Do S&P 500 Index Options Violate the Martingale Restriction?

NORMAN STRONG*
XINZHONG XU

The study tests Longstaff's martingale restriction on S&P 500 index options over the period 1990–1994. Assuming the S&P index follows a lognormal distribution results in systematic violations of the martingale restriction, the implied index value from options consistently overestimating the market value. Adopting a generalized distribution, allowing for nonnormal third and fourth moments, produces economically insignificant rejections of the martingale restriction. A simulation analysis supports the empirical results from the lognormal model in the presence of nonnormal skewness and kurtosis. Overall, the results support the conclusion that the no-arbitrage assumption coupled with the generalized distribution offers a good working model for S&P index options over the period studied. © 1999 John Wiley & Sons, Inc. Jrl Fut Mark 19: 499–521, 1999

The authors acknowledge the helpful comments and suggestions of one of the referees for this paper, and useful discussions with Mark Tippett, Nikolaos Panigirtzoglou, and Neil Cooper.
*Correspondence author, Manchester School of Accounting and Finance, The University of Manchester, Manchester, M13 9PL, ENGLAND.

- *Norman Strong is a Professor of Accounting and Finance at the University of Manchester in Manchester, England.*
- *Xinzhong Xu is a Senior Lecturer in Accounting and Finance at the University of Manchester in Manchester, England.*

The Journal of Futures Markets, Vol. 19, No. 5, 499–521 (1999)

CCC 0270-7314/99/050499-23

INTRODUCTION

Empirical tests of option pricing models involve examining the consistency of option prices with the price distribution of the underlying asset. Most studies focus on second moments, changes in second moments, and higher-order moments (Bates, 1996). Longstaff (1995) has recently suggested a simpler test that focuses exclusively on the first moment, corresponding to the observable market price of the underlying asset. The *martingale restriction* tested by Longstaff is that the value of the underlying asset implicit in option prices equals the market price of the asset.

Longstaff tests the martingale restriction on S&P 100 index options during 1988 and 1989. Because these are American options, his study only considers call options, where the early exercise feature is unlikely to have any effect. Assuming the Black-Scholes model and using all call options quotes in a daily five-minute window, Longstaff finds the implied S&P index value exceeds the actual index value for 442 out of 444 observations. The mean percentage pricing difference is 0.465 with a *t*-statistic of 31.52. Relaxing the Black-Scholes lognormal distributional assumption by allowing the risk-neutral measure to come from the four-parameter Edgeworth expansion family only confirms the results. Inverting these four parameters from the four nearest at the money call options with a common maturity results in 812 observations. Of these, 759 are positive, and the mean percentage pricing error is 0.4 with a *t*-statistic of 31.08. Several sensitivity tests of the results from the Black-Scholes model confirm their robustness. To examine whether market frictions can explain the pricing bias, Longstaff regresses the Black-Scholes percentage pricing differences on variables capturing the transactions costs of options, option market liquidity, and recent stock market movements. Each of these is significant in explaining the pricing difference. Longstaff concludes that general equilibrium models that relax the martingale restriction and accomodate market frictions hold more promise than no-arbitrage valuation models.

Aparicio and Hodges (1996) address the logic of Longstaff's martingale restriction test, pointing out that assuming the Black-Scholes model when the actual distribution has nonnormal skewness or kurtosis shifts the mean of the implied risk-neutral distribution. They argue that rejection of the martingale restriction is better interpreted as rejecting the assumed option pricing model than as rejecting the no-arbitrage assumption of no market frictions.

This study applies the martingale restriction test to S&P 500 (SPX) index options over the period 1990–1994. Using S&P 500 index options avoids the American and wild-card features that compromise studies of

S&P 100 index options. It offers a reexamination of the no-arbitrage assumption for a more recent period in a market that probably more closely satisfies the assumptions of standard option pricing models. The study also uses both put and call options. The results show that, as in Longstaff's study, the joint hypothesis that S&P 500 index options priced by the Black-Scholes model satisfy the martingale restriction can be rejected. However, unlike Longstaff's study, assuming a more general form for the risk-neutral distribution of the index that accommodates nonnormal higher moments and better describes the time-series properties of the index, results in less clear rejections. Applying the martingale restriction test either to calls and puts separately or to calls and puts combined, although giving some statistically significant violations, produces violations that are economically insignificant, being an order of magnitude smaller than those Longstaff finds. The study concludes that for the options contracts and period examined, the assumption of no-arbitrage is a reasonable working approximation.

OPTION PRICING AND THE MARTINGALE RESTRICTION

The fundamental theorem of asset pricing (Dybvig & Ross, 1987) states that the absence of arbitrage opportunities is equivalent to the existence of a linear pricing rule that, when applied to an asset's future payoffs, gives its current market price. No-arbitrage and a linear pricing rule also imply, and are implied by, the existence of an equivalent martingale measure, which is a probability measure, such that an asset's price equals its expected future payoff with respect to the equivalent martingale measure discounted at the riskless rate. Harrison and Kreps (1979) introduced the term "equivalent martingale measure," showing that with a zero riskless rate an asset's price process is a martingale under the equivalent martingale measure.[1]

Applied to options this gives,

$$G_0 = B_T E_Q[F(X_T)], \tag{1}$$

where G_0 is the current option price, B_T is the current price of a pure discount bond with maturity T, $F(X_T)$ is the payoff to the option at maturity T based on the realizations, X_T, of the underlying asset at T, and E_Q denotes expectation with respect to the equivalent martingale mea-

[1]With a nonzero riskless rate, the discounted price process is a martingale.

sure, Q. This corresponds to the risk-neutral valuation approach of Cox and Ross (1976) with Q as the risk-neutral measure.

Applying the same rule to the realizations of the underlying asset similarly must give its market price,

$$X_0 = B_T E_Q[X_T]. \tag{2}$$

Longstaff's insight is to combine these two results. If eqs. (1) and (2) hold, the discounted expected payoff from the underlying asset under the risk-neutral measure implicit in option prices equals the current asset price. Equivalently, the price of the underlying asset implied by option prices equals its actual market value. Because this effectively imposes a restriction on the mean of the implied risk-neutral measure, Longstaff calls this the martingale restriction and the corresponding test the martingale restriction test.

Any test of the martingale restriction inevitably involves a joint test, not only of the no-arbitrage assumption, but of all the assumptions underlying the assumed option pricing model. Rejection under one model need not necessarily imply rejection of the no-arbitrage assumption. To examine whether violations of the martingale restriction may be due to misspecification of the option pricing model, this study considers two alternative assumptions about the distribution of the SPX index. First is the standard assumption that the index follows a lognormal distribution, leading to the Black-Scholes model. A large empirical literature shows that the Black-Scholes model does not fully explain observed option prices. Strike price and maturity biases in implied volatility estimates from the Black-Scholes model suggest that a lognormal distribution does not give a good fit for the higher moments of the empirical distribution function of the SPX index. A second assumption that accommodates a generalized lognormal distribution for the SPX index is therefore used as an alternative to the Black-Scholes model. The generalization originates with Jarrow and Rudd (1982), who show that an option pricing formula given by the sum of the Black-Scholes price plus adjustment terms for non-normal skewness and kurtosis of the underlying asset distribution gives an accurate approximation for a wide class of distributions.

