac{1}{\sqrt{24}}\left[\sigma^4(T - t)^2xe^{\mu(T-t)}N(d_1(0)) + \frac{3\sigma^3(T - t)^{3/2}xe^{\mu(T-t)}}{\sqrt{2\pi}}e^{-d_1^2(0)/2} - \frac{3\sigma^2(T - t)xe^{\mu(T-t)}}{\sqrt{2\pi}}d_1(0)e^{-d_1^2(0)/2} + \frac{\sigma(T - t)xe^{\mu(T-t)}}{\sqrt{2\pi}}(d_1^2(0) - 1)e^{-d_1^2(0)/2}\right]$$

Of course x has to be replaced by $S_t - PVD_t$ and σ by $MMIV$ to recover the parameters in eq. (18) and minimization (19).

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