DATA AND EMPIRICAL RESULTS

Data

The raw data for this study comprises S&P 500 index option prices and contemporaneous index levels from the transactions price history provided by the Chicago Board Options Exchange. This is supplemented by S&P 500 index futures data provided by the Futures Industry Institute. At any time, SPX index options trade for six expiry dates, the March, June, September, and December quarterly dates with the addition of two nearby months. The S&P 500 futures contract trades only on the quarterly dates, with trading heavily concentrated in the nearest maturity.

In contrast to S&P 100 index American-style options, the SPX contract is the world's leading traded European-style index option. Rubinstein (1994) argues that the SPX contract provides the best conditions for the practical application of the Black-Scholes formula. The data covers the five years 1990 to 1994, during which time trading volume is significantly higher than during the 1980s, when Longstaff conducted his study on S&P 100 index options. This more recent sample period should remove any relative market illiquidity disadvantages of using the SPX contract.

Following Longstaff's sampling procedure, quotations for call and put options are taken from the five-minute window, 14.00–14.05. This avoids contamination by market opening and closing effects and ensures an identical time window throughout the period. Using alternative five-minute windows to check robustness gives essentially identical results. The option quotations are the midpoint of the first pair of bid–ask quotes for each option contract during the window. The actual index value is the average across option transactions within maturity bands (see below) during the five-minute window. Testing the martingale restriction also requires the riskless interest rate and the expected dividend stream for the S&P 500 index. The Treasury bill rate, taken from Datastream, provides the riskless interest rate. As a sensitivity check, using a Eurocurrency rate gives similar results. Following standard practice, the actual dividends on the index, from S&P Corporation, are used for the expected dividend stream. The index is adjusted by subtracting the *present value of expected dividends* (*PVD*) over the life of the option, given as,

$$PVD = \sum_{i=1}^{n} D_i e^{-r_i t_i}, \tag{3}$$

where

D_i is the value of dividends in index points at date t_i; and r_i is the Treasury bill rate.[2]

[2]Treasury bill rates are collected for one, three, six, and twelve month maturities. The one-month rate is used for ex-dividend dates falling within one month, and linear interpolation between successive rates is used for other ex-dividend dates.

For options expiring in a month or less, the delay between announcement and payment of dividends likely eliminates the difference between expected and actual dividends. The discrepancy likely remains for longer maturities. The percentage pricing difference is calculated as the difference between the implied and actual dividend-adjusted index values as a percentage of the actual dividend-adjusted index value. As a sensitivity check of the results to assumptions about dividends, the empirical analysis below also reports on tests of the martingale restriction using the S&P 500 index futures level. The study drops option quotations that violate standard arbitrage bounds.

The subsequent analysis tests the martingale restriction separately on option contracts with different maturities. The reason for distinguishing different maturities is that, although the option pricing models used in the study assume volatility is nonstochastic, volatility is likely to be time-varying even if it is deterministic. Several studies document maturity effects on implied volatility estimates.[3] If volatility varies with time, the variance in option pricing formulas is the average variance over the option's maturity.

When testing on calls and puts together a requirement of at least five option prices for each option type and maturity is imposed. When testing the restriction on calls alone or puts alone, a higher minimum requirement of at least seven option prices for each option-maturity combination in the five minute window is imposed.[4] In total, the tests are carried out on 799 days and 1,588 option-maturity combinations in the case of calls and puts together. For calls alone, the figures are 930 days and 2,120 call-maturity combinations, and for puts alone the figures are 783 days and 1,599 put-maturity combinations.

Table I gives summary statistics on daily sample sizes and moneyness for calls alone (Panel A) and puts alone (Panel B). Moneyness is defined in terms of the forward price and the strike price,

$$\ln\left[\frac{(S - PVD)e^{rt}}{X}\right]. \tag{4}$$

For option-maturity combinations satisfying the minimum restriction of seven strike prices available, there are slightly more available call option-maturity contracts traded in the five-minute sample window than put

[3]For evidence on the effect of maturity on stock index option implied volatility see Diz and Finucane (1993), Harvey and Whaley (1992), and Stein (1989).

[4]The lesser minimum requirement of five option prices gives qualitatively identical results to those reported here. Excluding observations with less than seven prices excludes several cases for longer maturity contracts where the percentage pricing difference exceeds an imposed upper limit of 10%.

TABLE I

Summary Statistics for the Research Sample

	Number of Options	*Minimum Moneyness*	*Average Moneyness*	*Maximum Moneyness*
Panel A: Call options				
Mean	11.3821	−0.0353	0.0489	0.1648
Standard deviation	3.1569	0.0385	0.0306	0.0854
Minimum	7.0000	−0.2170	−0.0520	0.0010
5%	7.0000	−0.1050	0.0030	0.0670
10%	8.0000	−0.0830	0.0140	0.0760
1st Quartile	9.0000	−0.0530	0.0280	0.0940
Median	11.0000	−0.0300	0.0460	0.1440
3rd Quartile	13.0000	−0.0120	0.0680	0.2300
90%	16.0000	0.0070	0.0891	0.2900
95%	17.0000	0.0210	0.1040	0.3250
Maximum	23.0000	0.0970	0.1590	0.4520
Panel B: Put options				
Mean	10.4497	−0.1087	−0.0256	0.0637
Standard deviation	3.1266	0.0534	0.0357	0.0604
Minimum	7.0000	−0.2720	−0.1320	−0.0620
5%	7.0000	−0.2040	−0.0900	−0.0290
10%	7.0000	−0.1920	−0.0760	−0.0010
1st Quartile	8.0000	−0.1485	−0.0480	0.0270
Median	10.0000	−0.1000	−0.0220	0.0560
3rd Quartile	12.0000	−0.0715	−0.0030	0.0870
90%	15.0000	−0.0508	0.0180	0.1350
95%	16.0000	−0.0310	0.0300	0.1841
Maximum	22.0000	0.0790	0.1290	0.4020

Notes: The table reports summary statistics for the sample of SPX call and put options used in the study. Panel A reports the statistics for call options, Panel B reporting statistics for put options. The statistics—given in each row—are for the time-series of daily options used in testing the martingale restriction. The table gives summary statistics for four variables: the number of options each day; the daily minimum moneyness; the daily average moneyness; and the daily maximum moneyness. Equation (4) defines moneyness.

option-maturity contracts. The median number is 11 for calls and 10 for puts with a maximum of 23 for calls and 22 for puts. These figures compare with corresponding figures from Longstaff's study of a median value of 11 calls and a maximum value of 20 calls. Looking at the average moneyness across option-maturity combinations, in the case of calls the mean contract is almost 5% out-of-the-money, whereas the mean contract for puts is approximately 2.6% out-of-the-money. For over 95% of the option-maturities the average moneyness for calls is positive (out-of-the-money), whereas the average moneyness for puts is negative (out-of-the-money) for over 75% of the sample.

Over the sample period the average dividend-adjusted SPX index value shows a 62.5% variation (roughly from 294 to 478). Over all option-maturity combinations the difference between the maximum and minimum index values averages 0.16. Comparing this with the index value and a mean option spread of 0.64 suggests that taking the actual index value as the average across corresponding option transactions should not result in any significant inaccuracies.

Empirical Tests of the Martingale Restriction

Testing the martingale restriction involves inverting the relevant option pricing model, for each call-maturity, put-maturity, or put/call-maturity combination, to give the implied dividend-adjusted SPX index value and the implied volatility (and implied higher moments where required). The criterion function in solving for these implied parameters is to minimize the sum of squared pricing errors between the model option prices and the market prices. This criterion function is widely used in the literature. However, it is likely to place heavier emphasis on in-the-money options within option-maturity combinations.

Assuming first that the dividend-adjusted SPX is lognormally distributed, for each set of option-maturity prices, the implied (dividend-adjusted) index value, *S-PVD*, and the implied variance, σ^2, are calculated by numerical computation from the Black-Scholes formulas for calls and puts,

$$c = (S - PVD)\, N(d) - Ke^{-r(T-t)}\, N(d - \sigma\sqrt{T - t}), \tag{5}$$

$$p = Ke^{-r(T-t)}\, N(-d - \sigma\sqrt{T - t}) - (S - PVD)\, N(-d), \tag{6}$$

to minimize the sum of squared pricing errors between the model option prices and the observed market prices,

$$\min_{\widehat{S-PVD},\hat{\sigma}} \sum_{j=1}^{J} [g_j(\widehat{S - PVD}, \hat{\sigma}) - g_m]^2. \tag{7}$$

The notation used here is standard for the Black-Scholes formula, and g_j and g_m denote the model and market prices for options (calls, puts, or both combined).

The alternative to the Black-Scholes formula, using a generalized distribution, follows several other studies of option pricing models that expand the normal distribution for log returns to approximate distributions with nonnormal higher moments. The application of this technique to option pricing originates with Jarrow and Rudd (1982), who show how a given probability measure can be approximated by an arbitrary distribution in terms of an Edgeworth series expansion involving higher-order cumulants. Longstaff (1995) uses this technique, with the series truncated after the third term, as a test of the martingale restriction on a general no-arbitrage option pricing model. Corrado and Su (1995) use a closely related Gram-Charlier (moment-based) expansion to estimate coefficients of skewness and kurtosis implied by S&P 500 index options.[5] Other studies using generalized distributions in option pricing include Corrado and Su (1996) and Backus *et al.* (1997).[6]

Using a generalized distribution results in the following call option pricing formula,

$$c' = c + q_3\mu_3 + q_4\,(\mu_4 - 3), \tag{8}$$

$$q_3 = \frac{1}{3!}\,(S - PVD)\,\sigma\sqrt{T - t}$$
$$[(2\sigma\sqrt{T - t} - d)\,n(d) - \sigma^2\,(T - t)\,N(d)],$$

$$q_4 = \frac{1}{4!}\,(S - PVD)\,\sigma\sqrt{\mathrm{T} - \mathrm{t}}$$
$$([d^2 - 1 - 3\sigma\sqrt{\mathrm{t}}(d - \sigma\sqrt{T - t})]\,n(d) + \sigma^3\,(T - t)^{3/2}\,N(d)),$$

where c is the Black-Scholes option value, μ_3 and μ_4 are the skewness and kurtosis of log returns, $n(.)$ is the standard normal density function, and other terms are as before. The corresponding put option formula combines eq. (8) with the put-call parity relation.

The implied dividend-adjusted index value, and implied volatility, skewness, and kurtosis are the values minimizing the sum of squared differences between the observed market price and the extended formula price,

[5]Corrado and Su (1995) truncate the Gram-Charlier expansion to exclude terms beyond the fourth moment, resulting in an equivalent procedure to Longstaff (1995)—compare eq. (1) in Corrado and Su (1995) with eq. (8) in Longstaff (1995). This study follows an identical approach.

[6]One potential problem in using either Edgeworth or Gram-Charlier expansions is that for certain parameter values they can assign negative probabilities to intervals in the tails of the distribution being approximated. Cont (1997) suggests this drawback should not be viewed as prohibitive, but advises caution in using these methods to price options far out-of-the-money. An analysis of the problem in the current study suggests negative probabilities are possible, but these are not significant for the parameter values found.

$$\min_{\widehat{S-PVD},\hat{\sigma},\hat{\mu}_3,\hat{\mu}_4} \sum_{j=1}^{J} \{g_{mj} - [g_j(\widehat{S-PVD}, \hat{\sigma}) + q_3\hat{\mu}_3 + q_4(\hat{\mu}_4 - 3)]\}^2, \quad (9)$$

where the g_{mj} are the observed market prices of SPX put and call options, J in total, and the g_j are the Black-Scholes option prices. This procedure results in maximum likelihood implied index estimates allowing for non-normal skewness and kurtosis.

Empirical Results

This subsection reports the empirical results assuming the lognormal distribution and the generalized distribution and presents a graphical comparison. It also reports a sensitivity check of the effect of any dividend assumptions, using S&P 500 index futures data.

The Lognormal Assumption

Table II reports the results of fitting the Black-Scholes model to the sample. The table gives statistics on the percentage pricing differences for five maturity bands and an average across all maturities. Panel A confirms the results for the S&P 500 index[7] that Longstaff finds for the S&P 100 index (see Longstaff, 1995, Table 2). The results are, if anything, stronger for the SPX index in rejecting the martingale restriction. Over the whole sample, 99.7% of call-maturity combinations give an implied value of the index above the actual index value. The associated Z-statistic for the null of an equal number of positive and negative pricing differences is 45.78. For all maturities exceeding two months the implied index value always exceeds the actual index value. At shorter maturities there are some negative differences, but over the whole sample only six out of 2120 call-maturity combinations give an implied index value below the actual value. The mean percentage pricing difference across the whole sample is 0.611 with an associated t-statistic of 46.99. These figures compare with Longstaff's results of 0.465 and 31.52. The median pricing differences are always less than the mean pricing differences, indicating a positive skew, but these statistics give consistent results.

Mean and median percentage pricing differences increase monotonically with time to expiry, although standard deviations also increase. The median percentage pricing difference increases from 0.18% for the short-

[7]For brevity, unless there is some possibility of confusion, when the text refers to the index, this means the *dividend-adjusted* index from now on.

TABLE II

Percentage Pricing Differences Assuming a Lognormal Distribution

Maturity (months)	*Proportion positive*	*Z-stat*	*Mean*	*Standard deviation*	*t-stat*	*Min*	*5%*	*Q1*	*Median*	*Q3*	*95%*	*Max*	*N*
Panel A: Call options													
0–1	0.991	22.70	0.222	0.1564	32.85	−0.250	0.051	0.108	0.180	0.314	0.495	0.997	535
1–2	0.998	22.07	0.423	0.2504	37.39	−0.032	0.129	0.247	0.369	0.540	0.900	1.950	491
2–3	1.000	17.35	0.569	0.2567	38.48	0.017	0.258	0.382	0.503	0.715	1.110	1.427	301
4–6	1.000	24.17	0.888	0.3914	54.85	0.224	0.410	0.609	0.826	1.085	1.600	2.953	584
7–12	1.000	14.46	1.338	0.5404	35.80	0.454	0.737	0.955	1.217	1.491	2.504	3.328	209
All	0.997	45.78	0.611	0.4699	59.91	−0.250	0.091	0.265	0.487	0.843	1.480	3.328	2,120
Panel B: Put options													
0–1	0.643	5.66	0.284	0.6759	8.32	−0.343	−0.128	−0.032	0.054	0.318	1.428	5.917	392
1–2	0.900	15.21	0.536	0.6454	15.79	−0.319	−0.053	0.144	0.391	0.729	1.459	4.496	361
2–3	0.844	10.71	0.712	1.3250	8.38	−0.176	−0.122	0.109	0.418	0.857	1.884	10.000	243
4–6	0.965	19.34	1.568	1.5851	20.56	−0.352	0.021	0.369	1.253	2.249	4.017	9.198	432
7–12	1.000	13.08	3.306	1.4699	29.41	0.943	1.223	2.230	3.012	4.173	5.859	9.181	171
All	0.857	28.53	1.076	1.4888	28.90	−0.352	−0.087	0.101	0.495	1.495	4.027	10.000	1,599
Panel C: Call and put options													
0–1	0.850	14.89	0.117	0.127	19.63	−0.272	−0.065	0.034	0.099	0.184	0.341	0.687	453
1–2	0.945	17.31	0.238	0.170	27.33	−0.552	−0.020	0.132	0.235	0.327	0.545	0.725	379
2–3	0.990	14.07	0.334	0.162	29.57	−0.083	0.111	0.213	0.319	0.400	0.608	0.901	206
4–6	0.992	18.77	0.456	0.167	52.01	−0.118	0.181	0.354	0.458	0.578	0.715	0.912	364
7–12	0.995	13.49	0.437	0.176	33.86	−0.103	0.154	0.319	0.433	0.565	0.703	1.030	186
All	0.940	35.08	0.289	0.208	55.50	−0.552	−0.006	0.131	0.275	0.434	0.642	1.030	1,588

Notes: The table reports summary statistics on percentage pricing differences assuming the dividend-adjusted SPX index follows a lognormal distribution, corresponding to the Black-Scholes model. The percentage pricing difference is the implied dividend-adjusted index value less the actual dividend-adjusted index value expressed as a percentage of the latter. Options are grouped into maturity bands to allow for deterministically time-varying volatility. A 0–1 month maturity corresponds to options with 7 to 30 days to expiry; 1–2 months corresponds to 31 to 60 days; 2–3 months corresponds to 61 to 90 days, 4–6 months corresponds to 91 to 180 days; and 7–12 months corresponds to over 180 days. Panel A reports results for call options, Panel B reports results for put options, and Panel C reports results for calls and puts combined. In Panels A and B the number of transactions on any day for a given maturity for inferring the index value and volatility must exceed six. The corresponding restriction in Panel C is at least five contracts for each option type. The summary statistics for percentage pricing differences are the proportion positive and associated *Z*-statistic for testing the null hypothesis that the proportion positive is 0.5; the mean, standard deviation and associated *t*-statistic; and the minimum, the 5th percentile, the 1st quartile, the median, the 3rd quartile, the 95th percentile (for calls and puts separately), and the maximum. The final column gives the number of observations on the implied index value within each maturity band over the sample period.

est maturity to 1.22% for the longest maturity. Longstaff (1995) does not provide results for separate maturity bands. However, he does report (Longstaff, 1995, Table 3) a regression coefficient for percentage pricing differences on time to maturity of 0.01421. Assuming a 20-day trading month suggests increases in percentage pricing differences are actually higher in Longstaff's study than here, although his range of maturities is shorter.[8] Aparicio and Hodges (1996) report a similar maturity effect to that found in the current study for CME options on the S&P 500 index futures over the post-Crash period until the end of 1992.

Overall, these figures are consistent with the lognormal distribution assumption of the Black-Scholes model not offering an accurate description of the SPX index distribution.

Longstaff was unable to examine put options on the S&P 100 index because of their early exercise feature. However, the put-call parity relation implies equivalent pricing errors for calls and puts with equal strikes and maturities. Panel B of Table II confirms this reasoning. Although there is greater dispersion in percentage pricing differences implied by put options, the results still strongly reject the martingale restriction in favor of positive pricing differences. The proportion of positive differences is 85.7% with a Z-statistic of 28.53. The median difference is 0.495, which is very similar to the value for calls. Again, mean and median percentage pricing differences increase monotonically with maturity, and the highest t-statistics occur in the longer maturities. Aparicio and Hodges (1996) find comparable results in their study.

Finally, Panel C of Table II reports summary statistics for percentage pricing differences for different times to expiry, combining calls and puts in calculating the implied index value and implied volatility. There are two main results. First, for all maturities, the median and mean percentage pricing differences are significantly lower for calls and puts combined than for calls or puts separately. Across all observations, the median percentage pricing difference is 0.275 and the mean is 0.289. Second, the proportion of positive pricing differences remains high at 94% with the associated Z-statistic being a highly significant 35.08.[9]

[8]The maximum maturity in Longstaff's study is 55 (trading) days with a mean and median maturity of 28 days.

[9]Neither result is affected by the different sets of options employed in the three panels. For example, using the same 1588 observations from Panel C for calls gives overall results of 99.56% positive pricing differences with a Z-statistic of 39.50. For puts, the figures are 87.59% positive with a Z-statistic 29.96. The major difference between the results is that for longer maturity puts, the implied index value from a small number of sets of options with less than seven available prices, gives a percentage pricing difference that hits the imposed upper limit of 10%.

TABLE III

Percentage Pricing Differences Assuming a Generalized Lognormal Distribution

Maturity (months)	*Proportion positive*	*Z-stat*	*Mean*	*Standard deviation*	*t-stat*	*Min*	*5%*	*Q1*	*Median*	*Q3*	*95%*	*Max*	*N*
Panel A: Call options													
0–1	0.834	15.43	0.046	0.216	4.94	−1.255	−0.468	0.029	0.081	0.139	0.266	0.517	535
1–2	0.794	13.04	0.080	0.316	5.57	−2.665	−0.395	0.035	0.111	0.203	0.370	2.239	491
2–3	0.721	7.67	0.077	0.292	4.59	−1.543	−0.475	−0.025	0.121	0.240	0.428	1.168	301
4–6	0.789	13.99	0.150	0.364	9.98	−2.070	−0.440	0.034	0.179	0.342	0.608	1.365	584
7–12	0.627	3.67	0.103	0.607	2.46	−1.880	−0.777	−0.136	0.082	0.287	1.262	2.282	209
All	0.776	25.41	0.093	0.348	12.25	−2.665	−0.475	0.020	0.108	0.226	0.486	2.282	2,120
Panel B: Put options													
0–1	0.434	−2.63	0.088	0.483	3.61	−0.647	−0.203	−0.085	−0.019	0.109	0.653	7.667	392
1–2	0.389	−4.23	−0.039	0.421	−1.77	−3.717	−0.524	−0.206	−0.050	0.095	0.693	1.661	361
2–3	0.346	−4.81	−0.150	1.345	−1.74	−10.000	−0.747	−0.209	−0.072	0.060	0.333	10.000	243
4–6	0.282	−9.05	−0.354	1.248	−5.89	−10.000	−1.326	−0.528	−0.205	0.041	0.746	2.537	432
7–12	0.170	−8.64	−0.498	0.835	−7.79	−2.983	−1.418	−0.822	−0.626	−0.332	0.825	4.038	171
All	0.339	−12.88	−0.159	0.952	−6.68	−10.000	−0.913	−0.308	−0.075	0.071	0.684	10.000	1,599
Panel C: Call and put options													
0–1	0.664	7.00	0.040	0.104	8.19	−0.361	−0.138	−0.021	0.036	0.100	0.227	0.390	453
1–2	0.623	4.78	0.030	0.126	4.69	−0.470	−0.176	−0.047	0.029	0.120	0.228	0.366	379
2–3	0.636	3.90	0.041	0.118	4.95	−0.342	−0.133	−0.040	0.043	0.107	0.241	0.357	206
4–6	0.643	5.45	0.034	0.145	4.51	−0.453	−0.221	−0.062	0.037	0.124	0.274	0.420	364
7–12	0.500	0.00	0.000	0.165	−0.01	−0.459	−0.286	−0.104	0.000	0.116	0.258	0.379	186
All	0.627	10.09	0.032	0.130	9.78	−0.470	−0.185	−0.045	0.033	0.112	0.240	0.420	1,588

Notes: The table reports summary statistics on percentage pricing differences assuming the dividend-adjusted SPX index follows a generalized lognormal distribution, allowing for nonnormal skewness and kurtosis. The percentage pricing difference is the implied dividend-adjusted index value less the actual dividend-adjusted index value expressed as a percentage of the latter. Options are grouped into maturity bands to allow for deterministically time-varying volatility. A 0–1 month maturity corresponds to options with 7 to 30 days to expiry; 1–2 months corresponds to 31 to 60 days; 2–3 months corresponds to 61 to 90 days; 4–6 months corresponds to 91 to 180 days; and 7–12 months corresponds to over 180 days. Panel A reports results for call options, Panel B reports results for put options, and Panel C reports results for calls and puts combined. In Panels A and B the number of transactions on any day for a given maturity for inferring the index value and volatility must exceed six. The corresponding restriction in Panel C is at least five contracts for each option type. The summary statistics for percentage pricing differences are the proportion positive and associated *Z*-statistic for testing the null hypothesis that the proportion positive is 0.5; the mean, standard deviation and associated *t*-statistic; and the minimum, the 5th percentile, the 1st quartile, the median, the 3rd quartile, the 95th percentile (for calls and puts separately), and the maximum. The final column gives the number of observations on the implied dividend-adjusted index value within each maturity band over the sample period.

The Generalized Distribution

Table III reports the results from assuming the SPX index follows a generalized distribution. Panel A shows that the proportion of positive percentage pricing differences for calls alone, although remaining significantly greater than 0.5, is significantly lower than reported in Table II for the Black-Scholes model. The median percentage pricing difference is 4.5 times lower than the median difference in Table II, and even the 95th percentile of the distribution of pricing differences is below the median pricing differences for the Black-Scholes model. For puts, Panel B shows that cases where the implied index value is below the actual index value are now significantly greater than cases where the reverse occurs. The median percentage pricing difference over all maturities is now -0.075, which in absolute terms is 6.6 times less than the corresponding value in Table II.[10] For puts with maturities of 31 to 60 days and 61 to 90 days the t-statistics for a mean value of zero are not significant. Finally, testing the martingale restriction on puts and calls combined, the median percentage pricing difference over all maturities is 0.033, with similar values within maturity bands up to 180 days; for the longest maturity options the median difference is zero, with an equal number of negative and positive errors. The overall median percentage pricing difference of 0.033 compares with a value of 0.275 in Table II, and with values reported by Longstaff of 0.410 for the Black-Scholes model and 0.311 for the general option pricing model (Longstaff, 1995, Tables 2 and 6). The mean value of 0.032 is statistically significant (t-statistic of 9.78), but economically insignificant. For example, taking the mean *percentage* pricing difference of 0.032 and an average index value over the sample period of around 400, a maximum option delta of 1.0 translates into an effect of 0.128 on option values. This is equivalent to one fifth of the mean option spread over the period. Further evidence of the economic insignificance of the percentage pricing error comes from combining the median (mean) value of 0.033 (0.032) with the average index value, giving a figure of 0.132 (0.128) index points, and comparing this with the overall mean value for the difference between the maximum and minimum SPX index values across different strike prices in option-maturity sets of 0.162.

Looking at median pricing errors for calls alone (Panel A) or calls and puts combined (Panel C) also shows that the positive time-to-maturity biases apparent in Table II disappear under the generalized distribution.

[10]Repeating the analysis on put options after deleting 25 outlying observations, results in a maximum percentage pricing difference of 2.42 and a median percentage pricing difference almost unchanged at -0.070.

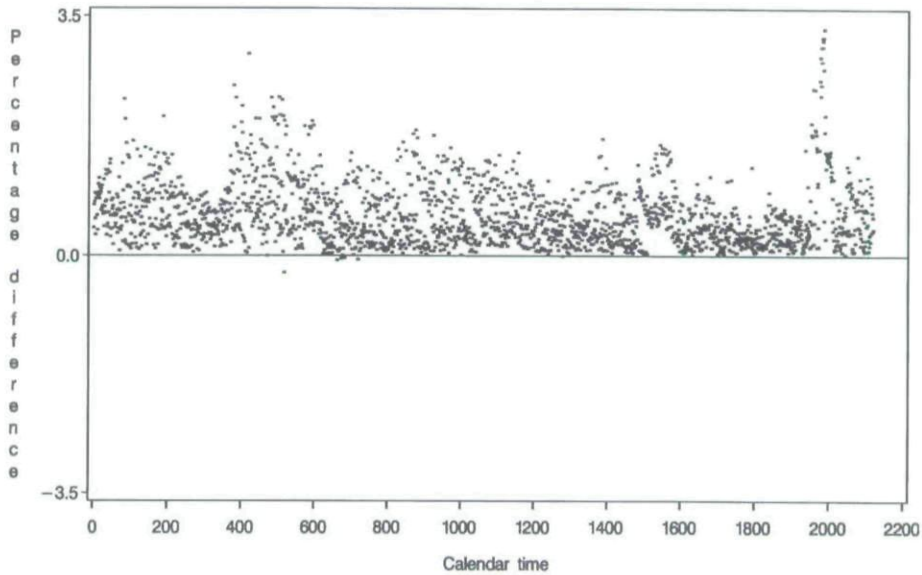

FIGURE 1
Time series of percentage differences between the implied value of the S&P index and the actual value of the index net of dividends: call options under the lognormal distribution.

Figures 1 and 2 graph the time series of percentage pricing errors associated with call option implied index values, assuming the lognormal and generalized distributions. The horizontal scale numbers the sample days in calendar time. Figure 1 provides a clear visual representation of the systematic positive bias in the implied index value under the lognormal assumption. By contrast, Figure 2 shows that the generalized distribution produces pricing errors more randomly and densely distributed around zero. Figures 3 and 4 show the corresponding pricing errors plotted against time to maturity. For the lognormal distribution Figure 3 shows a clear positive time-to-maturity bias, whereas there is no obvious expiration bias in Figure 4. Corresponding plots for calls and puts combined (not shown) tell a similar story.

Testing the Martingale Restriction with S&P 500 Index Futures

One possible criticism of the previous analysis is its dependence on assumptions about dividends. Although Longstaff (1995) reports the rejection of the martingale restriction on S&P 100 index options is not due to dividend uncertainty, there is a simple way to check for this in the case

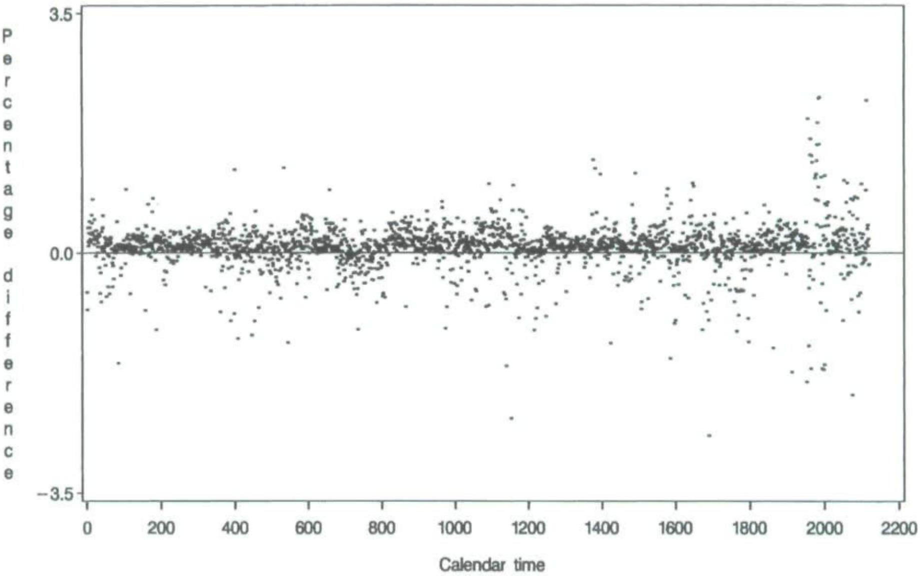

FIGURE 2
Time series of percentage differences between the implied value of the S&P index and the actual value of the index net of dividends: call options under the generalized distribution.

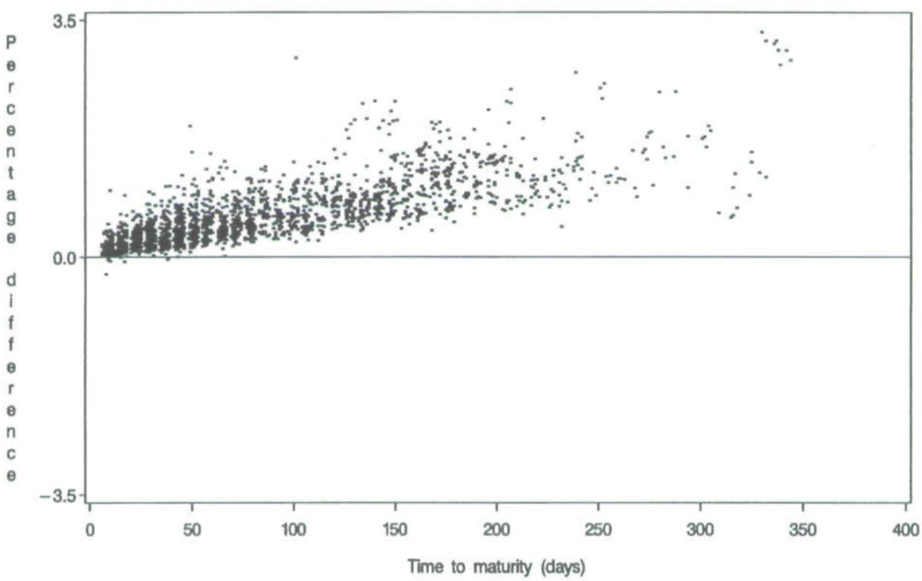

FIGURE 3
Percentage differences between the implied value of the S&P index and the actual value of the index net of dividends against time to maturity: call options under the lognormal distribution.

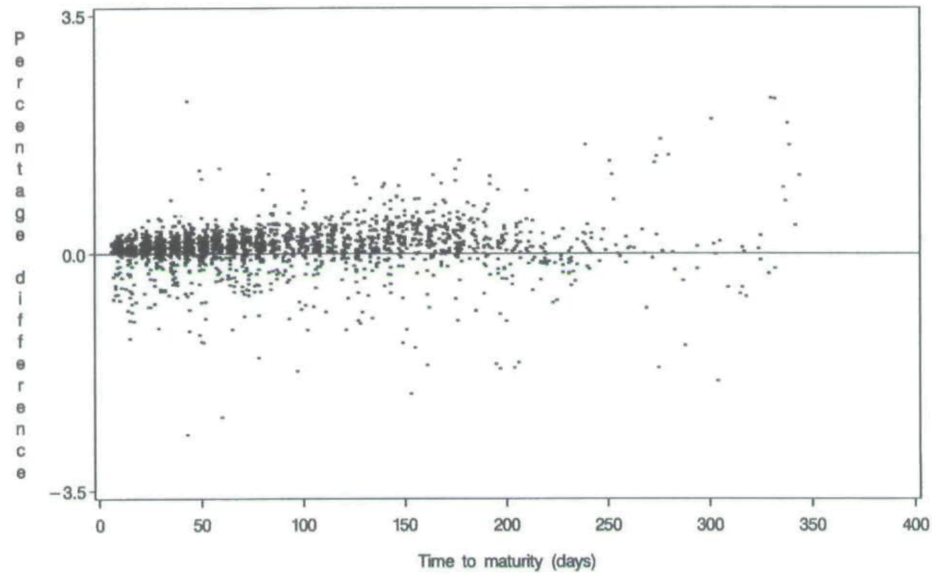

FIGURE 4
Percentage differences between the implied value of the S&P index and the actual value of the index net of dividends against time to maturity: call options under the generalized distribution.

of S&P 500 index options.[11] European S&P 500 index options can be considered as options on the S&P 500 futures contract. Traders use the futures contract to hedge index options. Therefore, implied index values can be compared directly with index futures prices dispensing with the need for any dividend adjustments. Table IV reports mean percentage pricing differences relative to S&P 500 index futures prices. The expiry dates of the S&P 500 index futures contract and the pattern of trading mean the analysis is restricted to maturities up to three months, and the numbers of observations within maturity bands and in total is less than in the previous analysis.[12] The mean percentage pricing differences from the previous analysis for the cash index level, but restricted to the same restricted set of option quotations are shown for comparison purposes.

Columns 2–4 of Table IV, labeled cash, compare the implied index level with the actual S&P 500 cash index level. Columns 5–7, labeled futures, compare the implied index level with the actual futures price

[11]A referee helpfully suggested this approach.

[12]A further complication is that, as pointed out by Dumas et al. (1998) SPX options expired at the market close until 24 August 1992, and expired at the market open thereafter. The S&P 500 index futures contract expires at the market open. Restricting the analysis to the period since 24 August 1992 gives similar figures to those reported in Table IV.

TABLE IV

Percentage Pricing Differences Relative to S&P 500 Index Futures

	Cash			Futures		
Maturity (months)	*Calls*	*Puts*	*Calls and Puts*	*Calls*	*Puts*	*Calls and Puts*
Panel A: Lognormal distribution						
0–1	0.193	0.283	0.101	0.139	0.265	0.062
	(22.441)	(4.411)	(12.771)	(15.116)	(4.171)	(11.614)
1–2	0.413	0.538	0.252	0.318	0.457	0.170
	(24.851)	(9.987)	(18.803)	(21.969)	(8.805)	(19.254)
2–3	0.596	0.829	0.382	0.461	0.755	0.269
	(31.972)	(6.972)	(27.927)	(25.610)	(6.475)	(28.382)
0–3	0.404	0.558	0.237	0.309	0.498	0.161
	(36.357)	(11.255)	(27.830)	(31.183)	(10.297)	(27.150)
Panel B: Generalized lognormal distribution						
0–1	0.055	0.095	0.033	0.000	0.077	−0.006
	(3.886)	(1.751)	(4.329)	(0.032)	(1.443)	(−1.265)
1–2	0.110	−0.020	0.046	0.015	−0.101	−0.037
	(6.977)	(−0.623)	(5.040)	(0.979)	(−3.160)	(−7.436)
2–3	0.109	−0.194	0.055	−0.026	−0.267	−0.057
	(6.089)	(−1.609)	(5.606)	(−1.458)	(−2.187)	(−9.350)
0–3	0.092	−0.043	0.044	−0.003	−0.102	−0.032
	(9.891)	(−0.949)	(8.638)	(−0.342)	(−2.220)	(−10.267)

Notes: The table reports mean percentage pricing differences relative to S&P 500 index futures data. Results are for calls, puts, and calls and puts combined. For comparison purposes, percentage pricing differences are presented for the cash index (cash) as well as index futures prices (futures), for a common set of option quotations. Figures in parenthesis are t-values. The number of observations within maturity bands ranges from 192 to 210 for calls with a total number of observations of 602. Corresponding numbers for puts are a range of 153 to 176 observations, with a total number of 497, and for calls and puts combined are a range of 141 to 173 with a total number of 482. Panel A gives results assuming the index follows a lognormal distribution; Panel B gives results assuming the index follows a generalized lognormal distribution. The numbers of observations are identical for both panels.

discounted by the appropriate Treasury bill rate.[13] For each option, the actual futures price is taken as the average across all transaction prices in the five-minute window 14.00–14.05. Panel A gives results assuming the lognormal distribution, Panel B gives results assuming the generalized lognormal distribution.

The results in Table IV confirm the analysis in Tables II and III. Columns 2–4 of Panels A and B give figures that are close to those in the first three rows of the panels of Tables II and III, showing that the sample of option quotations used in this analysis is representative of the original sample. Comparing corresponding columns for the cash index and the futures contract shows that the percentage pricing differences are slightly lower for the latter, but the overall effect is the same. The pattern of percentage pricing differences increasing with maturity still holds. More important, the results in columns 5–7 of Panels A and B give comparable reductions in mean percentage pricing differences in moving from the lognormal distribution in Panel A to the generalized lognormal distribution in Panel B for both cash and futures. For example, moving to a generalized lognormal distribution reduces the percentage pricing difference from 0.309 to −0.003 for the sample of calls and all maturities. Overall, these results confirm the previous analysis and suggest our treatment of dividends is appropriate.

SIMULATING THE BLACK-SCHOLES MODEL UNDER NONNORMAL HIGHER MOMENTS

One method of examining the effect on implied index values from assuming a lognormal distribution when the underlying asset distribution is not lognormal is to simulate a set of option prices for alternative values of skewness and kurtosis. Aparicio and Hodges (1996) show the effect of similar simulations on the value of the Black-Scholes implied volatility. This type of analysis offers some useful insights, although it should be interpreted as providing illustrative results, rather than as providing a precise calibration of pricing errors under the Black-Scholes model. For example, alternative choices for the range of strike prices change the magnitudes of the pricing bias, although the general pattern is unaffected. In addition, the alternative approaches used in this section to model the underlying asset distribution can only provide an approximation to the true underlying asset distribution.

[13]By the cost of carry formula, $F = (S - PVD)e^{rt}$, where F is the futures price. Columns 5–7 therefore compare the implied index level directly with Fe^{-rt}.

TABLE V

Simulated Implied Mean of the Risk-Neutral Distribution

Skewness	*Kurtosis*	*Calls*	*Puts*	*Calls and Puts*
−2.00	3.0	0.701	1.206	0.221
−1.75	3.0	0.631	1.018	0.193
−1.50	3.0	0.556	0.842	0.165
−1.25	3.0	0.476	0.678	0.137
−1.00	3.0	0.392	0.525	0.110
−0.75	3.0	0.303	0.381	0.082
−0.50	3.0	0.209	0.246	0.055
−0.25	3.0	0.108	0.120	0.027
−2.00	5.0	0.689	0.999	0.221
−1.75	5.0	0.624	0.830	0.193
−1.50	5.0	0.554	0.673	0.164
−1.25	5.0	0.481	0.526	0.135
−1.00	5.0	0.403	0.389	0.107
−0.75	5.0	0.321	0.259	0.079
−0.50	5.0	0.233	0.138	0.051
−0.25	5.0	0.139	0.024	0.023

Notes: The table gives values for the implied mean of the risk-neutral distribution as a percentage difference from the true underlying index value of 400. Option prices are calculated assuming a generalized lognormal distribution for moneyness ($\ln(Se^{rt}/X)$) ranging from -0.05 to $+0.05$ in increments of 0.01, assuming an underlying index value of 400, a riskless rate of 5%, volatility of 15%, a time to expiry of 1 month, and alternative values for skewness and kurtosis. The implied means and variances are then calculated, under the Black-Scholes lognormal assumption of zero skewness and zero excess kurtosis, by minimizing the sum of squared errors between the Black-Scholes model price and the "true" simulated option price.

The procedure is as follows. Assume an index value of 400 with no dividends, a riskless rate of 5 percent, a volatility of 15%, and a maturity of one month. A range of negative skewness values and kurtosis values are chosen as representative of the expected range of values for the SPX index. The true option prices are then calculated assuming a generalized lognormal underlying asset distribution for a range of strike prices chosen to give moneyness values from -0.05 to $+0.05$ in 0.01 increments. The Black-Scholes model is fitted to the true option prices, given the true riskless rate, maturity, and moneyness, and the implied index value and implied volatility are solved to minimize the sum of squared errors between the Black-Scholes prices and the true option prices, as in eq. (7).

Table V gives the implied values for the mean of the risk-neutral distribution as a percentage difference from the true underlying index value for two values of kurtosis and a range of negative skewness values. The results show positive pricing differences for all values of negative skewness and kurtosis. For a skewness value of -2 the maximum percentage pricing difference is 0.70 for calls, 1.21 for puts, and 0.22 for calls and puts combined. For skewness and kurtosis values most closely matching estimated moments for the actual sample period (skewness of -1.25 and kurtosis of 5), the percentage pricing differences are 0.48 (calls), 0.53 (puts), and 0.14 (calls and puts). These produce a similar pattern to those given in the previous section using actual data.

To corroborate these results, a similar exercise was conducted assuming a mixture of two lognormal distributions generates the underlying index values.[14] The mixture of lognormals was chosen to give the same range of skewness and kurtosis values as for the generalized lognormal distribution. For the same index value, interest rate, volatility, and range of strikes, this resulted in a similar pattern of positive pricing biases. For example, corresponding to a skewness value of -1.25 and kurtosis of 5, the percentage pricing differences were 0.51 (calls), 0.59 (puts), and 0.15 (calls and puts).

A final simulation examines the effect of time to expiry on the percentage pricing difference. This repeats the analysis producing Table V except skewness is fixed at -1.25 and kurtosis at 5.0 and alternative maturity dates are examined. The implied index value and implied volatility are then backed out of the Black-Scholes model. The text reports on these results. The results show that the percentage pricing difference increases monotonically with time to expiry for both calls alone and puts alone. For calls and puts combined the percentage pricing difference increases at first but then falls at longer maturities. At a maturity of three (six) months, the percentage pricing differences are 0.94 (1.34) for calls, 1.43 (2.26) for puts, and 0.13 (0.11) for calls and puts. These maturity effect results are also consistent with those given in the previous section.

SUMMARY AND CONCLUSIONS

A fundamental requirement of no-arbitrage option pricing models is that they satisfy the martingale restriction: the value of the underlying asset implied by option prices equals its market value. Crucial to any test of the martingale restriction is an appropriate option pricing model. There is no reason to expect the restriction to be satisfied if the data fails to satisfy the assumptions of the chosen option pricing model. Fitting the Black-Scholes model to S&P 500 index options over the period from 1990 to 1994 results in a clear rejection of the joint hypothesis that the martingale restriction and the Black-Scholes model fit the data. The Black-Scholes assumption that the S&P index follows a lognormal distribution

[14]For previous applications of this approach see Ritchey (1990), Melick and Thomas (1997), and Bahra (1997).

produces a systematic, positive pricing error between the implied and actual index values, accompanied by a positive time-to-maturity bias in the pricing error. Graphical representations provide clear visual evidence of these biases.

By contrast, assuming a generalized distribution for the S&P 500 index, and solving for the first four implied moments of the distribution, dramatically reduces the biases from the Black-Scholes model. Inferring these parameters from combined call and put options in particular produces pricing errors that are economically insignificant. Simulations using theoretical option prices generated from distributions representative of the volatility, skewness, and kurtosis of the S&P 500 index confirm the pricing biases of the Black-Scholes model. The conclusion of this study is that the assumption of no-arbitrage should not be jettisoned, but that it offers a good working assumption when used with an appropriate model of the underlying asset's distribution.

BIBLIOGRAPHY

Aparicio, S. D., & Hodges, S. (1996). Martingale restriction tests of option pricing models and their interpretation. Working paper, University of Warwick.

Backus, D., Foresi, S., Li, K., & Wu, L. (1997). Accounting for biases in Black-Scholes. Working paper, New York University.

Bahra, B. (1997). Implied risk-neutral probability density functions from option prices: theory and application. Working paper no. 66, Bank of England.

Bates, D. S. (1996). Testing option pricing models. In G. S. Maddala & C. R. Rao (Eds.), Handbook of statistics 14: Statistical methods in finance, The Netherlands: Elsevier Science.

Cont, R. (1997). Beyond implied volatility. In J. Kertesz & I. Kondor (Eds.), Econophysics, Dordrecht: Kluwer.

Corrado, C. J., & Su, T. (1995). Skewness and kurtosis in S&P 500 index returns implied by option prices. Journal of Financial Research, 19 (Summer), 175–192.

Corrado, C. J., & Su, T. (1996). S&P 500 index option tests of Jarrow and Rudd's approximate option valuation formula. The Journal of Futures Markets, 16, 611–629.

Cox, J., & Ross, S. A. (1976). The valuation of options for alternative stochastic processes. Journal of Financial Economics, 3, 145–166.

Diz, F., & Finucane, T. J. (1993). Do the options markets really overreact? The Journal of Futures Markets, 13, 298–312.

Dumas, B., Fleming, J., & Whaley, R. E. (1998). Implied volatility functions: empirical tests. Journal of Finance, 53 (December), 2059–2106.

Dybvig, P. H., & Ross, S. A. (1987). Arbitrage. In J. Eatwell, M. Milgate, & P. Newman (Eds.), The new Palgrave: Finance. London: Macmillan.

Harrison, J. M., & Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, 20, 381–408.

Harvey, C. R., & Whaley, R. E. (1992). Market volatility prediction and the efficiency of the S&P 100 index option market. Journal of Financial Economics, 31, 43–74.

Jarrow, R., & Rudd, A. (1982). Approximate option valuation for arbitrary stochastic processes. Journal of Financial Economics, 10 (November), 347–369.

Longstaff, F. A. (1995). Option pricing and the martingale restriction. Review of Financial Studies, 8 (Winter), 1091–1124.

Melick, W. R., & Thomas, C. P. (1997). Recovering an asset's implied PDF from option prices: an application to crude oil prices during the Gulf crisis. Journal of Financial and Quantitative Analysis, 32 (March), 91–115.

Ritchey, R. J. (1990). Call option valuation for discrete normal mixtures. Journal of Financial Research, 13(4), 285–296.

Rubinstein, M. (1994). Implied binomial trees. Journal of Finance, 69 (July), 771–818.

Stein, J. C. (1989). Overreactions in options markets. Journal of Finance, 44 (September), 1011–1023.